Summer Course

Linear System Theory

Control

&

Matrix Computations

Monopoli, Italy **September 8-12, 2008**

Lecture 4

Linear State Space Systems

&

Realization Theory

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Outline

- Finite dimensional linear systems (FDLSs)
- State stability
- State controllability and state observability
- The impulse response (IR) and the transfer function (TF)
- Minimal realizations & the state space isomorphism thm
- Realizability conditions based on the Hankel matrix
- Realization algorithms
- System norms

Finite dimensional linear systems

FDLSs

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The most studied class of dynamical systems are the FDLSs, represented by:

continuous time: $$\frac{d}{dt}x = Ax + Bu, \;\; y = Cx + Du$$

discrete time: $$\sigma x = Ax + Bu, \;\; y = Cx + Du$$

where σ is the *left shift*: $\sigma f(t) := f(t+1)$**.**

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Informal notation:

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \text{ **or** } x(t+1) = Ax(t) + Bu(t), \;\; y(t) = Cx(t) + Du(t)$$

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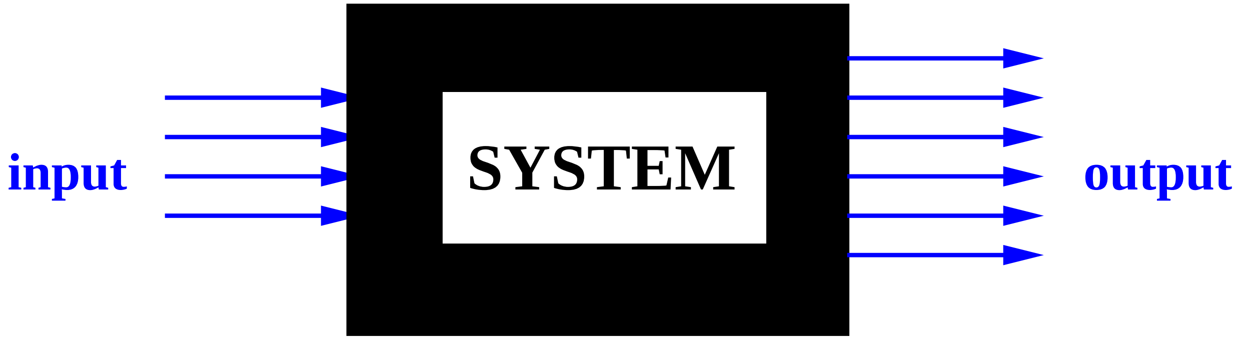
The time axis depends on the application.

For the continuous-time case, $\mathbb{T} = \mathbb{R}$ **or** $\mathbb{T} = \mathbb{R}_+ := [0, \infty)$**.**
For the discrete-time case, $\mathbb{T} = \mathbb{Z}$ **or** $\mathbb{T} = \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$**.**

The signal flow

$u : \mathbb{T} \to \mathbb{R}^{\mathtt{m}}$ **is the *input* (trajectory)**
$y : \mathbb{T} \to \mathbb{R}^{\mathtt{m}}$ **is the *output* (trajectory)**
$x : \mathbb{T} \to \mathbb{R}^{\mathtt{n}}$ **is the *state trajectory*.**



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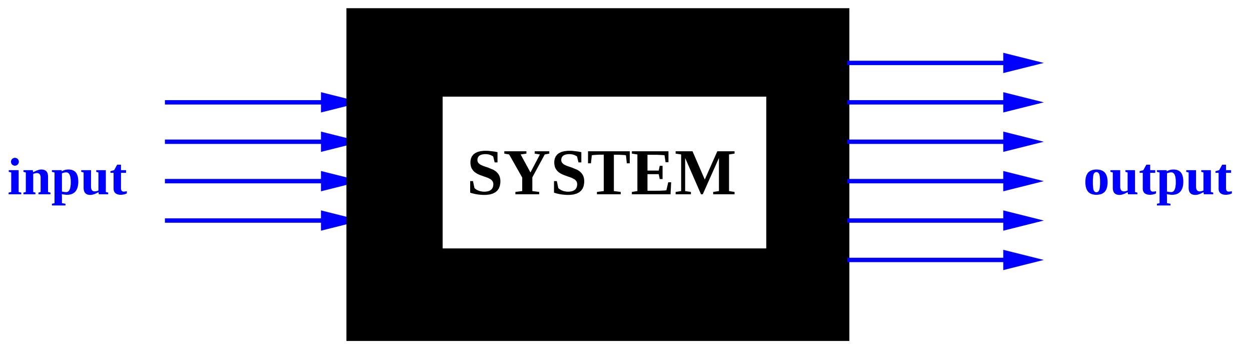

Often the states have a clear physical meaning, but in many applications they are introduced principally in order to give the equations a 'recursive' character.

In this lecture, we view states as latent variables, internal to the system, that serve to codify how external inputs are recursively transformed to external outputs.

Input and output space

Sometimes we denote the spaces where the input, output, and state take on their values by

$$\mathbb{U}\ (= \mathbb{R}^{\mathtt{m}}),\quad \mathbb{Y}\ (= \mathbb{R}^{\mathtt{p}}),\quad \mathbb{X}\ (= \mathbb{R}^{\mathtt{n}})$$

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$$\mathbb{U}\ (=\mathbb{R}^{\mathtt{m}}),\quad \mathbb{Y}\ (=\mathbb{R}^{\mathtt{p}}),\quad \mathbb{X}\ (=\mathbb{R}^{\mathtt{n}})$$

In the discrete-time case, the input space $\mathscr{U}$, and output space, $\mathscr{Y}$, are taken to be

$$\mathscr{U}=\mathbb{U}^{\mathbb{T}}\ \textbf{ and }\ \mathscr{Y}=\mathbb{Y}^{\mathbb{T}},$$

but, in the continuous-time case, we have to be a bit more conservative for the input. We can, for example, take

$$\begin{array}{llll} & \mathscr{U}=\mathscr{L}_1^{\text{local}}(\mathbb{R},\mathbb{R}^{\mathtt{m}}) & \textbf{and} & \mathscr{Y}=\mathscr{L}_1^{\text{local}}(\mathbb{R},\mathbb{R}^{\mathtt{p}}) \\ \textbf{or} & \mathscr{U}=\mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{m}}) & \textbf{and} & \mathscr{Y}=\mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{p}}) \end{array}$$

Notation

$A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}, C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}, D \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$

are the *system matrices*.

It is common to denote this system as

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad \text{or} \quad (A, B, C, D)$$

depending on the typographical constraints.

The dimension of the state space $\mathbb{X}$ is called the *order* of $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. It is a reasonable measure of the dynamic complexity of the system (important in lectures 7 and 8).

The external behavior

The input/output trajectories

The state trajectory corresponding to the input u **and the initial state** $x(0)$ **is given by**

$$x(t) = e^{At}x(0) + \int_0^t e^{At'}Bu(t-t')\,dt'.$$

The output y **depends on the input** u **and the initial state** $x(0)$ **as follows**

$$y(t) = Ce^{At}x(0) + Du(t) + \int_0^t Ce^{At'}Bu(t-t')\,dt'$$

Observe that $u \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **(or** $\mathscr{L}_1^{\text{local}}(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$**) and** $x(0) \in \mathbb{R}^{\mathtt{n}}$ **yield a unique output** $y \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{p}})$ **(or** $\mathscr{L}_1^{\text{local}}(\mathbb{R}, \mathbb{R}^{\mathtt{p}})$**).**

The corresponding LTIDS

It follows immediately from the elimination theorem that the set of (u,y) trajectories obtained this way is exactly equal to the solution set of a LTIDS

$$R\left(\frac{d}{dt}\right)\begin{bmatrix} u \\ y \end{bmatrix} = 0$$

for a suitable $R \in \mathbb{R}[\xi]^{\bullet \times (\mathtt{m}+\mathtt{p})}$.

From LTIDS to (A,B,C,D)

The converse is also true, in the following precise sense. For each $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$, there exists an permutation matrix $\Pi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and matrices $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}, C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}, D \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$ such that the w-behavior of

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \;\; y(t) = Cx(t) + Du(t), w = \Pi \begin{bmatrix} u \\ y \end{bmatrix}$$

is equal to the given $\mathscr{B}$.

From LTIDS to (A, B, C, D)

The permutation matrix Π corresponds to changing the order of the components of w. Hence every $\mathscr{B} \in \mathscr{L}^\bullet$ allows a representation as an input/state/output system $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, up to reordering of the components of w.

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Algorithms $R \;\longrightarrow\; \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **or** M, **or** $(R, M) \;\longrightarrow\; \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

are of much interest. (see lectures 2 and 12).

The external behavior

$\mathscr{L}^{\mathtt{w}}$, the LTIDSs, described by the differential equations

$$R\left(\frac{d}{dt}\right)w = 0,$$

and the FDLSs, described by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

model exactly the same class of systems.

