Summer Course

Linear System Theory

Control

&

Matrix Computations

Monopoli, Italy **September 8-12, 2008**

Lecture 7

Rational Symbols

Lecturer: Jan C. Willems

Outline

- Motivation
- Factorization of polynomial matrices
- Behaviors defined by rational symbols
- Distance between systems
- Model reduction without stability or i/o partition
- Left prime representations

Aim of this lecture

System theory is well developed for ODEs, especially LTIDSs

$$R\left(\frac{d}{dt}\right)w = 0 \qquad \textbf{or} \qquad R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell$$

and FDLSs

$$\frac{d}{dt}x = Ax + Bu, \; y = Cx + Du \qquad w \cong \begin{bmatrix} u \\ y \end{bmatrix}$$

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Recently, we have been able to incorporate

$$G\left(\frac{d}{dt}\right) = 0 \quad \textbf{with } G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}} \qquad \underline{\textbf{rational}}$$

firmly in the behavioral setting.

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Rational representations occur frequently in control, signal processing, etc. They play a very important role in recent research.

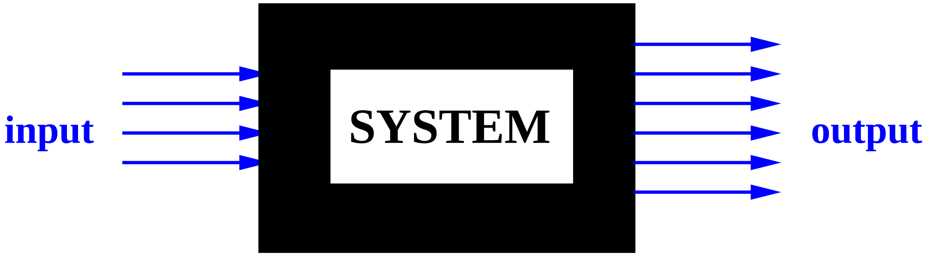
Joint research with

Yutaka Yamamoto
Kyoto University, Japan
born 1950

Rational symbols

Motivation

In system theory, it is customary to think of dynamical models in terms of inputs and outputs, viz.



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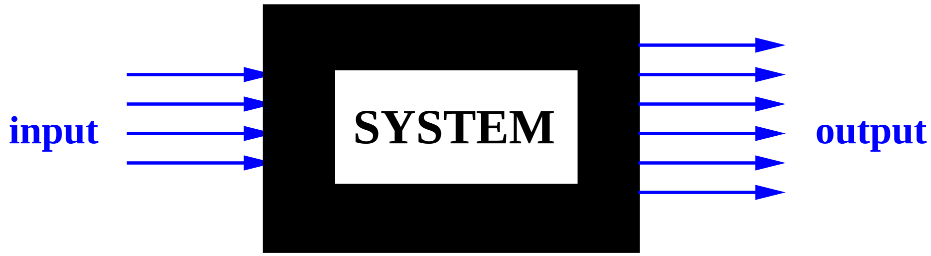

e.g. in linear time-invariant case,

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u, \quad \textbf{or} \quad \hat{y}(s) = F(s)\hat{u}(s)$$

with P, Q **polynomial matr., or** F **a matrix of rational f'ns.**

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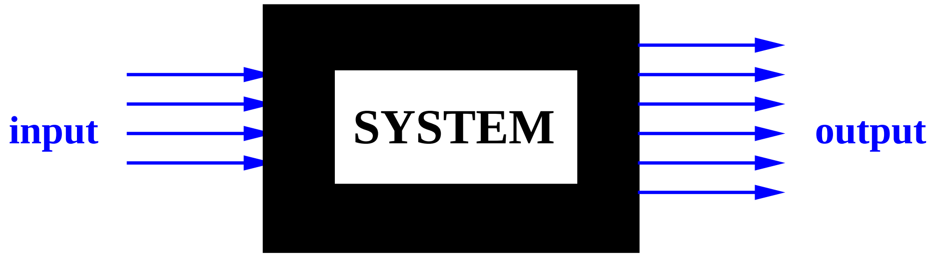

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In the present lecture, we

- **do not use an input/output partition**
- **interpret F, not in terms of Laplace transforms, but in terms of differential equations.**

LTIDSs

$(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ **where**

- $\mathbb{T} = \mathbb{R}$ **'time'**
- $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$ **'signal space'**
- **and 'behavior'** $\mathscr{B}$ **= the set of solutions of a system of**

linear constant coefficient ODEs

$\mathscr{B}$ **= the** $\mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$**-solutions of**

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{L}} \frac{d^{\mathtt{L}}}{dt^{\mathtt{L}}} w = 0$$

$$R\left(\frac{d}{dt}\right) w = 0$$

$R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$ **a matrix of real polynomials**

Differential equations with rational symbols

In signal processing, control, system ID, etc., we often meet models that involve rational functions. Cfr. transfer functions,

$$y = F(\text{‘}\mathbf{s}\text{’})u, \quad w \cong \begin{bmatrix} u \\ y \end{bmatrix}$$

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$$y = F(\text{‘}\mathbf{s}\text{’})u, \quad w \cong \begin{bmatrix} u \\ y \end{bmatrix}$$

Defining what a solution is for ODEs such as

$$R\left(\frac{d}{dt}\right)w = 0 \quad \textbf{or} \quad \frac{d}{dt}x = Ax + Bu,\ y = Cx + Du, w = \begin{bmatrix} u \\ y \end{bmatrix}$$

poses no difficulties worth mentioning, but rational functions $\rightsquigarrow$ Laplace transforms with domains of convergence, etc.

Differential equations with rational symbols

Let $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$, and consider the 'differential equation'

$$G\left(\frac{d}{dt}\right) w = 0$$

G is called the associated symbol

What do we mean by its solutions, i.e. by the behavior?

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Let (P, Q) **be a left coprime polynomial factorization of** G**. Then**

$$[\![G(\frac{d}{dt})w = 0]\!] \Leftrightarrow [\![P^{-1}Q(\frac{d}{dt})w = 0]\!] :\Leftrightarrow [\![\, Q\left(\frac{d}{dt}\right)w = 0\,]\!]$$

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By definition therefore, the behavior of $G(\frac{d}{dt})w = 0$ is equal to the behavior of $Q(\frac{d}{dt})w = 0$.
P is only of secondary importance.

Justification

1. G **proper. Let** (A,B,C,D) **be a controllable realization of the transfer function** G**. Consider the 'output nulling' inputs**

$$\frac{d}{dt}x = Ax + Bw, \quad 0 = Cx + Dw$$

This set of w**'s are exactly those that satisfy** $G\left(\frac{d}{dt}\right)w = 0$**.**

Analogous for $\frac{d}{dt}x = Ax + Bw, 0 = Cx + D\left(\frac{d}{dt}\right)w$**,** $\quad D \in \mathbb{R}[\xi]^{\bullet\times\bullet}$**.**

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2. View $G(s)$ **as a transfer f'n.**
Take your favorite definition of input/output pairs.

The output nulling inputs exactly those that satisfy
$G\left(\frac{d}{dt}\right)w = 0$**.**

3. ...

Justification

Note! With this definition, we can deal with transfer functions,

$$y = F(\frac{d}{dt})u, \quad \textbf{i.e.} \quad \left[F(\tfrac{d}{dt}) \;\vdots\; -I \right] \begin{bmatrix} u \\ y \end{bmatrix} = 0$$

with F a matrix of rational functions, and completely avoid Laplace transforms, domains of convergence, and such cumbersome, but largely irrelevant, mathematical traps.

