Solutions Lecture 11 and Lecture 15

Exercise 1

a) The left-upper block of the given LMI implies

$$X - A^T XA + \frac{1}{\gamma} C^T C \prec 0.$$

If $Ax = \lambda x$ with some complex $x \neq 0$, we infer

$$x^* Xx - \bar{\lambda}\lambda x^* Xx + \frac{1}{\gamma}\|Cx\|^2 < 0.$$

Since $x^* Xx > 0$, this implies

$$1 - |\lambda|^2 < 0$$

and hence $|\lambda| < 1$. Hence we have proved that all eigenvalues λ of A satisfy $|\lambda| < 1$, which just means that A is Schur.

Now observe that, for any $s \in \mathbb{C}$ not being an eigenvalue of A, we have

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix} \begin{pmatrix} (sI - A)^{-1}B \\ I \end{pmatrix} = \begin{pmatrix} (sI - A)^{-1}B \\ (A(sI - A)^{-1} + I)B \end{pmatrix} =$$
$$\begin{pmatrix} (sI - A)^{-1}B \\ (A + sI - A)(sI - A)^{-1}B \end{pmatrix} = \begin{pmatrix} (sI - A)^{-1}B \\ s(sI - A)^{-1}B \end{pmatrix} = \begin{pmatrix} I \\ sI \end{pmatrix} (sI - A)^{-1}B.$$

Therefore, left- and right-multiplying the LMI with the matrices $\begin{pmatrix} (sI - A)^{-1}B \\ I \end{pmatrix}^*$ and $\begin{pmatrix} (sI - A)^{-1}B \\ I \end{pmatrix}$ implies

$$((sI - A)^{-1}B)^* \begin{pmatrix} I \\ sI \end{pmatrix}^* \begin{pmatrix} -X & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} I \\ sI \end{pmatrix} (sI - A)^{-1}B +$$
$$+ \begin{pmatrix} (sI - A)^{-1}B \\ I \end{pmatrix}^* \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^T \begin{pmatrix} -\gamma I & 0 \\ 0 & \frac{1}{\gamma} I \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \begin{pmatrix} (sI - A)^{-1}B \\ I \end{pmatrix} \prec 0.$$

This reads as

$$((sI - A)^{-1}B)^* (-X + \bar{s}sX) (sI - A)^{-1}B +$$
$$+ \begin{pmatrix} I \\ T(s) \end{pmatrix}^* \begin{pmatrix} -\gamma I & 0 \\ 0 & \frac{1}{\gamma} I \end{pmatrix} \begin{pmatrix} I \\ T(s) \end{pmatrix} \prec 0$$

or as

$$((sI - A)^{-1}B)^* ((1 - |s|^2)X)(sI - A)^{-1}B + \frac{1}{\gamma} T(s)^* T(s) - \gamma I \prec 0.$$

Now choose $s = e^{i\omega}$ for any $\omega \in [0, 2\pi]$. Then $|s| = 1$ and hence

$$\frac{1}{\gamma}T(s)^*T(s) - \gamma I \prec 0$$

and hence

$$\sigma_{\max}(T(s))^2 \leq \gamma^2.$$

Since ω was arbitrary, we infer

$$\|T\|_{\mathscr{H}_\infty} = \sup_{\omega\in[0,2\pi]} \sigma_{\max}(T(e^{i\omega})) < \gamma,$$

where we use the fact that $[0, 2\pi]$ is compact and $\omega \to \sigma_{\max}(T(e^{i\omega}))$ is continuous.

b) We have

$$X \succ 0 \text{ and } \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^T \begin{pmatrix} -X & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^T \begin{pmatrix} -\gamma I & 0 \\ 0 & \frac{1}{\gamma} I \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0$$

iff

$$X \succ 0 \text{ and } \begin{pmatrix} -X & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A^T \\ B^T \end{pmatrix} X \begin{pmatrix} A & B \end{pmatrix} + \frac{1}{\gamma} \begin{pmatrix} C^T \\ D^T \end{pmatrix} \begin{pmatrix} C & D \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\gamma I \end{pmatrix} \prec 0$$

iff

$$X \succ 0 \text{ and } \begin{pmatrix} -X & 0 \\ 0 & -\gamma I \end{pmatrix} + \begin{pmatrix} A^T X \\ B^T X \end{pmatrix} X^{-1} \begin{pmatrix} XA & XB \end{pmatrix} + \frac{1}{\gamma} \begin{pmatrix} C^T \\ D^T \end{pmatrix} \begin{pmatrix} C & D \end{pmatrix} \prec 0$$

iff

$$X \succ 0 \text{ and } \begin{pmatrix} -X & 0 \\ 0 & -\gamma I \end{pmatrix} + \begin{pmatrix} XA & XB \\ C & D \end{pmatrix}^T \begin{pmatrix} X^{-1} & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix} \begin{pmatrix} XA & XB \\ C & D \end{pmatrix} \prec 0.$$