Linear time-invariant differential

$\equiv$ linear time-invariant finite dimensionsal

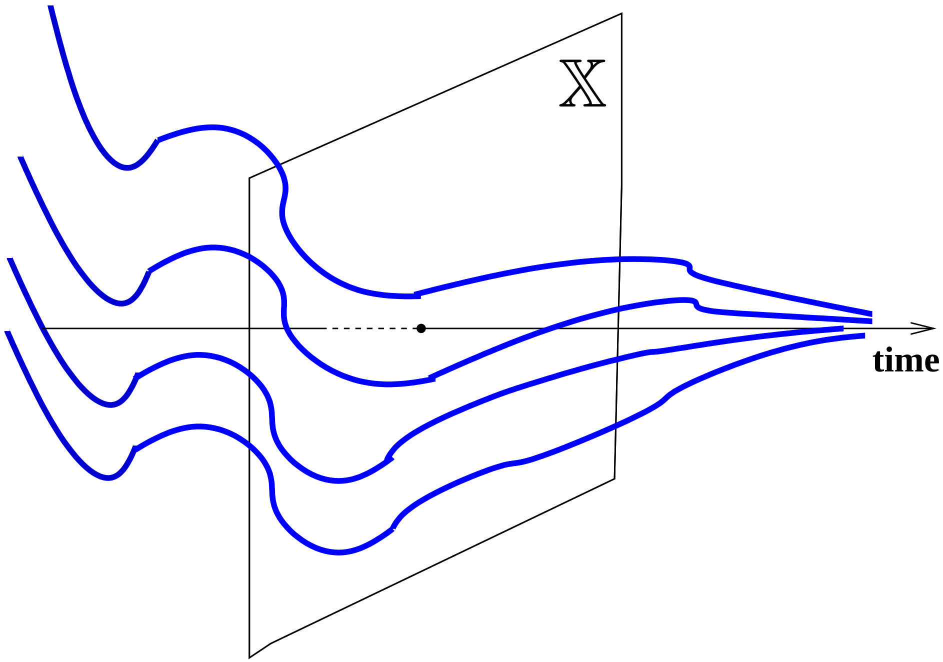
State stability

State stability

Many notions of stability pertain to $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$**.**

In particular, it is said to be *state stable* if, for $u = 0$**, every state trajectory** $x : \mathbb{R} \to \mathbb{R}^{\mathtt{n}}$ **converges to zero:**

$x(t) \to 0$ **as** $t \to \infty$



State stability

Obviously, state stability is equivalent to

$$e^{At} \longrightarrow 0 \textbf{ as } t \longrightarrow \infty.$$

It is easy to see that this holds iff

all the eigenvalues of A have negative real part

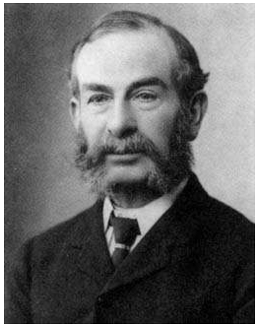
Square matrices with this property are called *Hurwitz*.

Hurwitz polynomials

The question of finding conditions on the coeff. of $p \in \mathbb{R}[\xi]$

$$p(\xi) = p_0 + p_1\xi + \cdots + p_{\mathtt{n}-1}\xi^{\mathtt{n}-1} + p_{\mathtt{n}}\xi^{\mathtt{n}}$$

so that its roots have negative real part (such pol's are called *Hurwitz*), known as the *Routh-Hurwitz problem*, has been the subject of countless articles ever since Maxwell raised first the question in 1868. There exist effective tests on $p_0, p_1, \cdots, p_{\mathtt{n}-1}, p_{\mathtt{n}}$**.**



Edward Routh
1831 – 1907

Adolf Hurwitz
1859 – 1919

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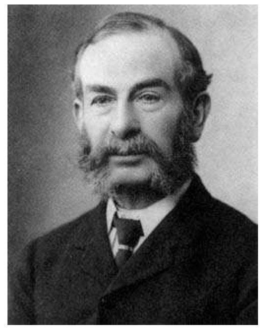

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Of course, $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is Hurwitz iff its characteristic polynomial, $\det(I\xi - A)$, is Hurwitz.
In principle this gives a test for checking state stability.

The following test on A is more in the spirit of this course.

The Lyapunov equation

The Lyapunov equation

Theorem

The following conditions on $A \in \mathbb{R}^{n \times n}$ are equivalent:

1. **A is Hurwitz**

The Lyapunov equation

Notation: $\succ$ ***positive definite***, $\prec$ ***negative definite***,
$\succeq$ ***positive semidefinite***, $\preceq$ ***negative semidefinite***.

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1. A **is Hurwitz**
2. **there exists a solution $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ to**

$$X = X^\top \succ 0, \quad A^\top X + XA \prec 0$$

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3. $\forall\, Y = Y^\top \prec 0, \exists\, X = X^\top \succ 0$ **s.t.** $\boxed{A^\top X + XA = Y}$

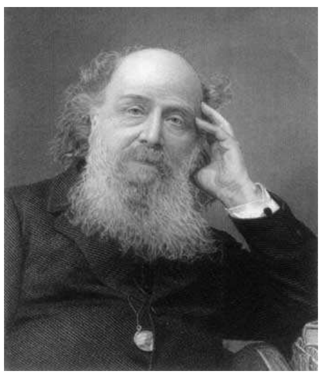
The equation in 3. is called *the Lyapunov equation*

The Lyapunov equation

The Lyapunov equation $A^\top X + XA = Y$

Alexandr Lyapunov
1857 – 1918

is a special case of the *Sylvester equation* $AX + XB = Y$**.**



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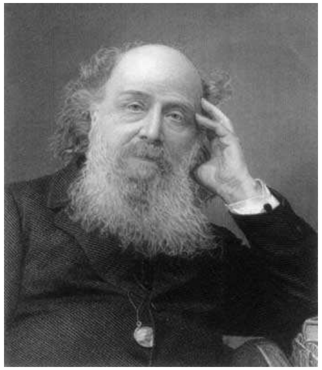

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The Lyapunov equation

2. is what is called an ‘LMI’ feasibility test (see lecture 6). It may be considered as an algorithm for verifying state stability.

3. can be used with, e.g. $Y = -I$, to compute X, and then verify definiteness of X (also considered an algorithm).

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Important in algorithms:

A Hurwitz implies that the map
$$X \in \mathbb{R}^{n\times n} \mapsto A^\top X + XA \in \mathbb{R}^{n\times n}$$
is bijective (surjective & injective).

Proof of the Lyapunov theorem

1. $\Rightarrow$ 3. A **Hurwitz implies that** $-\int_0^{+\infty} e^{A^\top t} Y e^{At}\, dt = X$ **converges, and yields a sol'n** X **of the Lyapunov eq'n for a given** Y**. Hence the map** $M \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}} \mapsto A^\top M + MA \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ **is surjective, and therefore injective. So,** X **is the only sol'n. Conclude that** $Y = Y^\top \prec 0$ **implies** $X = X^\top \succ 0$**.**

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3. $\Rightarrow$ 2. is trivial.

2. $\Rightarrow$ 1. **Let** $0 \neq a \in \mathbb{C}^{\mathtt{n}}$ **be an eigenvector of** A **with eigenvalue** λ**.** * **denotes complex conjugate transpose. Then**

$$0 > a^* A^\top X a + a^* X A a = (\overline{\lambda} + \lambda) a^* X a.$$

Since $a^* X a > 0$**, this implies** $(\overline{\lambda} + \lambda) < 0$**. Therefore,** A **is Hurwitz.**

State controllability

Definition of state controllability

The system $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **is said to be** ***state controllable*** **if for every** $x_1, x_2 \in \mathbb{R}^{\mathtt{n}}$**, there exists** $T \geq 0$ **and** $u : [0, T] \to \mathbb{R}^{\mathtt{m}}$ **(say** $u \in \mathscr{L}_1([0, T], \mathbb{R}^{\mathtt{m}})$ **or** $\mathscr{C}^{\infty}([0, T], \mathbb{R}^{\mathtt{m}})$**), such that the solution of**

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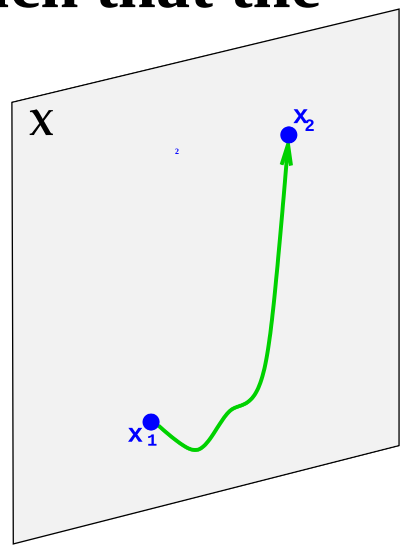

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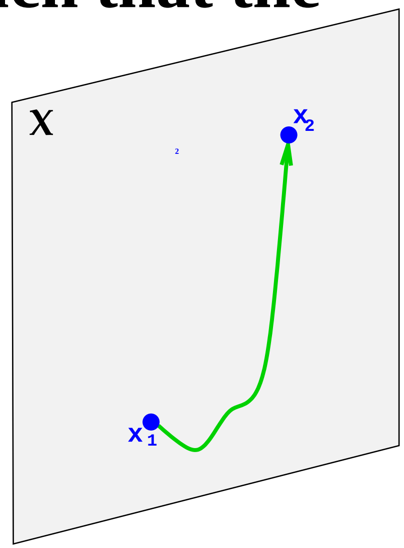


Observe: state controllability is equivalent to behavioral controllability, as defined in lecture 2, applied to

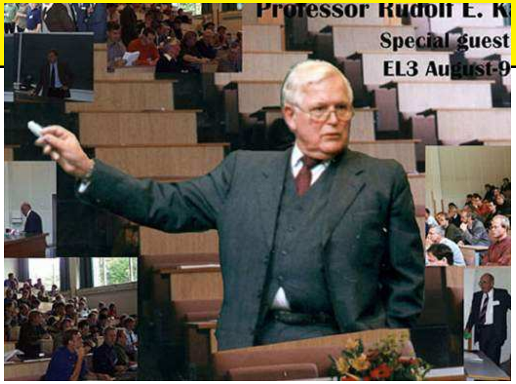
$$\frac{d}{dt}x = Ax + Bu \quad \text{both with } w = (x,u) \ \text{ or } \ w = x.$$

Test for state controllability

Theorem

$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **is state controllable iff**

$$\texttt{rank}\left(\begin{bmatrix} B & AB & A^2B & \cdots A^{\texttt{dimension}(\mathbb{X})-1}B \end{bmatrix}\right) = \texttt{dimension}(\mathbb{X})$$



Rudolf E. Kalman
born 1930

Proof of the controllability test

We give only an outline of the proof.