Pierre Simon Laplace
1749 – 1827

Caveats

$F\left(\frac{d}{dt}\right)$ is not a map!

Consider

$$y = F\left(\frac{d}{dt}\right) u$$

We now know what it means that $(u, y) \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ **satisfies this 'ODE'.**

Given u, $\exists$ **solution** y**, but not unique, unless** F **is polynomial**

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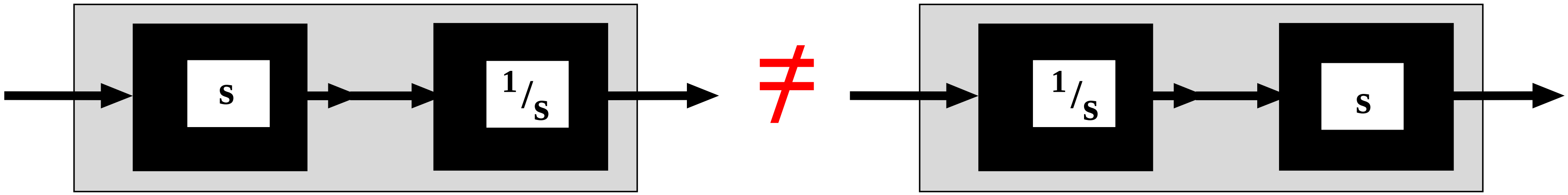
$F = P^{-1}Q$ **coprime fact.** $\Leftrightarrow$ $P^{-1}\begin{bmatrix} P & \vdots & -Q \end{bmatrix}$ **coprime fact.**

$$F = P^{-1}Q \quad \rightsquigarrow \quad y = F\left(\tfrac{d}{dt}\right) u \Leftrightarrow P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u$$

If $P \neq I$ **(better, not unimodular), there are many sol'ns** y **of this ODE for a given** u**.**

$$y = y_{\textbf{particular}} + y_{\textbf{homogeneous}} \qquad P\left(\frac{d}{dt}\right)y_{\textbf{homogeneous}} = 0$$

$G_1\left(\frac{d}{dt}\right)$ and $G_2\left(\frac{d}{dt}\right)$ need not commute



$$G_1(s) = \frac{1}{s} \text{ and } G_2(s) = s$$

do not commute.

$$y = \frac{1}{\frac{d}{dt}}v, \;\; v = \frac{d}{dt}u \quad \Rightarrow \quad y(t) = u(t) + \textbf{ constant}$$

$$y = \frac{d}{dt}v, \;\; v = \frac{1}{\frac{d}{dt}}u \quad \Rightarrow \quad y(t) = u(t)$$

Representations

LTIDSs $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ **have a representation**
$\mathscr{B} = \mathtt{kernel}\left(R\left(\frac{d}{dt}\right)\right)$ **for some** $R \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$ **by definition.**

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But we may as well take the representation $G\left(\frac{d}{dt}\right) w = 0$ **for some** $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$ **as the definition.**

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- R**: all poles at** ∞
- **we can take** G **with no poles at** ∞ **(**G **proper)**
- **or all poles in some non-empty set - symmetric w.r.t.** $\mathbb{R}$**.**

  'proper stable rational'

Representations

In particular:

Theorem: Every LTIDS has a representation

$$G\left(\frac{d}{dt}\right)w = 0$$

with $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$ **strictly proper stable rational.**

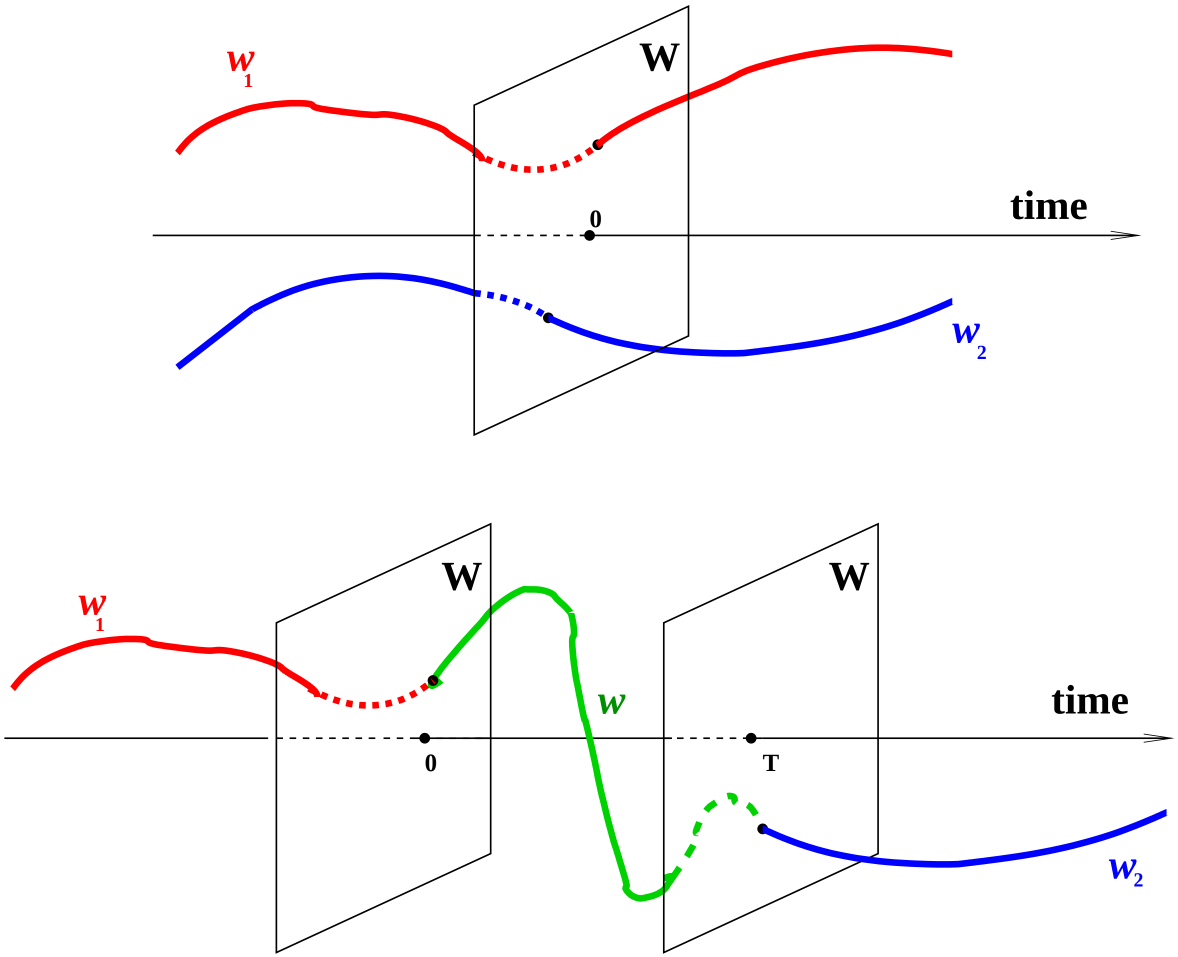
Proof: Take $G(\xi) = \frac{R(\xi)}{(\xi+\lambda)^{\mathtt{n}}}$**, suitable** $\lambda \in \mathbb{R}, \mathtt{n} \in \mathbb{N}$**.**

Controllability c.s.