iff

$$X \succ 0 \text{ and } \begin{pmatrix} X & 0 \\ 0 & \gamma I \end{pmatrix} - \begin{pmatrix} XA & XB \\ C & D \end{pmatrix}^T \begin{pmatrix} X^{-1} & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix} \begin{pmatrix} XA & XB \\ C & D \end{pmatrix} \succ 0.$$

iff (Schur and $\gamma > 0$)

$$\left(\begin{array}{cc|cc} X & 0 & XA & XB \\ 0 & \gamma I & C & D \\ \hline A^T X & C^T & X & 0 \\ B^T X & D^T & 0 & \gamma I \end{array}\right) \succ 0.$$

c) For the discrete-time linear system

$$\sigma x = Ax + B_1 d + Bu, \quad e = C_1 x + D_1 d + Eu$$

on $\mathbb{Z}_+$, find the gain F in the state-feedback law $u = Fx$ such that the resulting controlled system

$$\sigma x = (A + BF)x + B_1 d, \quad e = (C_1 + EF)x + D_1 d$$

is Schur and has a discrete-time $\mathscr{H}_\infty$-norm smaller than $\gamma > 0$.

Denote the transfer-matrix of the closed-loop system (as in the lectures) as

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right].$$

Then the spces are satisfied iff there exists some symmetric $\mathcal{X}$ with

$$\left(\begin{array}{cc|cc} \mathcal{X} & 0 & \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ 0 & \gamma I & \mathcal{C} & \mathcal{D} \\ \hline \mathcal{A}^T\mathcal{X} & \mathcal{C}^T & \mathcal{X} & 0 \\ \mathcal{B}^T\mathcal{X} & \mathcal{D}^T & 0 & \gamma I \end{array}\right) \succ 0.$$

Observe that

$$\left(\begin{array}{cc|cc} \mathcal{Y} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & \mathcal{Y} & 0 \\ 0 & 0 & 0 & I \end{array}\right) \left(\begin{array}{cc|cc} \mathcal{X} & 0 & \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ 0 & \gamma I & \mathcal{C} & \mathcal{D} \\ \hline \mathcal{A}^T\mathcal{X} & \mathcal{C}^T & \mathcal{X} & 0 \\ \mathcal{B}^T\mathcal{X} & \mathcal{D}^T & 0 & \gamma I \end{array}\right) \left(\begin{array}{cc|cc} \mathcal{Y} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & \mathcal{Y} & 0 \\ 0 & 0 & 0 & I \end{array}\right) =$$

$$= \left(\begin{array}{cc|cc} \mathcal{Y}^T\mathcal{X}\mathcal{Y} & 0 & \mathcal{Y}^T(\mathcal{X}\mathcal{A})\mathcal{Y} & \mathcal{Y}^T(\mathcal{X}\mathcal{B}) \\ 0 & \gamma I & \mathcal{C}\mathcal{Y} & \mathcal{D} \\ \hline \mathcal{Y}^T(\mathcal{A}^T\mathcal{X})\mathcal{Y} & \mathcal{Y}^T\mathcal{C}^T & \mathcal{Y}^T\mathcal{X}\mathcal{Y} & 0 \\ (\mathcal{B}^T\mathcal{X})\mathcal{Y} & \mathcal{D}^T & 0 & \gamma I \end{array}\right).$$

This formal congruence transformation shows that we arrive at an inequality in the blocks as they are required in the general procedure to step from analysis to synthesis (Lecture 11). Hence the substitutions for state-feedback as given in the lectures lead to the following synthesis inequalities:

$$\left(\begin{array}{cc|cc} Y & 0 & (AY + B_1M) & B_1 \\ 0 & \gamma I & (C_1Y + EM) & D_1 \\ \hline (AY + B_1M)^T & (C_1Y + EM)^T & Y & 0 \\ B_1^T & D_1^T & 0 & \gamma I \end{array}\right) \succ 0.$$

With any solution Y and M, a suitable state-feedback gain is obtained as $F = MY^{-1}$.

Exercise 2

a) Let us define $\eta = x - \xi$ to observe that

$$\dot{\eta} = Ax + Bd - A\xi - L(E\xi - y) = Ae + Bd - LE\xi + LEx + LFd = (A + LE)e + (B + LF)d$$

and

$$z - \hat{z} = Cx + Dd - C\xi = Ce + Dd.$$

b) $A + LE$ is Hurwitz and

$$\left\| \left[\begin{array}{c|c} A+LE & B+LF \\ \hline C & D \end{array} \right] \right\|_\infty < \gamma$$

iff there exists some X with

$$X \succ 0, \quad \begin{pmatrix} (A+LE)^T X + X(A+LE) & X(B+LF) & C^T \\ (B+LF)^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix} \prec 0.$$

c) The linearizing change of variables $\hat{L} = XL$ leads to the synthesis inequalities

$$X \succ 0, \quad \begin{pmatrix} (XA+\hat{L}E)^T + (XA+\hat{L}E) & XB+\hat{L}F & C^T \\ (XB+\hat{L}F)^T & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix} \prec 0$$

which are, obviously, affine in X and $\hat{L}$ (and also γ). After having found X, $\hat{L}$, a suitable observer gain is obtained by $L = X^{-1}\hat{L}$.