First observe that it suffices to consider the case $x_1 = 0$.

Denote the set of states reachable from $x(0) = 0$ over all u and $T \geq 0$ by $\mathscr{R}$. $\mathscr{R}$ is obviously a linear subspace of $\mathbb{R}^{\mathtt{n}}$.

Define

$$\mathscr{L}_{\mathtt{k}} = \mathtt{image}\left(\begin{bmatrix} B & AB & A^2B & \cdots A^{\mathtt{k}-1}B \end{bmatrix}\right), \mathtt{k} = 1, 2, \ldots$$

The rank test in the controllability thm requires $\mathscr{L}_{\mathtt{n}} = \mathbb{R}^{\mathtt{n}}$. We therefore need to prove that $\mathscr{R} = \mathbb{R}^{\mathtt{n}}$ iff $\mathscr{L}_{\mathtt{n}} = \mathbb{R}^{\mathtt{n}}$.

Proof of controllability theorem

(if) **All trajectories** x **of** $\frac{d}{dt}x = Ax + Bu$ **and hence their derivatives lie entirely in** $\mathscr{R}$**. This implies that** $\mathscr{R}$ **is** A**-invariant and contains** $\mathtt{image}(B)$**. Hence** $\mathscr{R}$ **contains** $\mathtt{image}(A^{\mathtt{k}}B)$ **for all** $\mathtt{k} \in \mathbb{Z}_+$**, and therefore** $\mathscr{R}$ **contains** $\mathscr{L}_{\mathtt{n}}$**. Conclude that the rank test implies controllability.**

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(only if) **Clearly** $\mathscr{L}_{\mathtt{k}} \subseteq \mathscr{L}_{\mathtt{k}+1}$**, and** $\mathscr{L}_{\mathtt{k}+1} = \mathscr{L}_{\mathtt{k}}$ **implies** $\mathscr{L}_{\mathtt{k}'} = \mathscr{L}_{\mathtt{k}}$ **for** $\mathtt{k}' \geq \mathtt{k}$**. The dimension of the** $\mathscr{L}_{\mathtt{k}}$**'s must go up by at least** 1**, or stay fixed forever. Therefore** $\mathscr{L}_{\mathtt{n}'} = \mathscr{L}_{\mathtt{n}}$ **for** $\mathtt{n}' \geq \mathtt{n}$**.**

If the rank condition is not satisfied, there exists $0 \neq f \in \mathbb{R}^{\mathtt{n}}$ **such that** $f^\top B = f^\top AB = \cdots = f^\top A^{\mathtt{n}-1}B = 0$**. Since** $\mathscr{L}_{\mathtt{n}'} = \mathscr{L}_{\mathtt{n}}$ **for** $\mathtt{n}' \geq \mathtt{n}$**, this implies** $f^\top A^{\mathtt{k}}B = 0$ **for all** $\mathtt{k} \in \mathbb{Z}_+$**. Therefore** $f^\top e^{At}B = 0$ **for all** $t \in \mathbb{R}$**, hence** $f^\top \mathscr{R} = 0$**. Therefore the system is not state controllable if the rank test is not satisfied.**

State observability

Definition of state observability

Define the 'internal' behavior $\mathscr{B}_{\text{internal}}$ **of** $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **by**

$$\mathscr{B}_{\text{internal}} := \{(u,y,x) \in \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{m}} \times \mathbb{R}^{\mathtt{p}} \times \mathbb{R}^{\mathtt{n}}) \mid \frac{d}{dt}x = Ax + Bu, y = Cx + Du\}$$

$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **said to be** ***state observable*** **if**

$$(u,y,x_1), (u,y,x_2) \in \mathscr{B}_{\text{internal}} \text{ implies } x_1 = x_2.$$

In other words, if knowledge of (u,y) **(and of the system dynamics) implies knowledge of** x**.**

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State observability is a special case of observability as defined in lecture 2, with (u,y) **the observed, and** x **the to-be-reconstructed variable.**

Definition of state observability

There are numerous variations of this definition, the most prevalent one that there exists $T > 0$ such that

$$(u(t), y(t)) \text{ for } 0 \leq t \leq T \text{ determines } x(0) \text{ uniquely}.$$

It is easily seen that this variation, and many others, are equivalent to the definition given.

Test for state observability

Theorem

$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **is state observable iff**

$$\mathtt{rank}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{\mathtt{dimension}(\mathbb{X})-1} \end{bmatrix}\right) = \mathtt{dimension}(\mathbb{X})$$

Proof of the observability test

We give only an outline of the proof.

Observe that state observability requires to find conditions for

$$[\![Ce^{At}x_1(0) = Ce^{At}x_2(0) \textbf{ for all } t \in \mathbb{R}]\!] \Leftrightarrow [\![x_1(0) = x_2(0)]\!]$$

Equivalently, for injectivity of the map

$$a \in \mathbb{R}^{\mathtt{n}} \mapsto L(a) \in \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{p}}) \textbf{ with } L(a) : t \in \mathbb{R} \mapsto Ce^{At}a \in \mathbb{R}^{\mathtt{p}}.$$

Now prove, using arguments as in the controllability case, that

$$[\![Ce^{At}f = 0 \textbf{ for all } t \in \mathbb{R}]\!] \Leftrightarrow [\![CA^{\mathtt{k}}f = 0 \textbf{ for } \mathtt{k} = 0, 1, 2, \cdots]\!]$$
$$\Leftrightarrow [\![Cf = CAf = CA^2f = CA^{\mathtt{n}-1}f = 0]\!].$$

Conclude that $a \mapsto L(a)$ is injective iff the rank test of the observability theorem holds.

System decompositions

Change of basis in state space

Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$**. Its trajectories are described by**

$$\frac{d}{dt}x = Ax + Bu, y = Cx + Du.$$

A change of basis in the state space means introducing $z = Sx$**, with** $S \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ **nonsingular. The dynamics become**

$$\frac{d}{dt}z = SAS^{-1}z + SBu, \quad y = CS^{-1}z + Du$$

Hence a change of basis corresponds to the transformation

$$(A,B,C,D) \xrightarrow{S} (SAS^{-1}, SB, CS^{-1}, D)$$

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A change of basis does not change the external behavior, and comes in handy to put certain system properties in evidence.

Controllable / uncontrollable decomposition

Decompose the state space $\mathbb{X} = \mathbb{R}^{\mathtt{n}}$ into

$$\mathbb{X} = \mathscr{R} \oplus \mathscr{S}$$

with $\mathscr{R} = \mathtt{image}([\, B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{\mathtt{n}-1}B \,])$ **and** $\mathscr{S}$ **any complement. Note that** $\mathscr{R}$ **is 'intrinsically' defined, but** $\mathscr{S}$ **is not, any complement will do.**

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Now choose the basis in the state space such that the first basis vectors span $\mathscr{R}$, and the last basis vectors span $\mathscr{S}$.

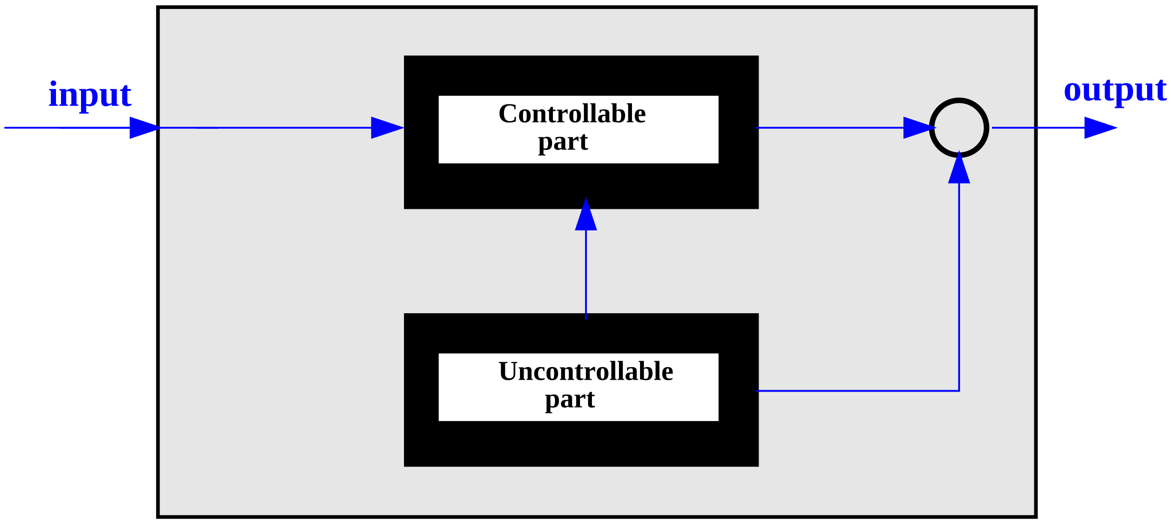
Controllable / uncontrollable decomposition

In this new basis the system dynamics take the form

$$\begin{aligned} \frac{d}{dt}z_1 &= A_{11}z_1 + A_{12}z_2 + B_1 u, \qquad y = C_1 z_1 + C_2 z_2 + Du \\ \frac{d}{dt}z_2 &= A_{22}z_2 \end{aligned}$$

with $\frac{d}{dt}z_1 = A_{11}z_1 + B_1 u$ **controllable.**

This decomposition brings out the controllability structure.