Controllability

$\mathscr{B}$ **is said to be controllable** $:\Leftrightarrow$

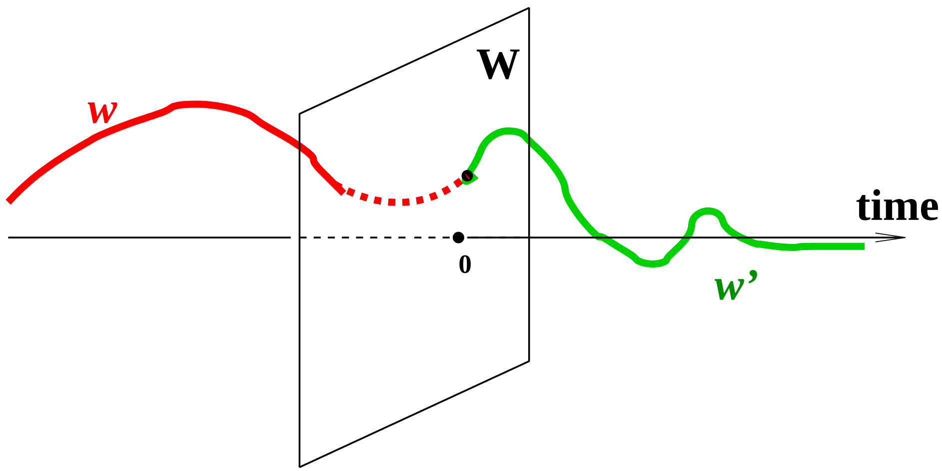
$\forall\, w_1, w_2 \in \mathscr{B},\ \exists\, T \geq 0$ **and** $w \in \mathscr{B}$ **such that ...**



Stabilizability

$\mathscr{B}$ **is said to be stabilizable** $:\Leftrightarrow$

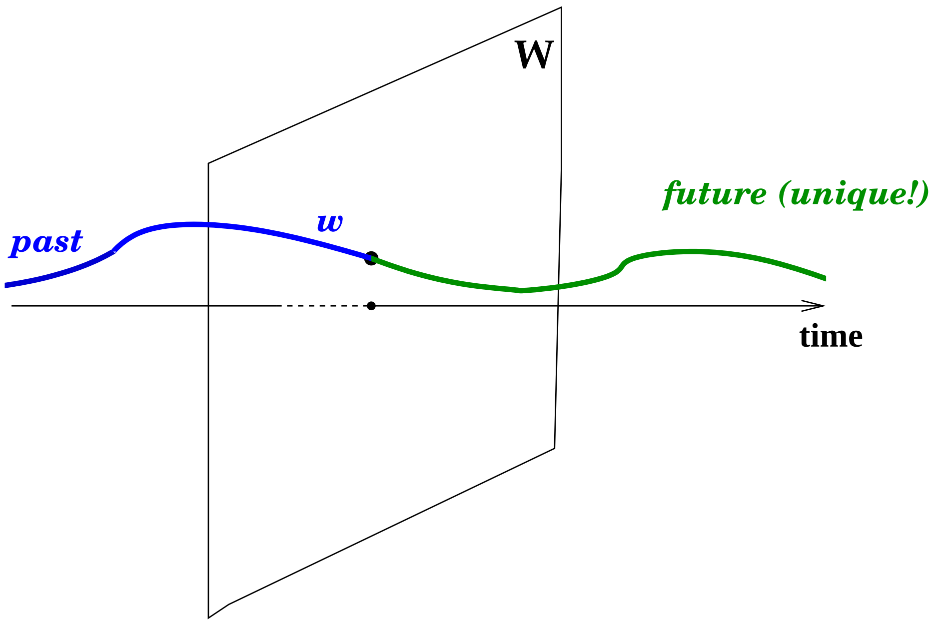
$$\forall\, w \in \mathscr{B},\ \exists\, w' \in \mathscr{B}$$ **such that ...**



Autonomous

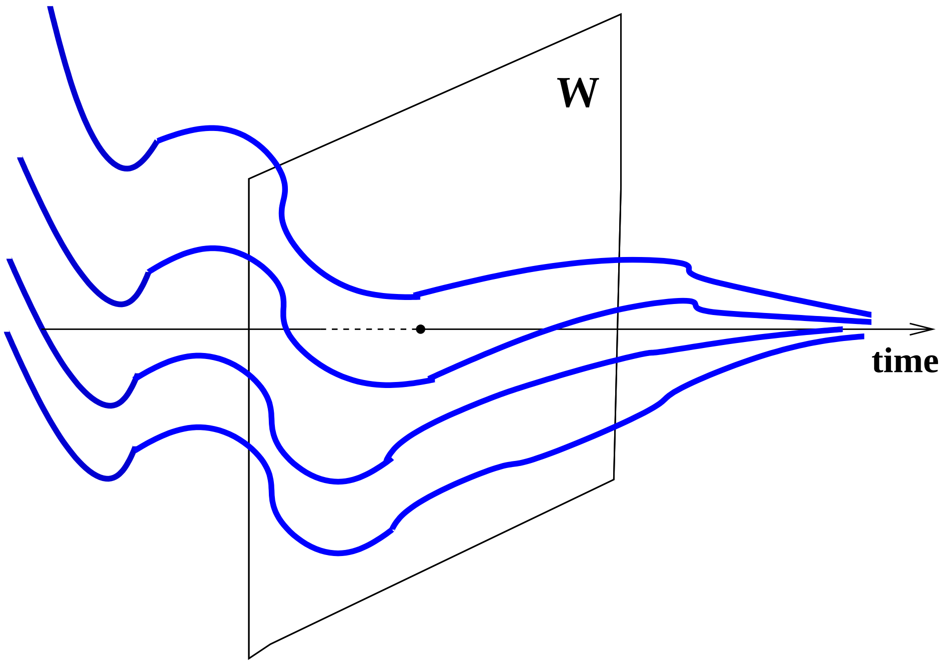
$\mathscr{B}$ **is said to be autonomous** $:\Leftrightarrow$

$$\forall\, w_- \in \mathscr{B}|_{\mathbb{R}_-},\ \exists\ (!)\ w_+ \in \mathscr{B}|_{\mathbb{R}_+}$$ **such that ...**



Stability

$\mathscr{B}$ **is said to be** **stable** $:\Leftrightarrow \quad [\![w \in \mathscr{B}]\!] \Rightarrow [\![w(t) \longrightarrow 0$ **as** $t \longrightarrow \infty]\!]$



for LTIDSs, stable $\Rightarrow$ autonomous

Stability in the sense of Lyapunov

Alexandr Lyapunov
1857 – 1918

Representations

What properties on G imply that the system with rational representation

$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$$

has any of these properties?

Under what conditions on G does $G\left(\frac{d}{dt}\right)w = 0$ define a controllable or a stabilizable system?

Representations

What properties on G imply that the system with rational representation

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Under what conditions on G does $G\left(\frac{d}{dt}\right)w = 0$ define a controllable or a stabilizable system?

Can a rational representation be used to put one of these properties in evidence?

Test for controllability

Theorem: The LTIDS

$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$$

is controllable if and only if

$$G(\lambda) \textbf{ has the same rank } \forall\lambda \in \mathbb{C}$$

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Interpret carefully in cases like

$$G(s) = \begin{bmatrix} s & 0 \\ 0 & \frac{1}{s} \end{bmatrix}, G(s) = \begin{bmatrix} s \\ \frac{1}{s} \end{bmatrix}, G(s) = \begin{bmatrix} s & \frac{1}{s} \end{bmatrix}$$

Test for stabilizability

Theorem: **The LTIDS**

$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$$

is stabilizable if and only if

$$G(\lambda) \text{ has the same rank } \forall\lambda \in \mathbb{C} \text{ with } \mathbb{R}\text{ealpart}(\lambda) \geq 0$$

Image representation

Theorem: **A LTIDS is controllable if and only if its behavior allows an image representation**

$$w = M(\tfrac{d}{dt})\ell \qquad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$$

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For example,

$$y = F(\frac{d}{dt})u \qquad \rightsquigarrow w = \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} \ell \\ F(\frac{d}{dt})\ell \end{bmatrix}$$

Systems defined by transfer functions are controllable

Transfer functions can only deal with controllable systems

Stabilizability

Theorem: **A LTIDS is stabilizable if and only if its behavior allows a kernel representation**

$$G(\tfrac{d}{dt})w = 0$$

with $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$ **left prime**
over the ring of (proper) stable rationals.