Exercise 3

Due to the particular structure of the system description, the closed-loop matrices are given by

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \left(\begin{array}{cc|c} A+BD_KC & 0 & B_1 \\ B_KC & A_K & B_KF \\ \hline C_1 - D_KC & -C_K & D_1 - D_KF \end{array} \right).$$

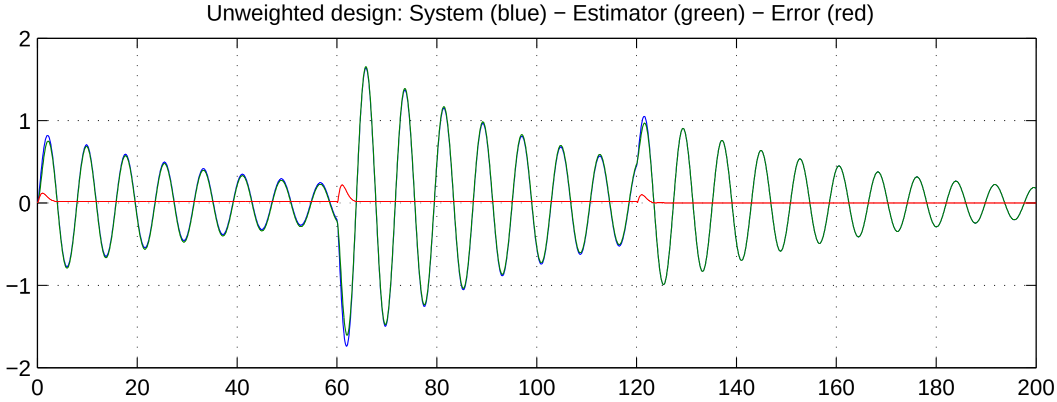
Hence stability of $\mathcal{A}$ implies stability of A_K which just means that the estimator is stable.

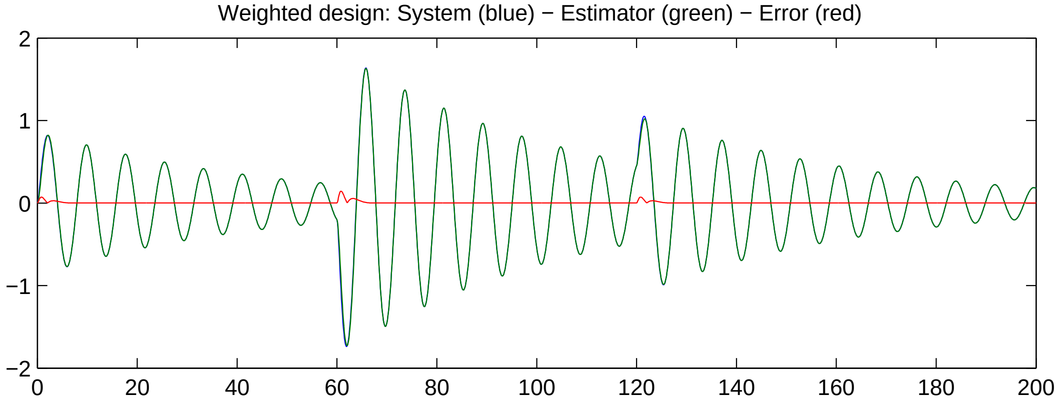
```
%Assumes that the following matrices are given:
%A  B1  B
%C1 D1  E
%C  F   0
%where B=0 and E=-I
ga=sdpvar(1,1);
X=sdpvar(n,n);
Y=sdpvar(n,n);
K=sdpvar(n,n);
L=sdpvar(n,ny);
M=sdpvar(nz,n);
N=sdpvar(nz,ny);
Av=[A*Y+B*M A+B*N*C;K X*A+L*C];
Bv=[B1+B*N*F;X*B1+L*F];
Cv=[C1*Y+E*M C1+E*N*C];
Dv=D1+E*N*F;
H=[Av+Av' Bv Cv';Bv' -ga*eye(nd) Dv';Cv Dv -ga*eye(nz)];
```