Unobservable / observable decomposition

Decompose the state space $\mathbb{X} = \mathbb{R}^{\mathtt{n}}$ into

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with $\mathscr{N} = \mathtt{kernel}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\mathtt{n}-1} \end{bmatrix}\right)$ **and** $\mathscr{S}$ **any complement.**

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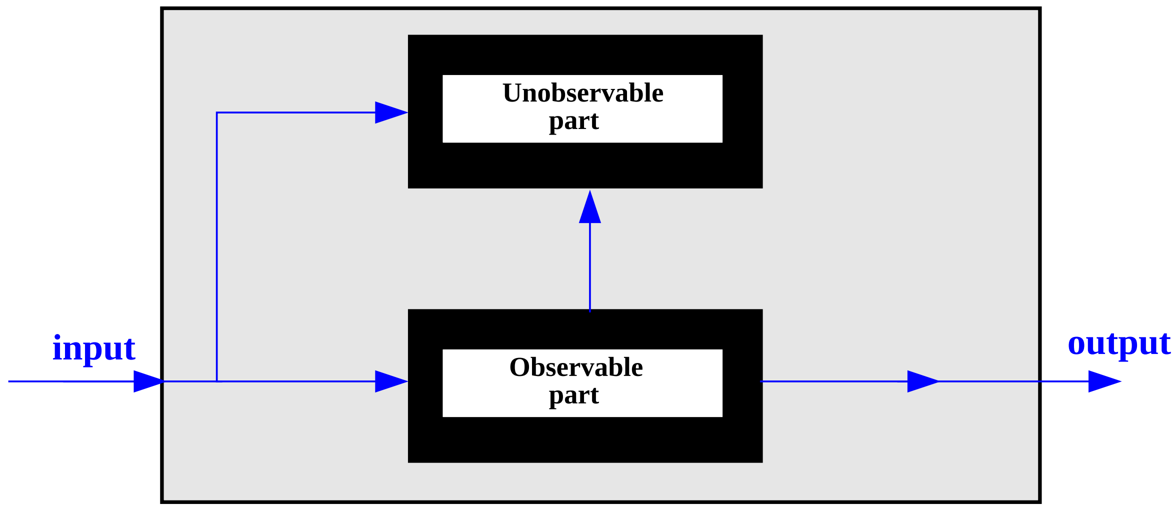
with $\frac{d}{dt}z_2 = A_{22}z_2 + B_2 u, y = C_2 z_2 + Du$ **observable.**

This decomposition brings out the observability structure.



4-way decomposition

We can combine the controllable / uncontrollable with the unobservable / observable decomposition,

$$\mathbb{X} = \overbrace{\mathscr{S}_1 \oplus \underbrace{\mathscr{R} \cap \mathscr{N}}_{}}^{\mathscr{R}} \oplus \mathscr{S}_3 \oplus \mathscr{S}_4,$$

$$\mathscr{R} = \mathscr{S}_1 \oplus (\mathscr{R} \cap \mathscr{N}), \quad \mathscr{N} = (\mathscr{R} \cap \mathscr{N}) \oplus \mathscr{S}_3$$

Again it should be noted that the $\mathscr{R} \cap \mathscr{N}, \mathscr{R},$ and $\mathscr{R}$ are 'intrinsic'. The complements $\mathscr{S}_1, \mathscr{S}_3, \mathscr{S}_4$ are not.

4-way decomposition

Choose the basis conformably, $Sx = z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \rightsquigarrow$

$$\begin{array}{rclcccccccc}
\frac{d}{dt}z_1 & = & A_{11}z_1 & & & & & + & A_{24}z_4 & + & B_1u \\
\frac{d}{dt}z_2 & = & A_{21}z_1 & + & A_{22}z_2 & + & A_{23}z_3 & + & A_{24}z_4 & + & B_2u \\
\frac{d}{dt}z_3 & = & & & & & A_{33}z_3 & + & A_{34}z_4 & & \\
\frac{d}{dt}z_4 & = & & & & & & & A_{24}z_4 & & \\
y & = & C_1z_1 & & & & & + & C_4z_4 & + & Du
\end{array}$$

with, in particular,

$$\frac{d}{dt}z_1 = A_{11}z_1 + B_1u, \;\; y = C_1z_1 + Du$$

controllable and observable.

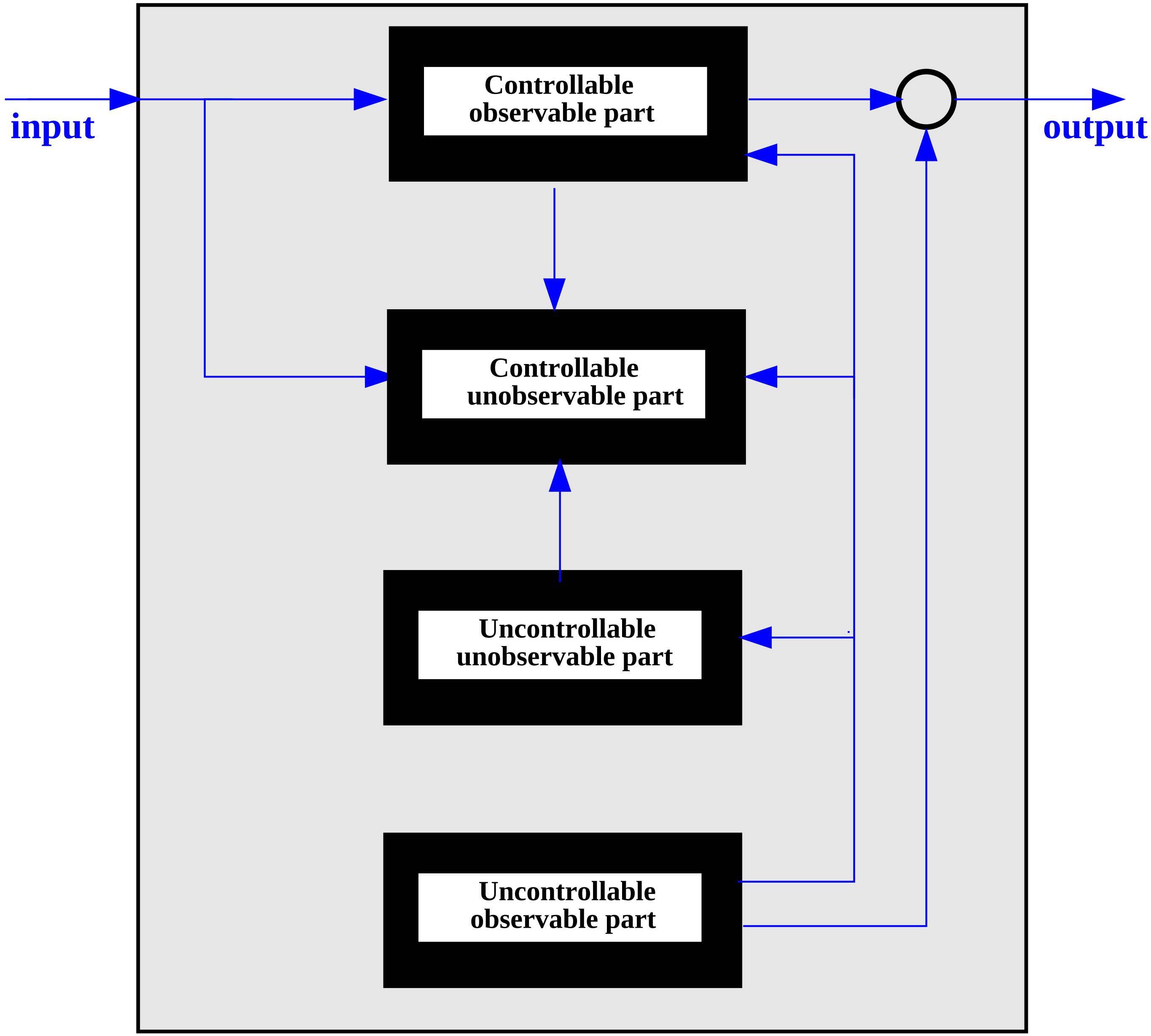
4-way Kalman decomposition

This leads to the 4-way decomposition (often called the Kalman decomposition) in the

1. **controllable / observable part (co)**
2. **controllable / unobservable part ($c\bar{o}$)**
3. **uncontrollable / unobservable part ($\overline{co}$)**
4. **uncontrollable / observable part ($\bar{c}o$)**

For systems that are controllable & observable, only the first part is present.

Kalman decomposition



Recapitulation

- The most studied representation of LTIDSs are the FDLSs $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. It combines the convenience of a state representation and an input/output partition.
- A LTIDS can be represented as an FDLS, up to reordering of the signal components.
- State stability $\Leftrightarrow$ A is Hurwitz.
  A central equation in stability questions is the Lyapunov equation.
- There exist explicit tests for state controllability and state observability.
- By choosing the basis in the state space appropriately, the controllable/uncontrollable and the observable/unobservable parts are put in evidence.

Discrete time

Discrete-time systems

The notions of state stability, state controllability, and state observability apply, *mutatis mutandis*, to discrete-time systems.

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The notions of state stability, state controllability, and state observability apply, *mutatis mutandis*, to discrete-time systems.

State stability in the discrete-time case requires that

all eigenvalues of A should be inside the unit circle

Matrices with this property are called *Schur matrices*

Issai Schur
1875 – 1941

Discrete-time systems

The analogue of the Lyapunov equation is $A^\top XA - X = Y$ **of called the 'discrete-time Lyapunov equation or the 'Stein equation'. It is actually a special case of the Stein equation which is** $A_1XA_2 - X = Y$**.**

Theorem

The following conditions on $A \in \mathbb{R}^{n\times n}$ **are equivalent:**

1. A **is Schur**
2. **there exists a solution** $X \in \mathbb{R}^{n\times n}$ **to**

$$X = X^\top \succ 0, \quad A^\top XA - X \prec 0$$

3. $\forall\, Y = Y^\top \prec 0, \exists\, X = X^\top \succ 0$ **s.t.** $\boxed{A^\top XA - X = Y}$

Discrete-time systems

The state controllability and state observability theorems and the decompositions apply unchanged in the discrete-time case.