We explain what this means later, and give a number of related results.

Recapitulation

- LTIDSs are defined in terms of polynomial symbols
  $R\left(\frac{d}{dt}\right)w = 0 \qquad R \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$,
- but can also be represented by rational symbols
  $G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$
- Sol'ns are defined in terms of a left coprime factorization of G
- This added flexibility is better adapted to certain applications

  e.g. (series, parallel, ...) interconnections

  e.g. distance between behaviors

  e.g. behavioral model reduction

  e.g. parametrization of the stabilizing controllers

  e.g. characterizing stabilizability

Distance between systems

Motivation

What is a good, computable, definition for the distance between two (LTID) systems?

Basic issue underlying model simplification, robustness, etc.

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What is a good, computable, definition for the distance between two (LTID) systems?

Basic issue underlying model simplification, robustness, etc.

- **Approximate a system by a simpler one.**
- **If a system has a particular property (e.g., stabilized by a controller), will this also hold for close-by systems?**
- **Does a sequence of systems converge?**

What is meant

by ‘approximate’, by ‘close-by’, by ‘converge’?

Distance between sets

A model is a behavior, a subset. Hence distance between models translates into distance between sets.

The common measure for distance between the subsets $\mathscr{B}_1, \mathscr{B}_2 \subset \mathscr{U}$ **, with** $\mathscr{U}$ **a metric space,**
is the Hausdorff distance defined as

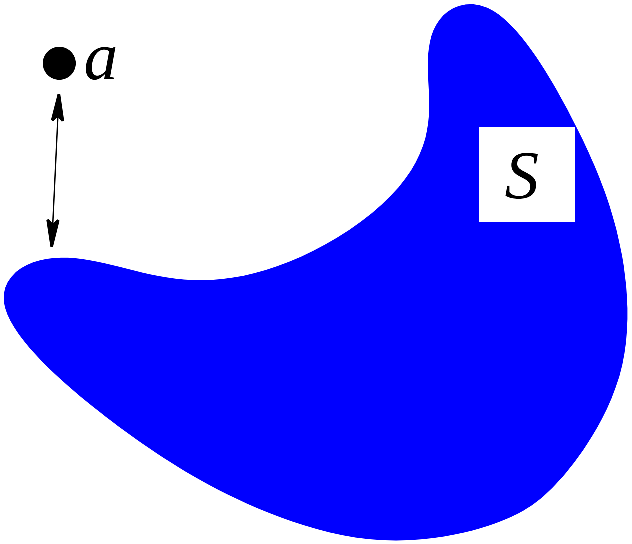
$$d_H(\mathscr{B}_1, \mathscr{B}_2) = \mathtt{max}\left(\overrightarrow{d}_H(\mathscr{B}_1, \mathscr{B}_2), \overrightarrow{d}_H(\mathscr{B}_2, \mathscr{B}_1)\right)$$

with

$$\overrightarrow{d}_H(S_1, S_2) = \sup_{w_1 \in S_1} \inf_{w_2 \in S_2} d(w_1, w_2)$$

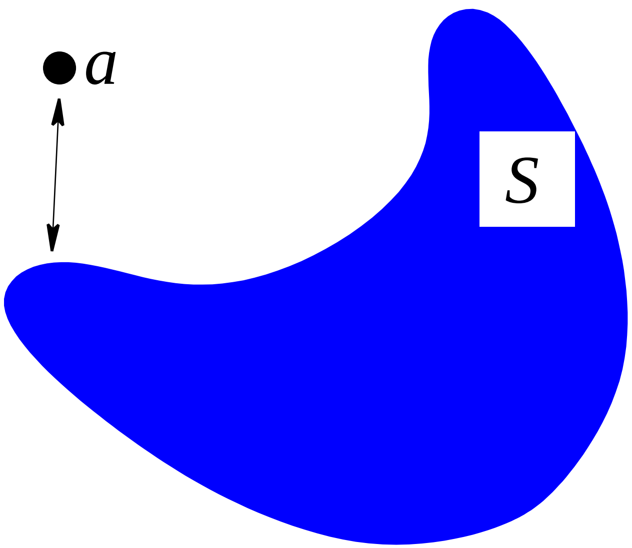
d_H **is a distance function on compact sets**

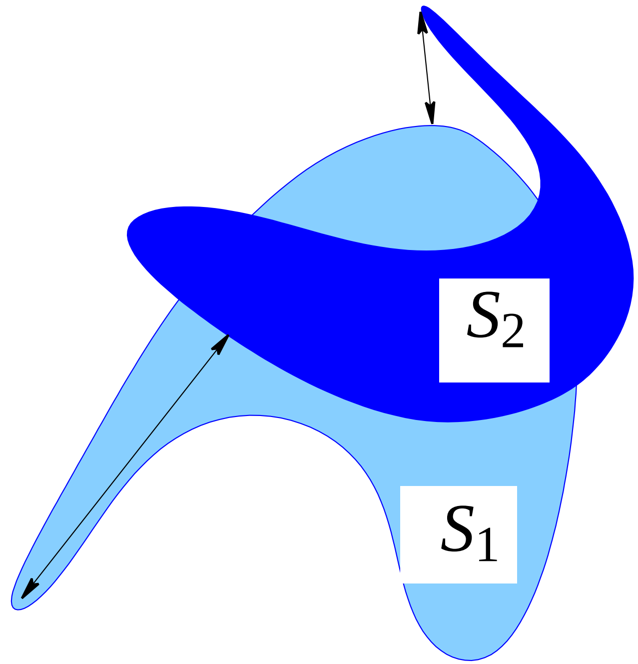
Distance between sets



Distance from a point to a set: closest distance

Distance between sets





Distance from a point to a set: closest distance

Distance between sets

Distance small $\Leftrightarrow$ close to every point of S_1, there is one of S_2
close to every point of S_2, there is one of S_1

The gap

Distance between linear subspaces

In the behavioral theory, we identify a dynamical system with its behavior, that is, with a subspace $\mathscr{B} \subseteq \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