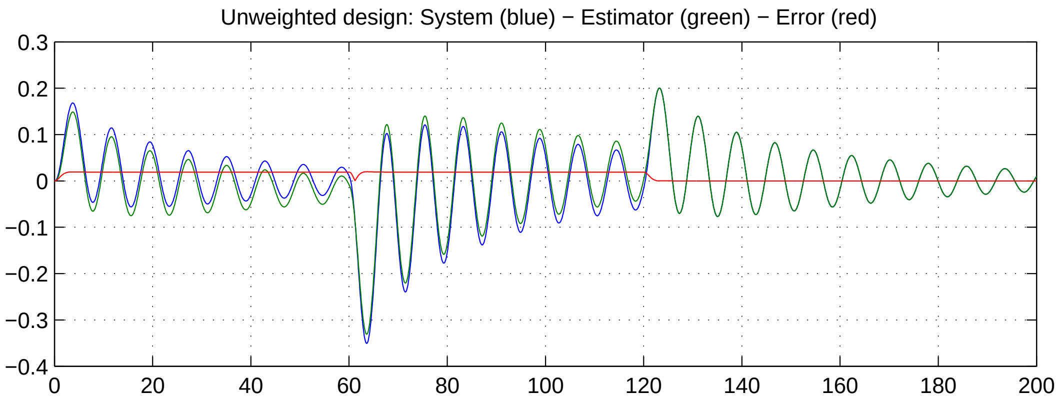
```
lmi=set([]);
lmi=lmi+set(H<0);
lmi=lmi+set([Y eye(n);eye(n) X]>0);
solvesdp(lmi,ga,sdpsettings('solver','sedumi'));
gan=double(ga)
Yv=double(Y);
Kv=double(K);
Mv=double(M);
Nv=double(N);
Xv=double(X);
Lv=double(L);
U=Xv;
V=inv(Xv)-Yv;
H1=inv([U Xv*B;zeros(nz,n) eye(nz)]);
H2=inv([V' zeros(n,ny);C*Yv eye(ny)]);
Mest=H1*[Kv-Xv*A*Yv Lv;Mv Nv]*H2;
est=ss(Mest(1:n,1:n),Mest(1:n,n+1:end),Mest(n+1:end,1:n),Mest(n+1:end,n+1:end));
```

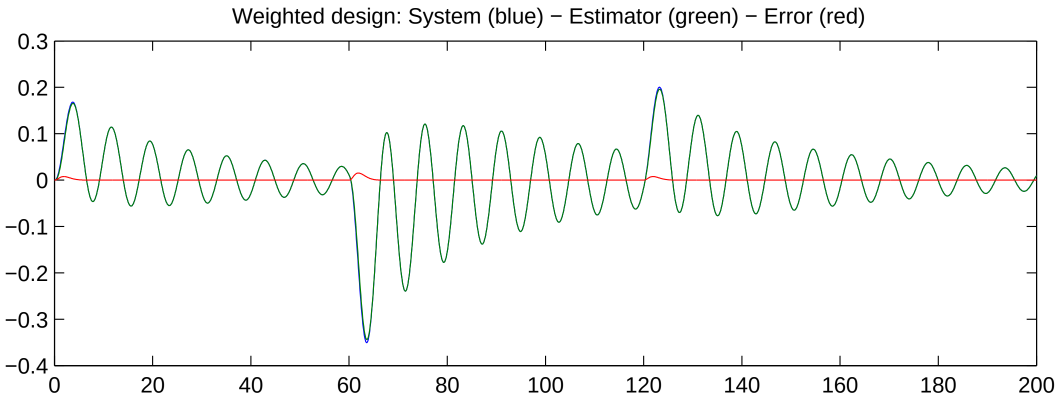
Exercise 4

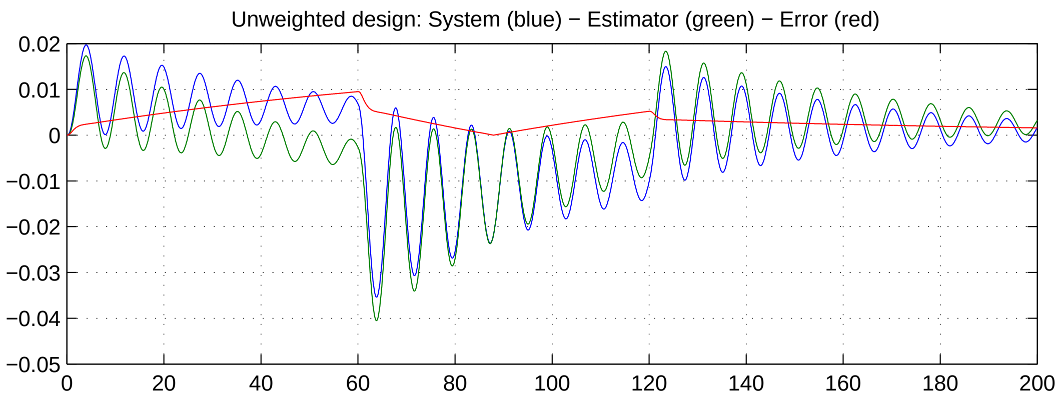
Here are the system's and estimator's outputs for the three parameters for the unweighted (part b)) and the weighted (part c)) synthesis:

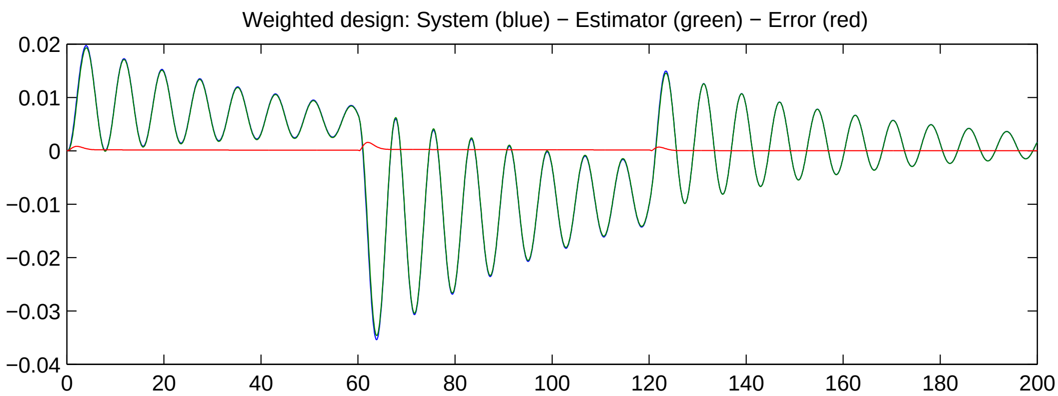




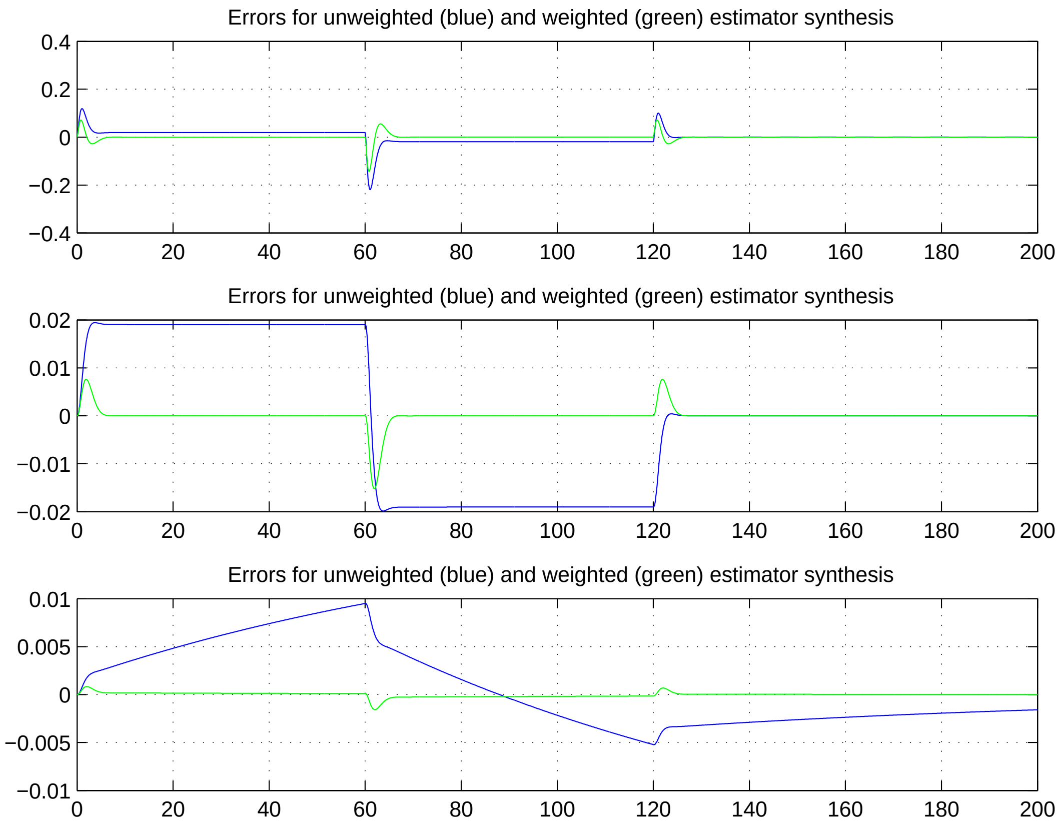






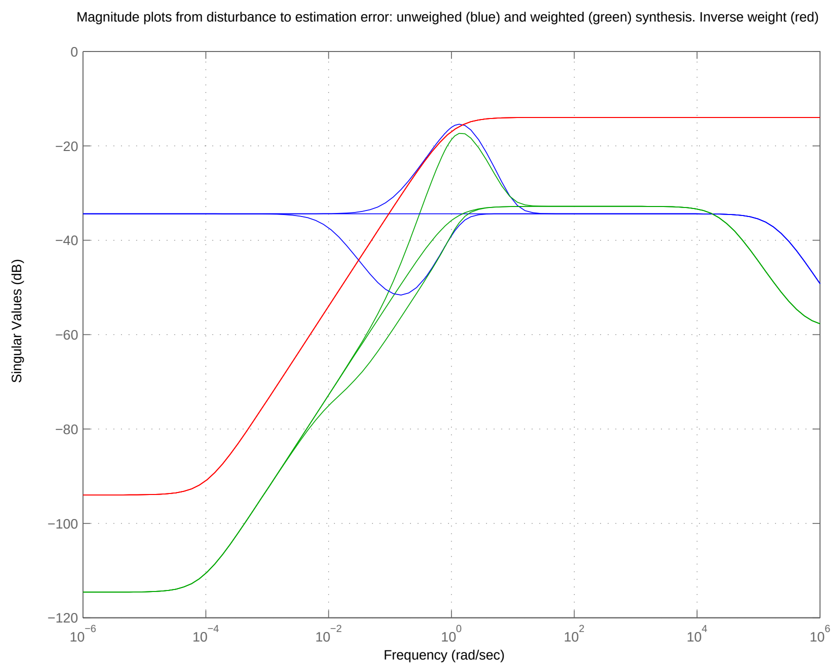


Let us also plot the estimation errors on a larger scale:



Discussion: The nominal design is rather sensitive with respect to variations in the uncertain parameter. Moreover, the unweighted design suffers from a steady-state error, which is taken care of in the weighted design. Still the step inputs incur some impulsive error behavior which is undesirable.