The impulse response & transfer function

The impulse response

$$D\delta + W$$

with δ **the** **δ-'function'**, **and** $W : [0, \infty) \to \mathbb{R}^{\mathtt{p} \times \mathtt{m}}$ **defined by**

$$W : t \mapsto Ce^{At}B$$

is called the ***impulse response*** **(IR) (matrix) of** $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$**.**

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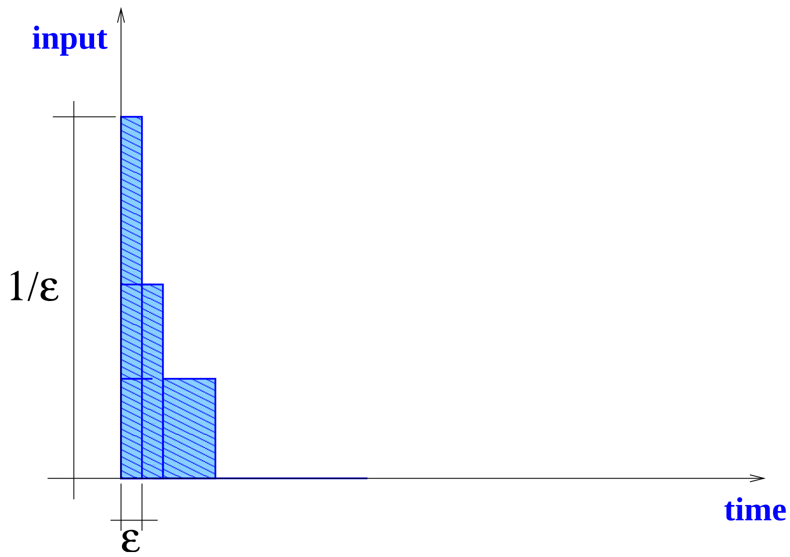
The response $y : \mathbb{R}_+ \to \mathbb{R}^{\mathtt{p}}$ **to the input** $u : \mathbb{R}_+ \to \mathbb{R}^{\mathtt{m}}$ **with zero initial condition** $x(0) = 0$ **is the convolution of the IR with** u**:**

$$y(t) = Du(t) + \int_0^t W(t')u(t-t')\,dt'$$

Informally: initial state $x(0^-) = 0$**, impulse input at** 0^+**, output for** $t \geq 0$**.**

An impulse

The entries of the IR matrix record channel-by-channel the response for $t \geq 0$ to an impulse input with initial state $x(0) = 0$.



$\varepsilon \to 0$ **illustrates an impulse**

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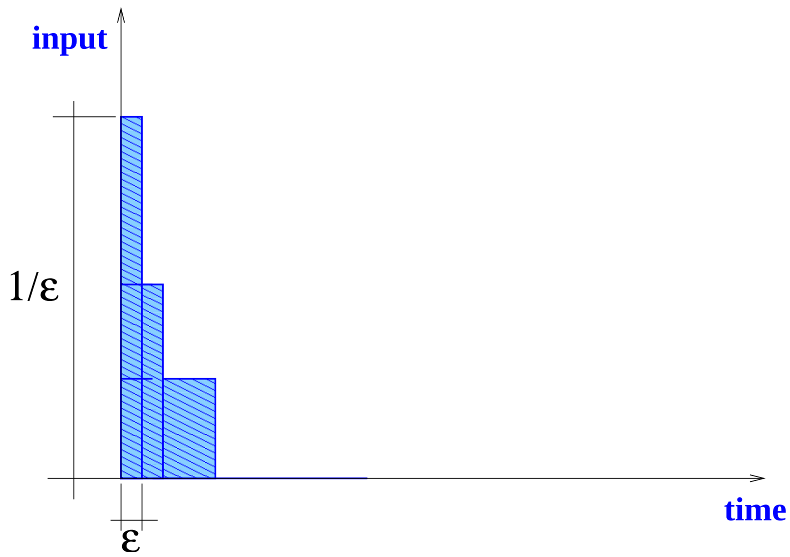

$\varepsilon \to 0$ **illustrates an impulse**

Note that $\int_0^t W(t')u_\varepsilon(t-t')\,dt' \underset{\varepsilon\to 0}{\longrightarrow} W(t)$ **for** $t \in \mathbb{R}_+$

The impulse response is Bohl

A function $f : \mathbb{R}_+ \to \mathbb{R}$ of the form

$$f(t) = \sum_{\mathtt{k}=1}^{\mathtt{n}} p_{\mathtt{k}}(t) e^{\lambda_{\mathtt{k}} t} \sin(\omega_{\mathtt{k}} t + \varphi_{\mathtt{k}})$$

with $\mathtt{n} \in \mathbb{Z}_+$, the $p_{\mathtt{k}}$'s real polynomials, and the $\lambda_{\mathtt{k}}, \omega_{\mathtt{k}}, \varphi_{\mathtt{k}}$'s real numbers is called a *Bohl function*. The set of Bohl functions is closed under addition and multiplication.

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Theorem

$D\delta + W$ **with** $D \in \mathbb{R}^{\mathtt{p} \times \mathtt{m}}$ **and** $W : \mathbb{R}_+ \to \mathbb{R}^{\mathtt{p} \times \mathtt{m}}$ **is the IR of a system** $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **iff**

W **is a matrix of Bohl functions**

Proof of the Bohl function theorem

(Outline) only if Observe that $t \in \mathbb{R} \mapsto e^{At} \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}$ **is a matrix of Bohl functions. Hence** $t \in \mathbb{R} \mapsto Ce^{At}B \in \mathbb{R}^{\mathtt{p} \times \mathtt{m}}$ **is.**

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if **First observe that the IR of** $\left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D_1 + D_2\right)$ **is the sum of the IRs of** (A_1, B_1, C_1, D_1) **and** (A_2, B_2, C_2, D_2)**.** **This implies that it suffices to consider the case** $\mathtt{m} = \mathtt{p} = 1$**.** **Next prove that** $t \mapsto t^{\mathtt{k}}e^{\lambda t}\cos(\omega t)$ **and** $t \mapsto t^{\mathtt{k}}e^{\lambda t}\sin(\omega t)$ **are IRs of FDLSs by considering**

$$A = \begin{bmatrix} A' & I_{2\times 2} & 0 & \cdots & 0 \\ 0 & A' & I_{2\times 2} & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & 0 & A' & I_{2\times 2} \\ 0 & \cdots & 0 & 0 & A' \end{bmatrix} \quad \textbf{with} \quad A' = \begin{bmatrix} \lambda & \omega \\ -\omega & \lambda \end{bmatrix}$$

and choosing B **and** C **appropriately.**

The transfer function

The matrix of rational functions

$$G(\xi) := D + C(I\xi - A)^{-1}B \in \mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{m}}$$

is called the *transfer function* (TF) of $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$**.**

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Consider a complex number $s \in \mathbb{C}$**, not an eigenvalue of** A**. Corresponding to the exponential input** $t \in \mathbb{R} \mapsto e^{st}, \mathtt{u} \in \mathbb{C}^{\mathtt{m}}$**,** $\mathtt{u} \in \mathbb{C}^{\mathtt{m}}$**, there is a unique exponential output** $t \in \mathbb{R} \mapsto e^{st}, \mathtt{y} \in \mathbb{C}^{\mathtt{p}}$ **with** $\mathtt{y} \in \mathbb{C}^{\mathtt{p}}$ **given in terms of** $\mathtt{u} \in \mathbb{C}^{\mathtt{m}}$ **by**

$$\mathtt{y}(s) = G(s)\mathtt{u}(s)$$

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$$\mathtt{y}(s) = G(s)\mathtt{u}(s)$$

This holds not only for those $s \in \mathbb{C}$ that are not eigenvalues of A, but for $s \in \mathbb{C}$ that are not poles of G. Poles of G are eigenvalues of A, but the converse is not necessarily trues (unless the system is state controllable and state observable) We return to this later.

The transfer function and Laplace tfms

Let $y : \mathbb{R}_+ \to \mathbb{R}^{\mathtt{p}}$ **be the output of** $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **for the input** $u : \mathbb{R}_+ \to \mathbb{R}^{\mathtt{m}}$ **and** $x(0) = 0$**. Assume that** u **is Laplace transformable.**

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Then y **is also Laplace transformable with domain of convergence the intersection of the domain of convergence of** u **and the half plane to the right of the poles on** G. **The Laplace transforms** $\hat{u}, \hat{y}$ **of** u **and** y **are related by**

$$\hat{y}(s) = G(s)\hat{u}(s)$$

The impulse response and the transfer function as Laplace tfms

Let $D\delta + W$ **be the IR and** G **be the TF of** $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$**.**

G **is the Laplace transform of** $D\delta + W$**:**

$$G(s) = D + \int_0^\infty W(t) e^{-st}\, dt$$

for all $s \in \mathbb{C}$ **to the right of the poles of** G**.**

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for all $s \in \mathbb{C}$ **to the right of the poles of** G**. Conversely,**

$$D = G(\infty) \text{ and } W(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} (G(s) - G(\infty))\, ds$$

where the integration is along a vertical line in $\mathbb{C}$ **to the right of the poles of** G**.** $G(\infty)$ **is the 'constant term', or the non-strictly proper term of** G**, say,** $G(\infty) := \lim_{\lambda \to \infty} G(\lambda)$**.**

The transfer function

Theorem

The TF of the system $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is

a matrix of proper real rational functions

Conversely, for any $\mathtt{p} \times \mathtt{m}$ matrix G of proper real rational functions, there exists a system $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ that has TF G.