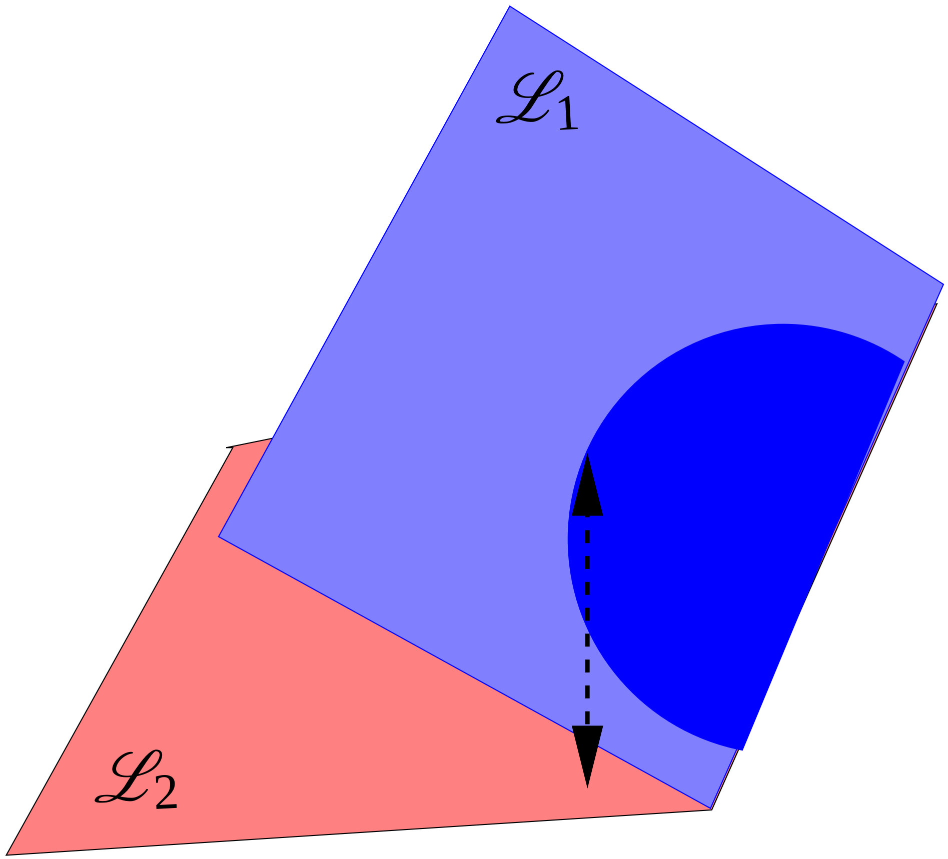
Distance between LTIDSs

$\cong$ distance between linear subspaces.

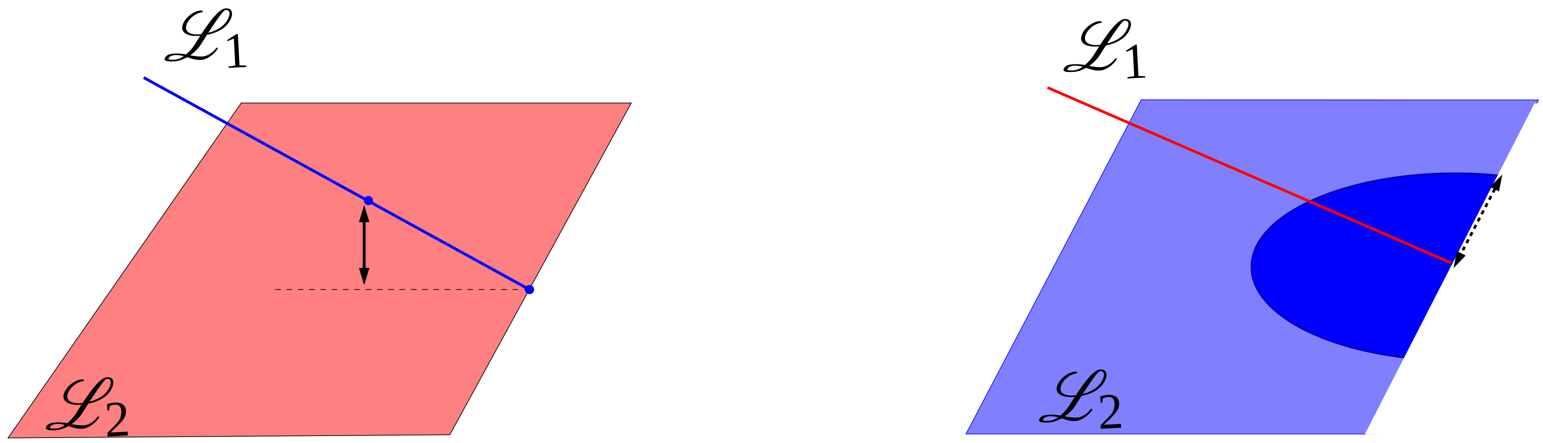
Linear subspaces of $\mathbb{R}^n$

$\mathscr{L}_1, \mathscr{L}_2 \subseteq \mathbb{R}^n$, **linear subspaces**

$$\overrightarrow{d}(\mathscr{L}_1, \mathscr{L}_2) := \max_{x_1 \in \mathscr{L}_1, ||x_1||=1} \ \min_{x_2 \in \mathscr{L}_2} ||x_1 - x_2||$$

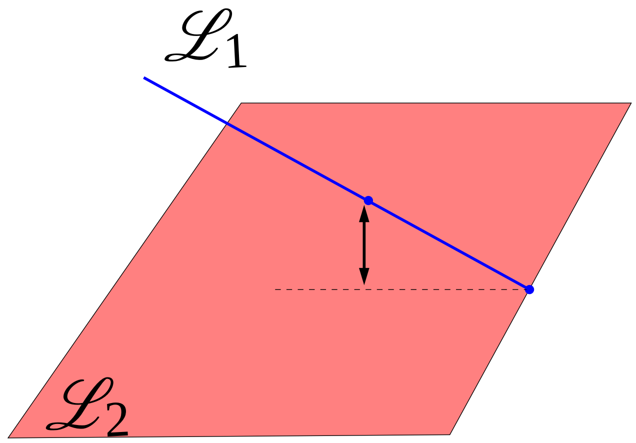


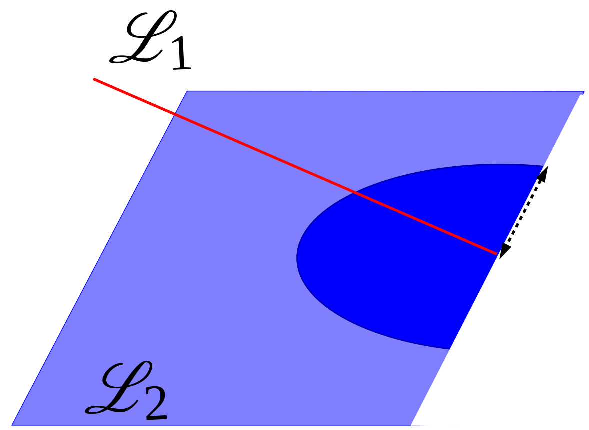
Linear subspaces of $\mathbb{R}^n$



Note again asymmetry of directed gap

Linear subspaces of $\mathbb{R}^n$





Note again asymmetry of directed gap

$$\mathtt{gap}(\mathscr{L}_1, \mathscr{L}_2) := \max\left(\left\{\overrightarrow{d}(\mathscr{L}_1, \mathscr{L}_2), \overrightarrow{d}(\mathscr{L}_2, \mathscr{L}_1)\right\}\right)$$

$$0 \le \mathtt{gap}(\mathscr{L}_1, \mathscr{L}_2) \le 1$$

$= 1$ **if** $\mathtt{dimension}(\mathscr{L}_1) \neq \mathtt{dimension}(\mathscr{L}_2)$

Formula for the gap

$P_{\mathscr{L}}$ $\perp$ **projection onto** $\mathscr{L}$

S_1, S_2 **matrices, columns orthonormal basis for** $\mathscr{L}_1, \mathscr{L}_2$

Note: $S_1 S_1^\top, S_2 S_2^\top$ **orthogonal projectors**

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$$
\begin{aligned}
\mathtt{gap}(\mathscr{L}_1, \mathscr{L}_2) &= \| P_{\mathscr{L}_1} - P_{\mathscr{L}_2} \| \\
&= \| S_1 S_1^\top - S_2 S_2^\top \| \\
&= \min_{\textbf{matrices } U} \| S_1 - S_2 U \| \\
&= \min_{U \textbf{ such that } U\mathscr{L}_1 = \mathscr{L}_2} \| I - U \|
\end{aligned}
$$