For those who understand frequency responses, let us consider the Bode magnitude plots of the transfer functions from d_1 to e for the different parameter values:

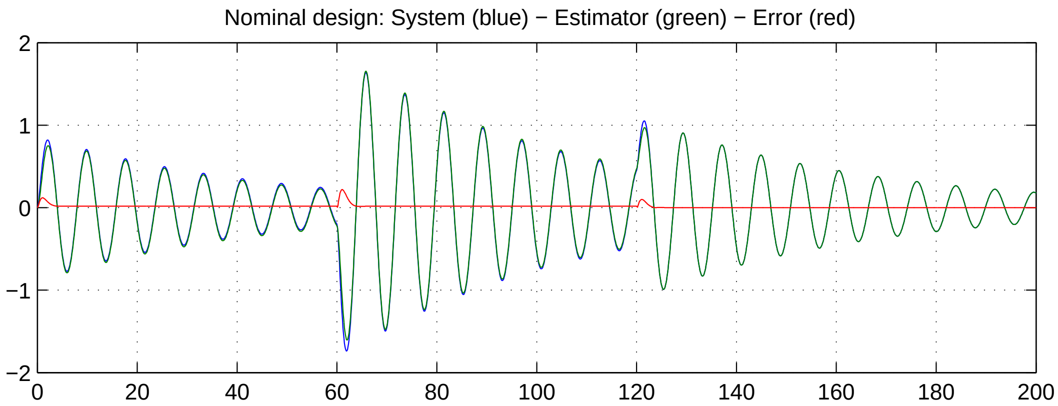


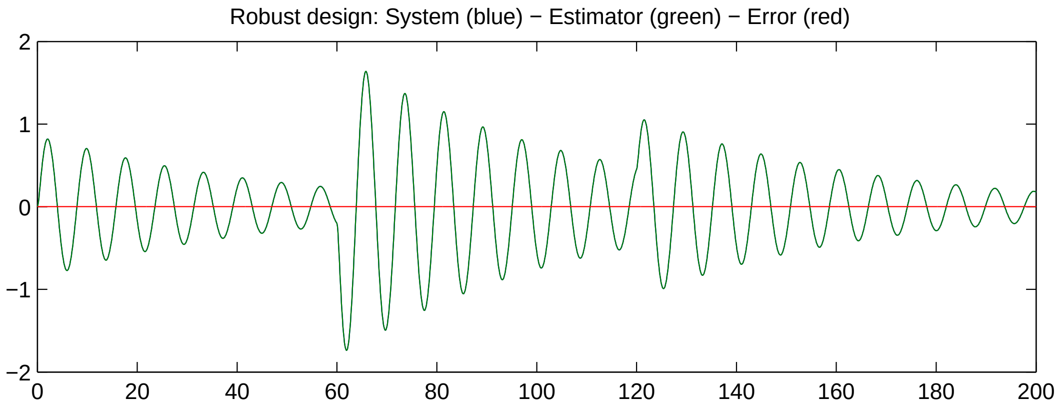
Exercise 5

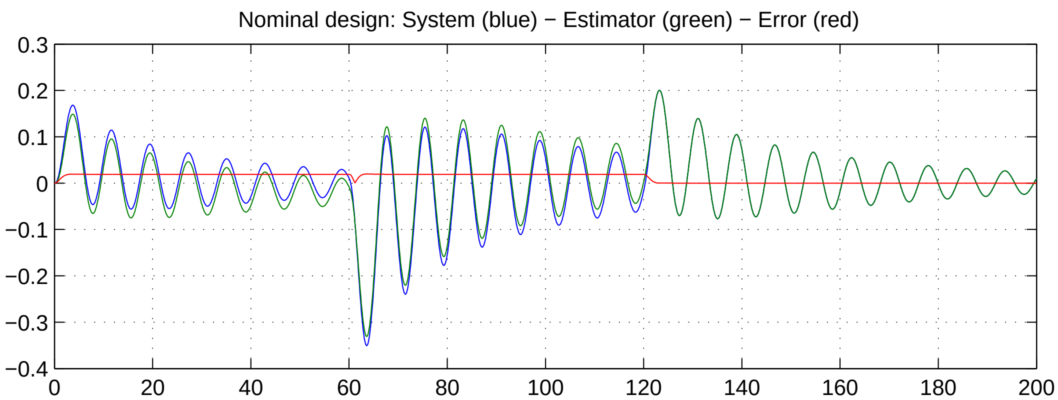
The code for robust synthesis is

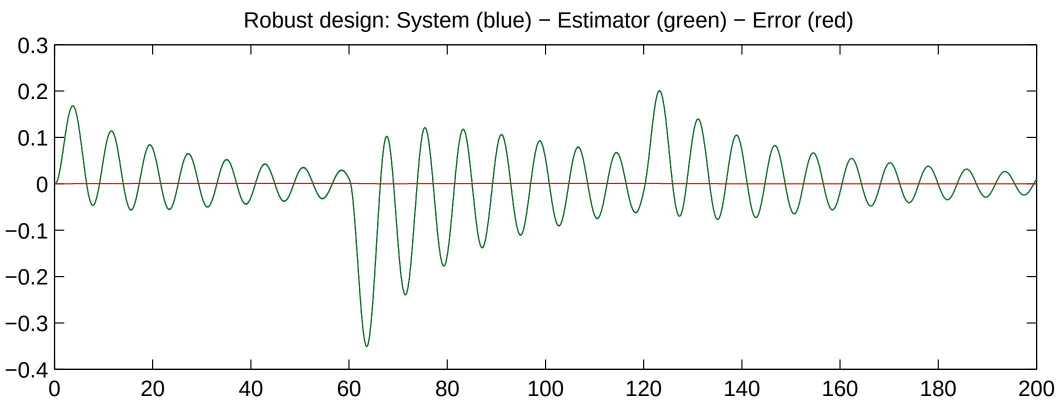
```
%Suppose we are given
%A  B1  B2
%C1 D1  D12
%C2 D21 D2
%C  F1  F2
X=sdpvar(n,n);
Yh=sdpvar(n,n);
Kh=sdpvar(n,n);
L=sdpvar(n,ny);
Mh=sdpvar(nz,n);
N=sdpvar(nz,ny);
Av=[Yh*A Yh*A;Kh X*A+L*C];
Bv=[Yh*B1 Yh*B2;X*B1+L*F1 X*B2+L*F2];
Of1=[zeros(nu,n+n) eye(nu) zeros(nu,nd);C1 C1 D1 D12];
Of2=[zeros(nd,n+n) zeros(nd,nu) eye(nd)];
Of3=[C2-Mh C2-N*C D21-N*F1 D2-N*F2];