Proof of the rational function theorem

(Outline) **only if** **Since** $(I\xi - A)^{-1}$ **is a matrix of proper real rational functions, so is** $D + C(I\xi - A)^{-1}B$**.**

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if **Use the addition property, explained in the proof of the IR case, to show that it suffices to consider** $\mathtt{m} = \mathtt{p} = 1$**.**
Use partial fraction expansion to reduce to the cases

$$G(\xi) = \frac{1}{(\xi+\lambda)^{\mathtt{k}}}, G(\xi) = \frac{1}{((\xi+\lambda)^2+\omega^2)^{\mathtt{k}}}, G(\xi) = \frac{\xi}{((\xi+\lambda)^2+\omega^2)^{\mathtt{k}}}.$$

Series connection $\rightsquigarrow \mathtt{k} = 1$**. Finally, contemplate the TFs of single-input / single-output system with**

$$A = -\lambda \text{ and } A = \begin{bmatrix} -\lambda & \omega \\ -\omega & -\lambda \end{bmatrix}$$

and suitable B, C**.**

Minimal realizations

Definition of minimal realization

A system $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **is called a** ***realization*** **of its IR** $D\delta + W$**,** $W : t \in \mathbb{R}_+ \to Ce^{At}B \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$ **and of its TF** $G, G(\xi) = D + C(I\xi - A)^{-1}B$**.**

Definition of minimal realization

A system $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is called a *realization* of its IR $D\delta + W$, $W : t \in \mathbb{R}_+ \rightarrow Ce^{At}B \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$ and of its TF $G, G(\xi) = D + C(I\xi - A)^{-1}B$.

It is called a *minimal realization* of its IR if the dimension of its state space is as small as possible among all realizations of its IR, and of its TF if the dimension of its state space is as small as possible among all realizations of its TF.

Conditions for minimality

Theorem

The following conditions on $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ are equivalent:

- it is a minimal realization of its impulse response
- it is a minimal realization of its transfer function
- it is

state controllable and state observable

The essence of this theorem is that minimality of a realization corresponds ***exactly*** to

state controllability & state observability combined.

Proof of the minimal realization theorem

1. If (A, B, C, D) is not contr. + obs., then (A_{11}, B_1, C_1, D) from the Kalman dec. gives a lower order realization with the same IR and TF. Hence minimality implies contr. + obs.

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2. Assume that (A,B,C,D), of order n, is a min. real. of its IR Let (A_1,B_1,C_1,D), of order n_1, have the same IR. Hence $Ce^{At}B = C_1e^{A_1t}B_1$ for $t \geq 0$. This implies $CA^{\mathtt{k}}B = C_1A_1^{\mathtt{k}}B_1$ for $\mathtt{k} \in \mathbb{Z}_+$. Hence

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\mathtt{n}-1} \end{bmatrix} \begin{bmatrix} B & AB & \cdots A^{\mathtt{n}-1}B \end{bmatrix} = \begin{bmatrix} C_1 \\ C_1A_1 \\ \vdots \\ C_1A_1^{\mathtt{n}-1} \end{bmatrix} \begin{bmatrix} B_1 & A_1B_1 & \cdots A_1^{\mathtt{n}-1}B_1 \end{bmatrix}$$

The LHS is the product of an injective $\times$ a surjective matrix. $\Rightarrow \mathtt{rank} = n$. The RHS is the product of two matrices with 'inner dimension' n_1. $\Rightarrow \mathtt{rank} \leq n_1$. Therefore, $n \leq n_1$.

Proof of the minimal realization theorem

3. If $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **realizes the TF** G**, then**

$$G(\xi) = D + \frac{CB}{\xi} + \frac{CAB}{\xi^2} + \frac{CA^2B}{\xi^3} + \cdots$$

Hence (A,B,C,D) **and** (A_1,B_1,C_1,D) **have the same TF iff** $CA^{\mathtt{k}}B = C_1A_1^{\mathtt{k}}B_1$ **for** $\mathtt{k} \in \mathbb{Z}_+$**. The proof now follows 2.**

The exponential response

If $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **is a minimal realization of its TF,** G**, then the eigenvalues of** A **are identical to the poles of** G **(including multiplicity).**

This may be used to show that corresponding to the exponential input $t \in \mathbb{R} \mapsto e^{st}\mathtt{u} \in \mathbb{C}^{\mathtt{m}}$, $\mathtt{u} \in \mathbb{C}^{\mathtt{m}}$**, there is a unique exponential output** $t \in \mathbb{R} \rightarrow e^{st}\mathtt{y} \in \mathbb{C}^{\mathtt{p}}$ **with** $\mathtt{y} = G(s)\mathtt{u}$ **for all** $s \in \mathbb{C}$ **that are not poles of** G**.**

Reduction algorithm

It is easy to see that the IR and TF of $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is equal to that of $\left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$, the controllable / observable part of the Kalman decomposition. The latter, being controllable and observable, is a minimal realization of the IR and TF.

This gives a method of obtaining a system that is a minimal realization of the IR or the TF of a non-minimal one.

State isomorphism theorem

Theorem

Assume that $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **is a minimal realization of its IR, equivalently, of its TF. Let** $\mathtt{n}$ **be the dimension of its state space. Denote the set of invertible elements of** $\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ **by** $\mathscr{G}\ell(\mathtt{n})$**. Then all minimal realizations are obtained by the transformation group**

$$(A,B,C,D) \xrightarrow{S\in\mathscr{G}\ell(\mathtt{n})} (SAS^{-1},SB,CS^{-1},D)$$

In other words, after fixing the IR or the TF, the choice of basis in the state space is all that remains in minimal realizations.

Proof of the state isomorphism theorem

All systems $(SAS^{-1}, SB, CS^{-1}, D)$ **obviously have the same IR.**

Proof of the state isomorphism theorem

All systems $(SAS^{-1}, SB, CS^{-1}, D)$ **obviously have the same IR.**

Let $(A_\mathtt{q}, B_\mathtt{q}, C_\mathtt{q}, D), \mathtt{q} = 1, 2$ **both be minimal** $\mathtt{n}$ **dim. realizations of the same IR or TF. Then**

$$C_1 A_1^{\mathtt{k}} B_1 = C_2 A_2^{\mathtt{k}} B_2 \quad \textbf{for} \quad \mathtt{k} \in \mathbb{Z}_+.$$

Define

$$W_\mathtt{q} = \begin{bmatrix} C_\mathtt{q} \\ C_\mathtt{q} A_\mathtt{q} \\ \vdots \\ C_\mathtt{q} A_\mathtt{q}^{\mathtt{n}-1} \end{bmatrix}, \quad R_\mathtt{q} = \begin{bmatrix} B_\mathtt{q} & A_\mathtt{q} B_\mathtt{q} & \cdots A_\mathtt{q}^{\mathtt{n}-1} B_\mathtt{q} \end{bmatrix}, \mathtt{q} = 1, 2.$$

By controllability and observability, $W_\mathtt{q}$ **is injective and** $R_\mathtt{q}$ **surjective for** $\mathtt{q} = 1, 2$**.**

Proof of the isomorphism theorem

Let $W_2^\dagger$ be a left inverse of W_2 and $R_2^\dagger$ a right inverse of R_2.

Define $\boxed{S = W_2^\dagger W_1}$**. Then**

- $[\![W_1 R_1 = W_2 R_2]\!] \Rightarrow [\![W_2^\dagger W_1 R_1 R_2^\dagger = I_{\mathtt{n}\times\mathtt{n}}]\!]$. **Hence** $S^{-1} = R_1 R_2^\dagger$.
- $[\![W_2^\dagger W_1 R_1 = R_2]\!] \Rightarrow [\![B_2 = S B_1]\!]$
- $[\![W_1 R_1 R_2^\dagger = W_2]\!] \Rightarrow [\![C_2 = C_1 S^{-1}]\!]$
- $[\![W_1 A_1 R_1 = W_2 A_2 R_2]\!] \Rightarrow [\![A_2 = W_2^\dagger W_1 A_1 R_1 R_2^\dagger = S A_1 S^{-1}]\!]$

This implies that

$$\boxed{(A,B,C,D) \xrightarrow{S \in \mathscr{G}\ell(\mathtt{n})} (SAS^{-1}, SB, CS^{-1}, D)}$$

generates all minimal realizations from one.

Relations with behavioral systems

The behavior respects the uncontrollable part of the system.
The impulse response and the transfer function ignore it.

The external behavior

We have see that any $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ allows a representation $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ in the following sense. There exists $\mathtt{m}, \mathtt{p} \in \mathbb{Z}_+$ with $\mathtt{m} + \mathtt{p} = \mathtt{w}$, a componentwise partition of w into $w = \begin{bmatrix} u \\ y \end{bmatrix}$, an $\mathtt{n} \in \mathbb{Z}_+$, and matrices A, B, C, D such that the external behavior equals $\mathscr{B}$.

With *componentwise partition* we mean that there exists a permutation Π such that $w = \Pi \begin{bmatrix} u \\ y \end{bmatrix}$.

The external behavior

The external behavior of (A,B,C,D) **is governed by**

$$P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u \qquad (*)$$

with $P \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{p}}, Q \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{m}}$**,** P **nonsingular,** $P^{-1}Q$ **proper.**

It is easy to see that $G = P^{-1}Q$**.**

The external behavior

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with $P \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{p}}, Q \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{m}}$, P nonsingular, $P^{-1}Q$ proper.

It is easy to see that $G = P^{-1}Q$. However, (P,Q) contains more information than $G = P^{-1}Q$. Since P and Q in $(*)$ may not be left coprime polynomial matrices (see lecture 2), since the system $(*)$ also models the uncontrollable part of $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$.

IRs and TFs do not capture the uncontrollable part

The uncontrollable part is important in many applications.

State construction for $\mathscr{L}^{\mathtt{w}}$ is covered in lecture 12.

The external behavior

Two systems $\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ **and** $\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$ **can have the same IR and TF, but can differ drastically because of the non-controllable part.**

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Example: Autonomous systems are very important in applications. In particular, all systems $\frac{d}{dt}x = Ax + 0u, y = Cx$ **have the same TF function** 0**.**

Recapitulation

- FDLS $\Leftrightarrow$ IR is Bohl $\Leftrightarrow$ TF is proper rational.
- A realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ of a TF or an IR is minimal $\Leftrightarrow$ it is state controllable + state observable.
- All minimal realizations are equivalent up to a choice of basis in the state space.
- The IR and the TF capture only the controllable part of a system.