'`gap`**', 'aperture'**

Formula for the gap

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$$

'$\mathtt{gap}$**', 'aperture'**

Therefore, $d(\mathscr{L}_1, \mathscr{L}_2) = ||S_1 S_1^\top - S_2 S_2^\top|| \leq ||S_1 - S_2||$

Distance between LTIDSs

Association of a Hilbert space to a controllable behavior

$\min \to \inf, \max \to \sup$**, etc., readily generalized to linear subspaces of Hilbert space, ... and to LTIDSs.**

But, $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **is not a subspace of a Hilbert space. Which subspace of which Hilbert space should we associate with a LTIDS with behavior** $\mathscr{B} \subseteq \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$**?**

Association of a Hilbert space to a controllable behavior

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$$\boxed{\mathscr{B} \mapsto \mathscr{B}_{\mathscr{L}_2}} := \left(\mathscr{B} \cap \mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})\right)^{\text{closure}}$$

For kernel representations, corresponds to

$$\mathscr{B}_{\mathscr{L}_2} = \{w \in \mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{d}{dt}\right) w = 0, \ \textbf{distributionally}\}$$

Association of a Hilbert space to a controllable behavior

$$\mathscr{B} \mapsto \mathscr{B}_{\mathscr{L}_2}$$

defines a $1 \leftrightarrow 1$ **relation between controllable systems and certain closed subspaces of** $\mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$**.**

Moreover,

$$[\![\mathscr{B}1_{\text{controllable}} = \mathscr{B}2_{\text{controllable}}]\!] \Leftrightarrow [\![\mathscr{B}1_{\mathscr{L}_2} = \mathscr{B}2_{\mathscr{L}_2}]\!]$$

Distance between controllable behaviors

Define the distance between two controllable behaviors as

$$d(\mathscr{B}_1, \mathscr{B}_2) := \mathtt{gap}\left(\mathscr{B}_{1\,\mathscr{L}_2}, \mathscr{B}_{2\,\mathscr{L}_2}\right)$$

So, we consider the $\mathscr{L}_2$-trajectories for measuring distance.

Henceforth, keep notation $\mathscr{B}$ for $\mathscr{B}_{\mathscr{L}_2}$ = $\left(\mathscr{B} \cap \mathscr{L}_2\left(\mathbb{R}, \mathbb{R}^{\mathtt{w}}\right)\right)^{\text{closure}}$

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- **How to compute the gap?**
- **Model reduce according to the gap!**

Norm-preserving representations

Let $\mathscr{B}$ be the behavior of a controllable LTIDS. Then it allows a rational symbol based image representation

$$w = M(\frac{d}{dt})\ell \quad \textbf{with} \quad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet} \;\; \& \;\; M(-\xi)^\top M(\xi) = I$$

i.e., $||\ell||^2_{\mathscr{L}_2(\mathbb{R},\mathbb{R}^\bullet)} = ||w||^2_{\mathscr{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}})}$ **norm preserving image repr.**

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$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} ||M(i\omega)\hat{\ell}(i\omega)||^2\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} ||\hat{\ell}(i\omega)||^2\, d\omega = \int_{-\infty}^{+\infty} ||\ell(t)||^2\, dt$$

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Note: M **cannot be polynomial, it must be rational**
Obviously M must be proper. Can also make it stable.

Proof of existence of a norm-preserving image representations

Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ be controllable. Then it allows an image representation

$$w = M\left(\frac{d}{dt}\right)\ell$$

with $M \in \mathbb{R}[\xi]^{\mathtt{w}\times\bullet}$ right prime.

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Hence ('spectral factorization') there exists $H \in \mathbb{R}[\xi]^{\mathtt{w}\times\mathtt{w}}$ **with** $\mathtt{determinant}(H)$ **Hurwitz, such that**

$$M(-\xi)^\top M(\xi) = H(-\xi)^\top H(\xi)$$

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Define $\tilde{M} := MH^{-1}$**, and observe that**

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- **and norm preserving, since**

$$\tilde{M}^\top(-i\omega)\tilde{M}(i\omega) = I.$$

Norm-preserving representations

$\mathscr{B}_1 \mapsto M_1, \mathscr{B}_2 \mapsto M_2$, **both norm preserving & stable, then**

$$\texttt{gap}(\mathscr{B}_1, \mathscr{B}_2) \quad = ||M_1(i\omega)M_1(-i\omega)^\top - M_2(i\omega)M_2(-i\omega)^\top||_{\mathscr{L}_\infty}$$

$$\leq ||M_1(i\omega) - M_2(i\omega)||_{\mathscr{H}_\infty}$$

Model reduction

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There are effective methods (balancing, AAK) with good error bounds (in terms of the $\mathscr{H}_\infty$ norm) for approximating $\mathscr{B}$ by a (stable) system with a lower dimensional state space.

Keith Glover born in 1945

Error bound

Let F be the transfer function of the original system, and F_{reduced} of the reduced system.

Balanced model reduction $\Rightarrow$

$$||F(i\omega) - F_{\text{reduced}}(i\omega)||_{\mathscr{H}_\infty} \leq 2\left(\sum_{\textbf{neglected Hankel SVs}} \sigma_k\right)$$

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$F(s)$ **proper stable rational $\Rightarrow$ reducible**

with an $\mathscr{H}_\infty$ error bound.

¡¡ Extend this to situations where we do not make a distinction between inputs and outputs, and to unstable systems.

Model reduction by balancing for behavioral systems

Start with $\mathscr{B}$. Take representatation

$$w = M(\frac{d}{dt})\ell \quad \textbf{with} \quad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$$

norm preserving, stable

Model reduction by balancing for behavioral systems

Start with $\mathscr{B}$. Take representatation

$$w = M(\frac{d}{dt})\ell \quad \textbf{with} \quad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$$ **norm preserving, stable**

Now model reduce $w = M(\frac{d}{dt})\ell$ (viewed as a stable input/output system) using, for example, balancing

$$\rightsquigarrow \quad w = M_{\mathtt{reduced}}(\frac{d}{dt})\ell$$

and an error bound

$$||M - M_{\mathtt{reduced}}||_{\mathscr{H}_\infty} \leq 2 \left(\sum_{\textbf{neglected SVs of } M} \sigma_{\mathtt{k}}\right)$$

Behavioral error bound

Start with stable norm preserving representation of $\mathscr{B}$

$$w = M(\frac{d}{dt})\ell \quad \textbf{with} \quad M \in \mathbb{R}(\xi)^{\mathtt{w} \times \bullet}$$

Model reduce using balancing $\rightsquigarrow \quad w = M_{\mathtt{reduced}}(\frac{d}{dt})\ell$**.**
Call behavior $\mathscr{B}_{\text{reduced}}$**.**

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Call behavior $\mathscr{B}_{\text{reduced}}$**. Error bound**

$$\begin{aligned} \mathrm{gap}(\mathscr{B}, \mathscr{B}_{\text{reduced}}) &= \|MM^\top - M_{\mathtt{reduced}}M_{\mathtt{reduced}}^\top\|_{\mathscr{L}_\infty} \\ &\leq \|M - M_{\mathtt{reduced}}\|_{\mathscr{H}_\infty} \\ &\leq 2\left(\Sigma_{\textbf{neglected SVs of } M}\, \sigma_{\mathtt{k}}\right) \end{aligned}$$

Behavioral error bound

Start with stable norm preserving representation of $\mathscr{B}$

$$w = M(\frac{d}{dt})\ell \quad \textbf{with} \quad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$$