ga=sdpvar(1,1);
P1=sdpvar(nu,nu,'full');
P=[-P1-P1' P1-P1';P1'-P1 P1+P1'];
H=[Av+Av' Bv;Bv' zeros(size(Bv,2))];
H=H+Of1'*P*Of1-ga*Of2'*Of2;
H=[H Of3';Of3 -ga*eye(size(Of3,1))];
lmi=set([]);
lmi=lmi+set(P1+P1'>0);
lmi=lmi+set(H<0);
lmi=lmi+set([Yh Yh;Yh X]>0);
solvesdp(lmi,ga,sdpsettings('solver','sedumi'));
gar=double(ga)
Yv=inv(double(Yh));
Kv=double(Kh)*Yv;
Mv=double(Mh)*Yv;
Nv=double(N);
Xv=double(X);
Lv=double(L);
U=Xv;
V=inv(Xv)-Yv;
Mest=[inv(U)*Kv-A*Yv inv(U)*Lv;Mv Nv]*inv([V' zeros(n,ny);C*Yv eye(ny)]);
res=ss(Mest(1:n,1:n),Mest(1:n,n+1:end),Mest(n+1:end,1:n),Mest(n+1:end,n+1:end));
```

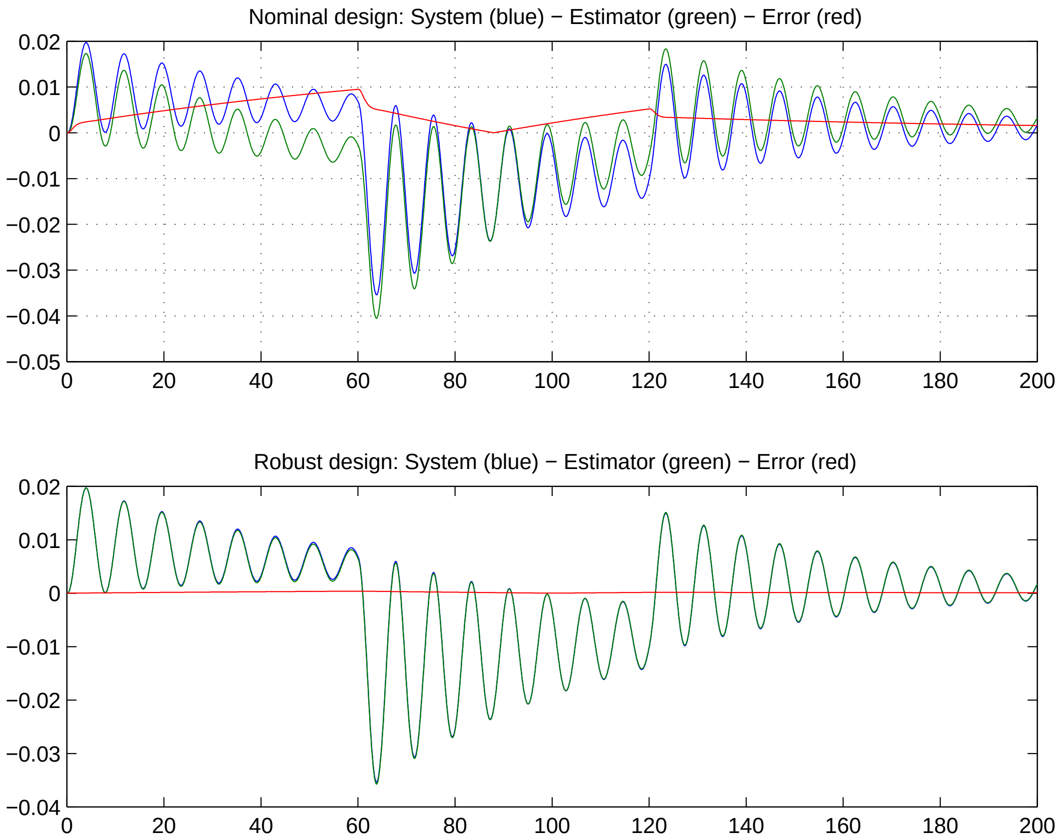
Let us now compare the un-weighted nominal and robust synthesis. The outputs are