Discrete-time convolutions

The impulse response matrix

We now look at a particular class of systems for which very concrete realization algorithms have been derived. Notably, discrete-time systems

$$y(t) = \sum_{t'=0}^{t} H(t')u(t-t') \qquad \textbf{for } t \in \mathbb{Z}_+.$$

The impulse response matrix

Call $H : \mathbb{Z}_+ \rightarrow \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$ **the** ***IR (matrix).***

Origin: consider the 'impulse' input $u_{\mathtt{k}} : \mathbb{Z}_+ \rightarrow \mathbb{R}^{\mathtt{m}}$ $u_{\mathtt{k}} = \delta e_{\mathtt{k}}$ **with** $e_{\mathtt{k}}$ **the** $\mathtt{k}$**-th basis vector in** $\mathbb{R}^{\mathtt{m}}$**, and** $\delta : \mathbb{Z}_+ \rightarrow \mathbb{R}$ **the 'pulse'**

$$\delta(t) := \begin{cases} 1 \textbf{ for } t = 0 \\ 0 \textbf{ for } t > 0 \end{cases}$$

The corresponding output $\mathtt{y}_k =$ **the** $\mathtt{k}-$**th column of** H**.**
Arrange as a matrix $\rightsquigarrow$ **the 'IR' matrix.**

The realization question

The question studied now:

Go from the IR to a minimal state representation

i.e. to a recursive model.

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Go from the IR to a minimal state representation

i.e. to a recursive model.

When and how can the convolution

$$y(t) = \sum_{t'=0}^{t} H(t')u(t-t')$$

be represented by

$$x(t+1) = Ax(t) + Bu(t), \;\; y(t) = Cx(t) + Du(t), \;\; x(0) = 0 \;?$$

Realizability equations

¡¡ Construct $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **from** H**, such that for all inputs** $u : \mathbb{Z}_+ \to \mathbb{R}^{\mathtt{m}}$**, the output responses** $y : \mathbb{Z}_+ \to \mathbb{R}^{\mathtt{p}}$ **are equal !!**

$H : \mathbb{Z}_+ \to \mathbb{R}^{\mathtt{p} \times \mathtt{m}}$ **is given, and the matrices** A, B, C, D
(!! including $\mathtt{n} = \dim(\mathbb{X}) = \dim(A)$**) are the unknowns.**

Realizability equations

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(!! including $\mathtt{n} = \dim(\mathbb{X}) = \dim(A)$**) are the unknowns.**

The IR matrix of $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **is equal to**

$$D, CB, CAB, \ldots, CA^{t-1}B, \ldots$$

$\leadsto$ **realization iff**

$$D = H(0) \quad \textbf{and} \quad CA^{t-1}B = H(t) \quad \textbf{for } t \in \mathbb{N}$$

Given H**, find** $(A, B, C, D)!$ $\leadsto$ **nonlinear equations, there may not be a sol'n, if there is one, not unique!**

Realizability equations

Notation: $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **or** $(A, B, C, D) \Rightarrow H$

Realizability equations

Notation: $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **or** $(A,B,C,D) \Rightarrow H$

$$\mathtt{n}_{\min}(H) := \mathtt{min}\{\mathtt{n} \mid \ \exists \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \text{of order } \mathtt{n},\ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \Rightarrow H\}$$

The corr. $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **is called a *minimal realization* of** H**.**

The central role in this theory and the related algorithms is played by the *Hankel matrix*.

The Hankel matrix

$$\mathscr{H}_H := \begin{bmatrix} H(1) & H(2) & H(3) & \cdots & H(t'') & \cdots \\ H(2) & H(3) & H(4) & \cdots & H(t''+1) & \cdots \\ H(3) & H(4) & H(5) & \cdots & H(t''+2) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ H(t') & H(t'+1) & H(t'+2) & \cdots & H(t'+t''-1) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}$$

plays the lead role in realization theory and model reduction.

Toeplitz and Hankel: close cousins

The Toeplitz matrix

We show how the Hankel matrix arises, by contrasting it with the *Toeplitz matrix*. Consider $y(t) = \sum_{t'=0}^{t} H(t')u(t-t')$. Write the input and output as 'long' column vectors

$$\mathsf{u}_+ = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t) \\ \vdots \end{bmatrix} \quad \mathsf{y}_+ = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(t) \\ \vdots \end{bmatrix}.$$

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$$\leadsto \quad \mathsf{y}_+ = \begin{bmatrix} H(0) & 0 & 0 & \cdots & 0 & \cdots \\ H(1) & H(0) & 0 & \cdots & 0 & \cdots \\ H(2) & H(1) & H(0) & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ H(t) & H(t-1) & H(t-2) & \cdots & H(0) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix} \mathsf{u}_+.$$

In shorthand,

$$y_+ = \mathscr{T}_H \mathsf{u}_+$$

The Toeplitz matrix

The matrix $\mathfrak{T}_H$ has special structure: the elements in lines *parallel to the diagonal* are identical block matrices.

Such (finite or infinite) matrices are called *Toeplitz* — *block Toeplitz* perhaps being more appropriate.

Otto Toeplitz
1881 – 1940

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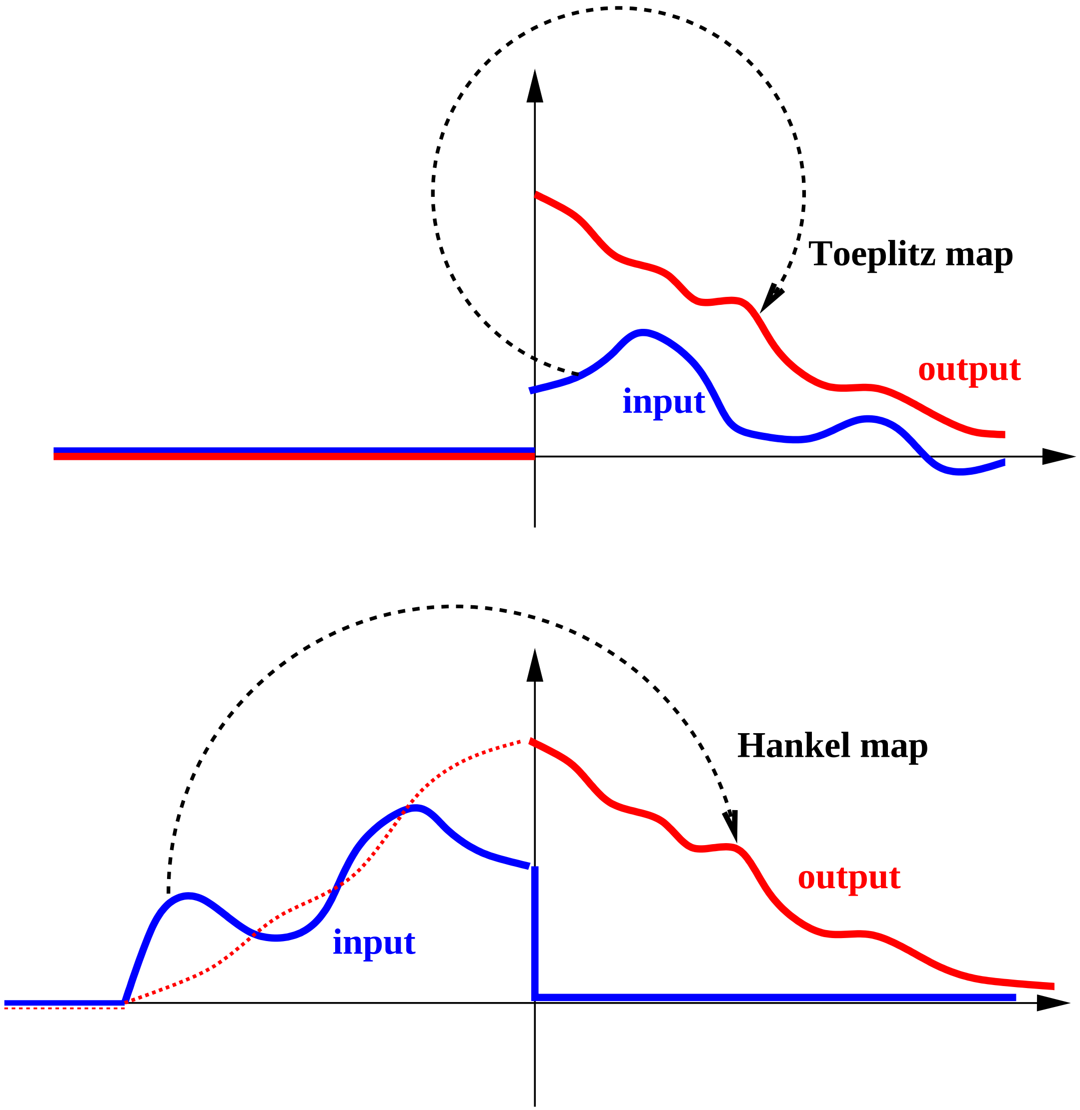
1881 – 1940

The Toeplitz matrix tells what the output corresponding to an input is.

That it comes up in linear system theory is evident: it codifies convolution.

Toeplitz and Hankel maps



The Hankel matrix

Consider an input that starts at some time in the past and 'ends' at $t = 0$. We are only interested in the response for $t \geq 0$.