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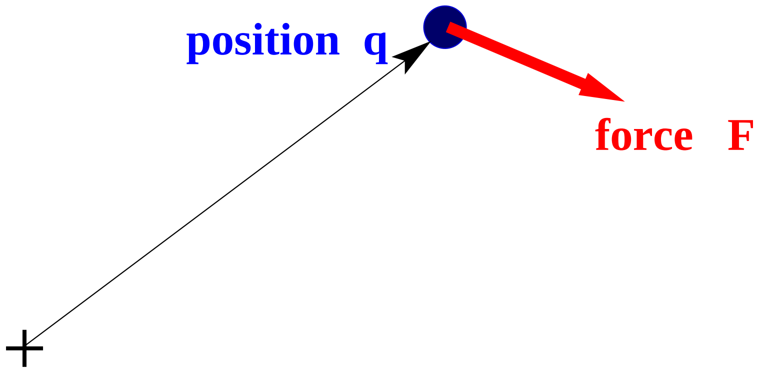
Call behavior $\mathscr{B}_{\text{reduced}}$. Error bound

$$\begin{aligned} \mathrm{gap}(\mathscr{B}, \mathscr{B}_{\text{reduced}}) &= ||MM^\top - M_{\mathtt{reduced}} M_{\mathtt{reduced}}^\top||_{\mathscr{L}_\infty} \\ &\leq ||M - M_{\mathtt{reduced}}||_{\mathscr{H}_\infty} \\ &\leq 2 \left(\sum_{\textbf{neglected SVs of } M} \sigma_{\mathtt{k}}\right) \end{aligned}$$

$\forall w \in \mathscr{B} \exists w' \in \mathscr{B}_{\text{red}}$ **such that** $||w - w'|| \leq 2(\sum_{\textbf{neglected SVs}} \sigma_{\mathtt{k}})||w||$

and vice-versa.
$\sum_{\textbf{neglected SVs of } M} \sigma_{\mathtt{k}}$ **small** $\Rightarrow$ **good approximation in the gap.**

Example



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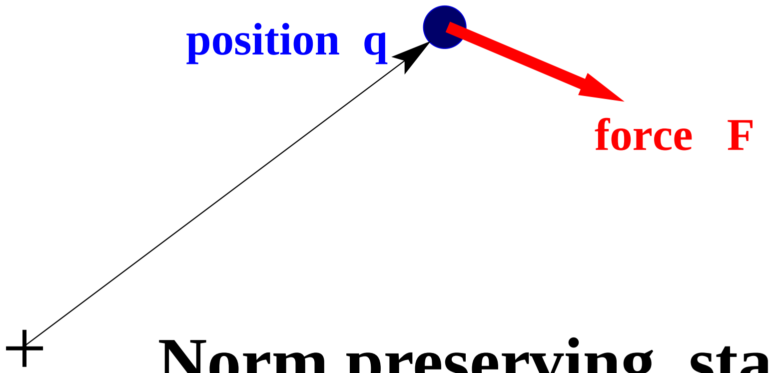

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Norm preserving, stable

$$\begin{bmatrix} F \\ q \end{bmatrix} \cong \begin{bmatrix} \frac{\xi^2}{\xi^2+\sqrt{2}\xi+1} \\ \frac{1}{\xi^2+\sqrt{2}\xi+1} \end{bmatrix} \ell$$

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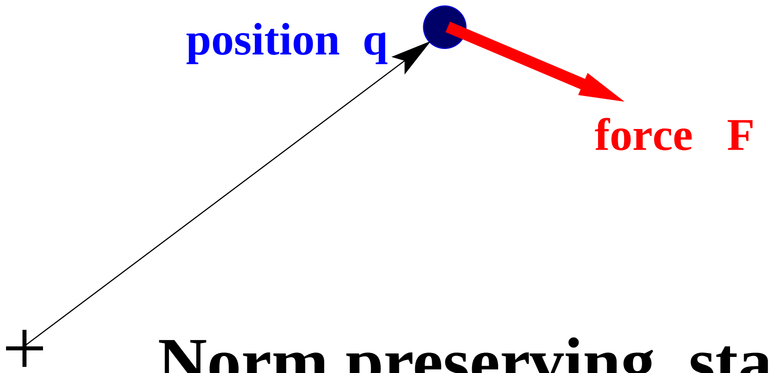

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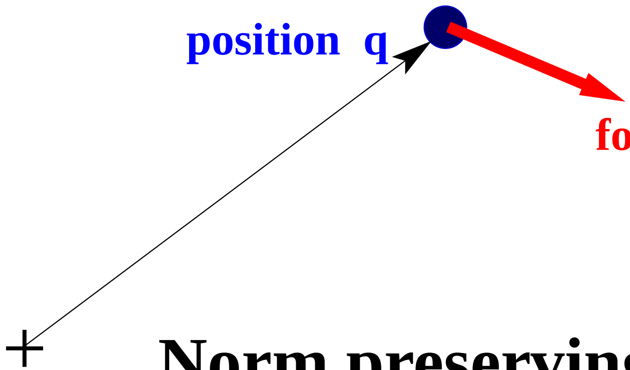

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$F = \frac{d^2}{dt^2} q$ **first order approximation** $\frac{1}{2} F = \frac{d}{dt} q - \frac{1}{2} q$

Recapitulation

- The gap is a measure of the distance between closed linear subspaces of a Hilbert space.
- Through the $\mathscr{L}_2$ behavior, the gap gives a good measure of distance between controllable LTIDSs.
- A controllable LTIDS admits a stable norm preserving image representation.
- Norm preserving image representations of LTIDSs allow to compute that gap,
- and lead to a model reduction algorithm for a controllable $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.

$\mathbb{R}(\xi)$ and some of its other subrings

Relevant rings

Field of (real) rationals

Subrings of interest

polynomials

proper rationals

stable rationals

proper stable rationals

Each of these rings has $\mathbb{R}(\xi)$ as its field of fractions !

Relevant rings

unimodularity $:\Leftrightarrow$ **invertibility in the ring**

Field of (real) rationals **nonzero**

Subrings of interest

- **polynomials** **nonzero constant**
- **proper rationals** **biproper**
- **stable rationals** **miniphase**
  $:=$ **poles and zeros in** $\mathbb{R}\text{eal}(\lambda) < 0$
- **proper stable rationals** **biproper & miniphase**

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Unimodularity

unimodularity of square matrices over rings
$\Leftrightarrow$ `determinant` **unimodular**

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$$:\Leftrightarrow \left[\!\left[[\![M = FM']\!] \Rightarrow [\![F \textbf{ unimodular}]\!] \right]\!\right]$$

$$\Leftrightarrow \exists \textbf{ matrix } M \textbf{ such that } FM = I$$

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Left coprime factorizability of $G \in \mathbb{R}(\xi)^{\mathtt{n}_1 \times \mathtt{n}_2}$ **holds over each of these rings**

Representability

The LTIDS $\mathscr{B}$ admits a representation that is left prime over

- rationals: always
- proper rationals: always
- stable rationals: iff $\mathscr{B}$ is stabilizable
- proper stable rationals: iff $\mathscr{B}$ is stabilizable
- polynomials: iff $\mathscr{B}$ is controllable

Left prime representations over subrings allow to express certain system properties...