Nominal design: System (blue) − Estimator (green) − Error (red)
2
1
0
−1
−2
0
20
40
60
80
100
120
140
160
180
200

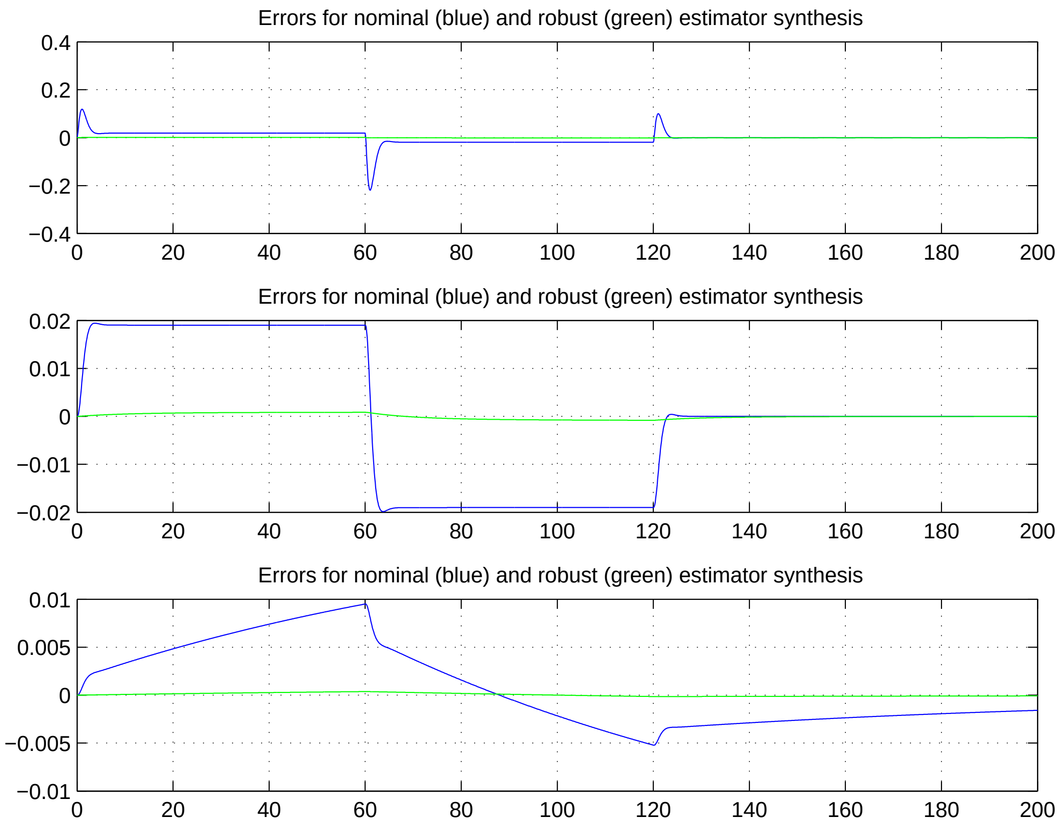

Robust design: System (blue) − Estimator (green) − Error (red)
2
1
0
−1
−2
0
20
40
60
80
100
120
140
160
180
200


Nominal design: System (blue) − Estimator (green) − Error (red)
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
0
20
40
60
80
100
120
140
160
180
200


Robust design: System (blue) − Estimator (green) − Error (red)
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
0
20
40
60
80
100
120
140
160
180
200



while a more detailed look at the errors is given by



Discussion. We observe a substantial improvement in terms of sensitivity with respect to the parameter variations. Also, the steady-state error for the robust synthesis is much smaller, despite the fact that we did not employ a weight on the estimation error.

The Bode-magnitude plots nicely illustrate how the robust estimator synthesis renders the transfer from d_1 to e considerably less sensitive against parameter variations.