Stabilizality

The LTIDS $\mathscr{B}$ admits a representation that is left prime over

- stable rationals: iff $\mathscr{B}$ is stabilizable
- proper stable rationals: iff $\mathscr{B}$ is stabilizable

$\mathscr{B}$ stabilizable $\Leftrightarrow \exists G$, matrix of rational functions, such that

(i) $\mathscr{B} = \mathtt{kernel}\left(G\left(\frac{d}{dt}\right)\right)$

(ii) G is proper (no poles at ∞)

(iii) $G^{\infty} := \mathtt{limit}_{\lambda\to\infty} G(\lambda)$ has full row rank (no zeros at ∞)

(iv) G has no poles in $\mathbb{C}_{+} := \{\lambda \in \mathbb{C} \mid \mathtt{real}(\lambda \geq 0\}$

(v) $G(\lambda)$ has full row rank $\forall\, \lambda \in \mathbb{C}_{+}$ (no zeros in $\mathbb{C}_{+}$)

Controllability

The LTIDS $\mathscr{B}$ admits a representation that is left prime over

- **polynomials: iff $\mathscr{B}$ is controllable**

$\mathscr{B}$ controllable $\Leftrightarrow \exists R$, matrix of polynomials, such that

(i) $\mathscr{B} = \texttt{kernel}\left(R\left(\frac{d}{dt}\right)\right)$

(ii) $R(\lambda)$ **full row rank** $\forall\, \lambda \in \mathbb{C}$

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Summary of Lecture 7

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End of lecture 7

Mathematical Appendix

Factorization of rational matrices

A bit of mathematics

Polynomials

A (one-variable) polynomial over the ring R is an expression as

$$p(\xi) = p_0 + p_1\xi + \cdots + p_{\mathtt{n}}\xi^{\mathtt{n}}$$

with the $\mathtt{p_k}$'s elements of R. The variable ξ is called the *indeterminate*. Its power $\xi^{\mathtt{k}}$ should in first instance be viewed as a placeholder to specify the element $p_{\mathtt{k}} \in R$.
We can think of a polynomial as a sequence

$$p \cong (p_0, p_1, \ldots, p_{\mathtt{n}}, \ldots)$$

of elements of R such that only a finite number of elements of the sequence are non-zero.

Polynomials

Addition and multiplication of polynomials over R are defined in the obvious way, the latter by multiplying term by term, multiplying the corresponding coefficients, adding the corresponding powers of the indeterminate, and collecting equal powers. Note that this corresponds to *convolution* of the corresponding coefficient sequences.

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The set of polynomials over the ring R is denoted as $R[\xi]$.

When $R = \mathbb{R}$, we call a corresponding polynomial *real*, and *complex* if $R = \mathbb{C}$.

$\mathbb{R}[\xi]$ and $\mathbb{C}[\xi]$ are clearly commutative rings.

A polynomial induces a map

The element $p \in \mathbb{R}[\xi], p(\xi) = p_0 + p_1\xi + \cdots + p_{\mathtt{n}}\xi^{\mathtt{n}} \in \mathbb{R}$ **stands in one-to-one relation with polynomial maps** $x \in \mathbb{R} \mapsto p_0 + p_1 x + \cdots + p_{\mathtt{n}} x^{\mathtt{n}} \in \mathbb{R}$. **The one-to-one relation follows from the derivatives of the map at** $x = 0$**.**

Similarly $p \in \mathbb{C}[\xi], p(\xi) = p_0 + p_1\xi + \cdots + p_{\mathtt{n}}\xi^{\mathtt{n}}$**, stands in one-to-one relation with the map** $x \in \mathbb{C} \mapsto p_0 + p_1 x + \cdots + p_{\mathtt{n}} x^{\mathtt{n}} \in \mathbb{C}$**.**

Often, therefore, a polynomial is viewed as a map.

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Thinking of p as a formal expression (rather than as a map) and of ξ as an indeterminate (rather than as a real or complex number) is exceedingly important.

To illustrate this point, consider $p \in \mathbb{R}[\xi]$. Then we can substitute for ξ any expression such that real scalar multiples of its powers and their sums are well defined. This holds, for example, for any element of a real algebra.

This implies in particular that for $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $p(A)$ is a well-defined element of $\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Similarly, $p\left(\frac{d}{dt}\right)$ becomes a differential operator, p induces a map from $\mathbb{C}$ to $\mathbb{C}$, etc.

This point of view is used very frequently, for example, in the Cayley-Hamilton theorem, in the fundamental theorem of algebra, in our discussion of LTIDSs, etc.

Rational functions

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Formally $\mathbb{R}(\xi)$ **is defined as the field of fractions of** $\mathbb{R}[\xi]$
This means the following.

Consider the set

$$(\mathbb{R}[\xi] - \{0\}) \times \mathbb{R}[\xi]$$

with the equivalence relation

$$[\![(d_1, n_1) \sim (d_2, n_2]\!] :\Leftrightarrow [\![d_1 n_2 = d_2 n_1]\!]$$

Then $\mathbb{R}(\xi)$ **is defined as the set of equivalence classes obtained this way. With the obvious definition of addition and multiplication,** $\mathbb{R}(\xi)$ **becomes a field.**

Rational functions

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- By definition, we can *ad libitum* cancel or add common factors in n and d, without changing the rational function $f = \frac{n}{d}$.
- Each rational function $f \in \mathbb{R}(\xi)$ equals $f = \frac{n}{d}$, for some $n, d \in \mathbb{R}[\xi]$ coprime (meaning that n and d have no common roots), and d monic (meaning that the highest power in ξ with a non-zero coefficient has coefficient 1.

Prime polynomial matrices

$M \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$ **is said to be left prime if** $M = FM'$ **with** $F \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_1}, M'\mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$ **implies that** F **is unimodular.**

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The following are equivalent for $M \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$**:**

- M **is left prime**
- $M(\lambda)$ **has full row rank (i.e. its rank is** $\mathtt{n}_1$**) for all** $\lambda \in \mathbb{C}$**.**
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Right prime is defined and characterized completely analogously.

Prime polynomial matrices

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Étienne Bézout
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For $\mathtt{n}_1 = 1, \mathtt{n}_2 = 2$ **this states:**
given $m_1, m_2 \in \mathbb{R}[\xi]$**, there exist** $x_1, x_2 \in \mathbb{R}[\xi]$
such that

$$m_1 x_1 + m_2 x_2 = 1$$

iff m_1 **and** m_2 **have no common root.**

Factorization of matrices of rational functions

A left coprime polynomial factorization (or a left coprime factorization over $\mathbb{R}[\xi]$) of $M \in \mathbb{R}(\xi)^{\mathtt{n}_1 \times \mathtt{n}_2}$ is a pair (P, Q) such that

- $P \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_1}$, $Q \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$
- $\mathtt{determinant}(P) \neq 0$
- $\begin{bmatrix} P & Q \end{bmatrix}$ **is left prime**
- $P^{-1}Q = M$

Every $M \in \mathbb{R}(\xi)^{\mathtt{n}_1 \times \mathtt{n}_2}$ admits a left coprime fact. over $\mathbb{R}[\xi]$

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(P_1, Q_1) and (P_2, Q_2) are both left coprime factorizations of M over $\mathbb{R}[\xi]$ iff $\exists$ a unimodular $U \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_1}$ such that

$$P_2 = UP_1, Q_1 = UQ_2$$

Factorization of matrices of rational functions

For example, in the scalar case,

$$M = \begin{bmatrix} m_1 & m_2 \cdots m_{\mathtt{n}} \end{bmatrix}, \text{ with the } m_{\mathtt{k}}\text{'s } \in \mathbb{R}(\xi)$$

is factored as

$$M = \frac{1}{p} \begin{bmatrix} q_1 & q_2 \cdots q_{\mathtt{n}} \end{bmatrix},$$

with $p, q_1, q_2, \ldots, q_{\mathtt{n}} \in \mathbb{R}[\xi]$ **coprime polynomials (that is, they have no common roots).**

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