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Consider an input that starts at some time in the past and 'ends' at $t = 0$. We are only interested in the response for $t \geq 0$. Then

$$y(t) = \Sigma_{t' \in \mathbb{Z}_+} H(t') u(t - t'), \quad t \in \mathbb{Z}_+.$$

Write the past input and future output as 'long' column vectors

$$\mathsf{u}_- = \begin{bmatrix} u(-1) \\ u(-2) \\ \vdots \\ u(-t) \\ \vdots \end{bmatrix} \qquad \mathsf{y}_+ = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(t) \\ \vdots \end{bmatrix}.$$

The Hankel matrix

The relation between the 'past' input u_- and the 'future' output y_+ is

$$\mathrm{y}_+ = \begin{bmatrix} H(1) & H(2) & H(3) & \cdots & H(t'') & \cdots \\ H(2) & H(3) & H(4) & \cdots & H(t''+1) & \cdots \\ H(3) & H(4) & H(5) & \cdots & H(t''+2) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ H(t') & H(t'+1) & H(t'+2) & \cdots & H(t'+t''-1) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix} \mathrm{u}_- .$$

Note: since u has 'compact support', no convergence pbms.

The Hankel matrix

In shorthand, in terms of the (infinite) Hankel matrix $\mathscr{H}_H$,

$$\mathsf{y}_+ = \mathscr{H}_H \mathsf{u}_-$$

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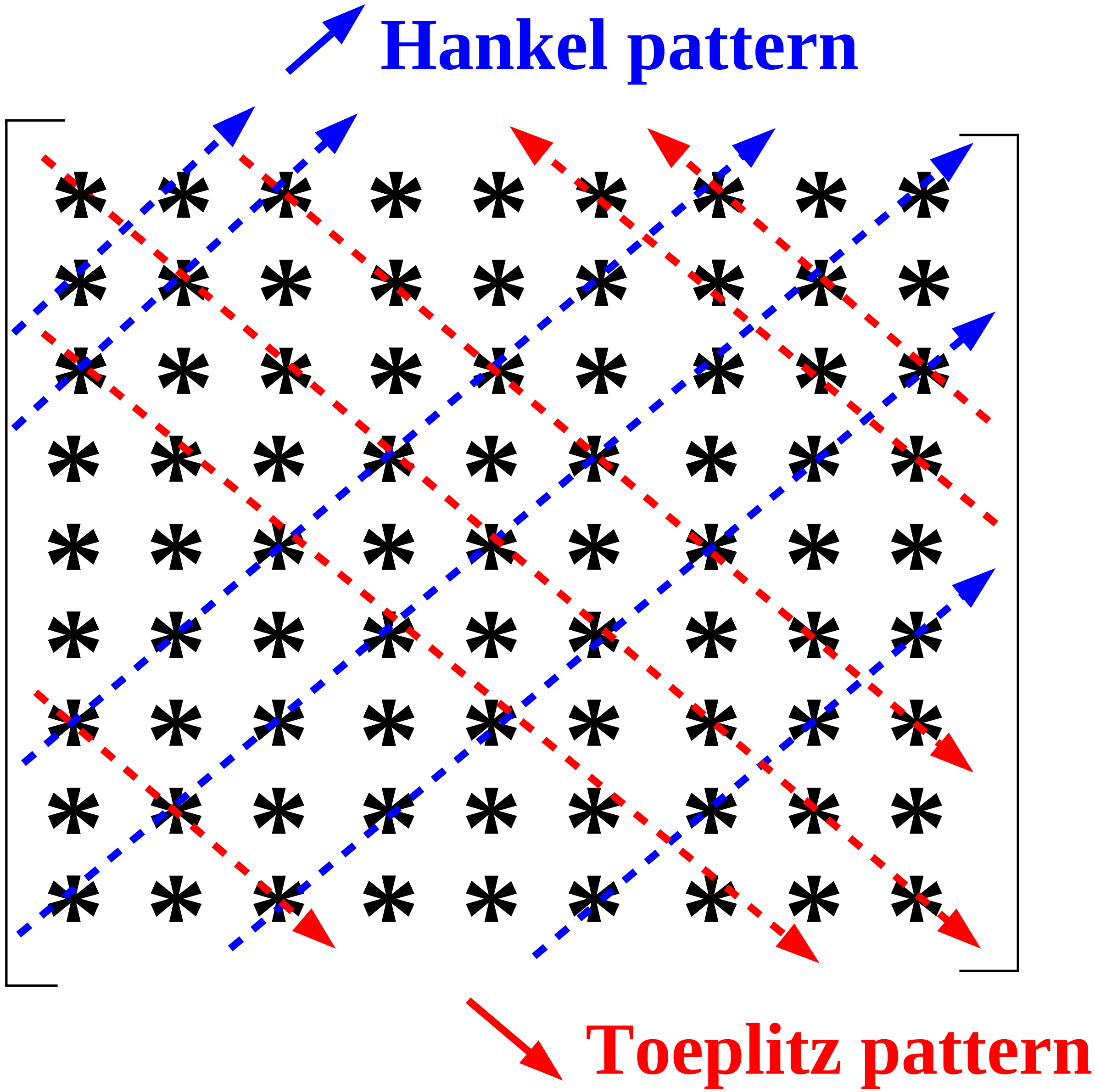
Such matrices (finite or infinite) are called *Hankel* — *block Hankel* being perhaps more appropriate.

Herman Hankel
1839 – 1873

Hankel matrices: central role in realization problems and in model reduction

Toeplitz and Hankel matrices



The shifted Hankel matrix

We will also meet the *shifted Hankel matrix*

$$\mathscr{H}_{\sigma H} := \begin{bmatrix} H(2) & H(3) & H(4) & \cdots & H(t''+1) & \cdots \\ H(3) & H(4) & H(5) & \cdots & H(t''+2) & \cdots \\ H(4) & H(5) & H(6) & \cdots & H(t''+3) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ H(t'+1) & H(t'+2) & H(t'+3) & \cdots & H(t'+t'') & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix},$$

obtained by deleting the first block row (equivalently, block column) from $\mathscr{H}_H$,

The truncated Hankel matrix

and the *truncated Hankel matrix*

$$\mathscr{H}_H^{t',t''} := \begin{bmatrix} H(1) & H(2) & H(3) & \cdots & H(t'') \\ H(2) & H(3) & H(4) & \cdots & H(t''+1) \\ H(3) & H(4) & H(5) & \cdots & H(t''+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H(t') & H(t'+1) & H(t'+2) & \cdots & H(t'+t''-1) \end{bmatrix}$$

The rank of a Hankel matrix

Define the `rank` **of an infinite matrix in the obvious way: as the supremum of the ranks of its submatrices.**
The rank of an infinite matrix can hence be ∞.

Note that

$$\mathtt{rank}(\mathscr{H}_H) \quad = \quad \sup_{t',t''\in\mathbb{N}}\{\mathtt{rank}(\mathscr{H}_H^{t',t''})\}.$$

Realizability

The realization theorem

Theorem

Let $H : \mathbb{Z}_+ \to \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$ **be an IR matrix.**

- $[\![\exists\ (A, B, C, D) \Rightarrow H]\!] \Leftrightarrow [\![\mathtt{rank}(\mathscr{H}_H) < \infty]\!]$

The realization theorem

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- **The order of the realization** (A,B,C,D) **is** $\mathtt{n}_{\min}(H)$ **iff it is controllable and observable**
- **All minimal realizations of** H **are generated from one by the transformation group**

$$(A,B,C,D) \xrightarrow{S\in\mathscr{G}\ell(\mathtt{n}_{\min}(H))} (SAS^{-1},SB,CS^{-1},D)$$

Proof of the realization theorem

Algorithms

A general realization algorithm

Step 1:

Find a sub-matrix M **of** $\mathscr{H}_H$ **with**

$$\texttt{rank}(M) = \texttt{rank}(\mathscr{H}_H) \quad (= \texttt{n}_{\min}(\mathscr{H}_H)).$$

Say M **is formed by the elements in rows** $\texttt{r}_1, \texttt{r}_2, \ldots, \texttt{r}_{\texttt{n}'}$ **and columns** $\texttt{k}_1, \texttt{k}_2, \ldots, \texttt{k}_{\texttt{n}''}$. **Whence** $M \in \mathbb{R}^{\texttt{n}' \times \texttt{n}''}$**.**

A general realization algorithm

Step 2:

Let $\sigma M \in \mathbb{R}^{\mathtt{n}' \times \mathtt{n}''}$ **be the sub-matrix of** $\mathscr{H}_{\sigma H}$ **formed by the elements in rows** $\mathtt{r}_1, \mathtt{r}_2, \ldots, \mathtt{r}_{\mathtt{n}'}$ **and columns** $\mathtt{k}_1, \mathtt{k}_2, \ldots, \mathtt{k}_{\mathtt{n}''}$.

Equivalently, by the Hankel structure, the sub-matrix of $\mathscr{H}_H$ **formed by the elements in rows** $\mathtt{r}_1 + \mathtt{p}, \mathtt{r}_2 + \mathtt{p}, \ldots, \mathtt{r}_{\mathtt{n}'} + \mathtt{p}$ **and columns** $\mathtt{k}_1, \mathtt{k}_2, \ldots, \mathtt{k}_{\mathtt{n}''}$.
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Let $R \in \mathbb{R}^{\mathtt{m} \times \mathtt{n}''}$ **be the sub-matrix of** $\mathscr{H}_H$ **formed by the elements in the first** $\mathtt{p}$ **rows and columns** $\mathtt{k}_1, \mathtt{k}_2, \ldots, \mathtt{k}_{\mathtt{n}''}$**.**

A general realization algorithm

Step 2:

Let $\sigma M \in \mathbb{R}^{\mathtt{n}' \times \mathtt{n}''}$ **be the sub-matrix of** $\mathscr{H}_{\sigma H}$ **formed by the elements in rows** $\mathtt{r}_1, \mathtt{r}_2, \ldots, \mathtt{r}_{\mathtt{n}'}$ **and columns** $\mathtt{k}_1, \mathtt{k}_2, \ldots, \mathtt{k}_{\mathtt{n}''}$.

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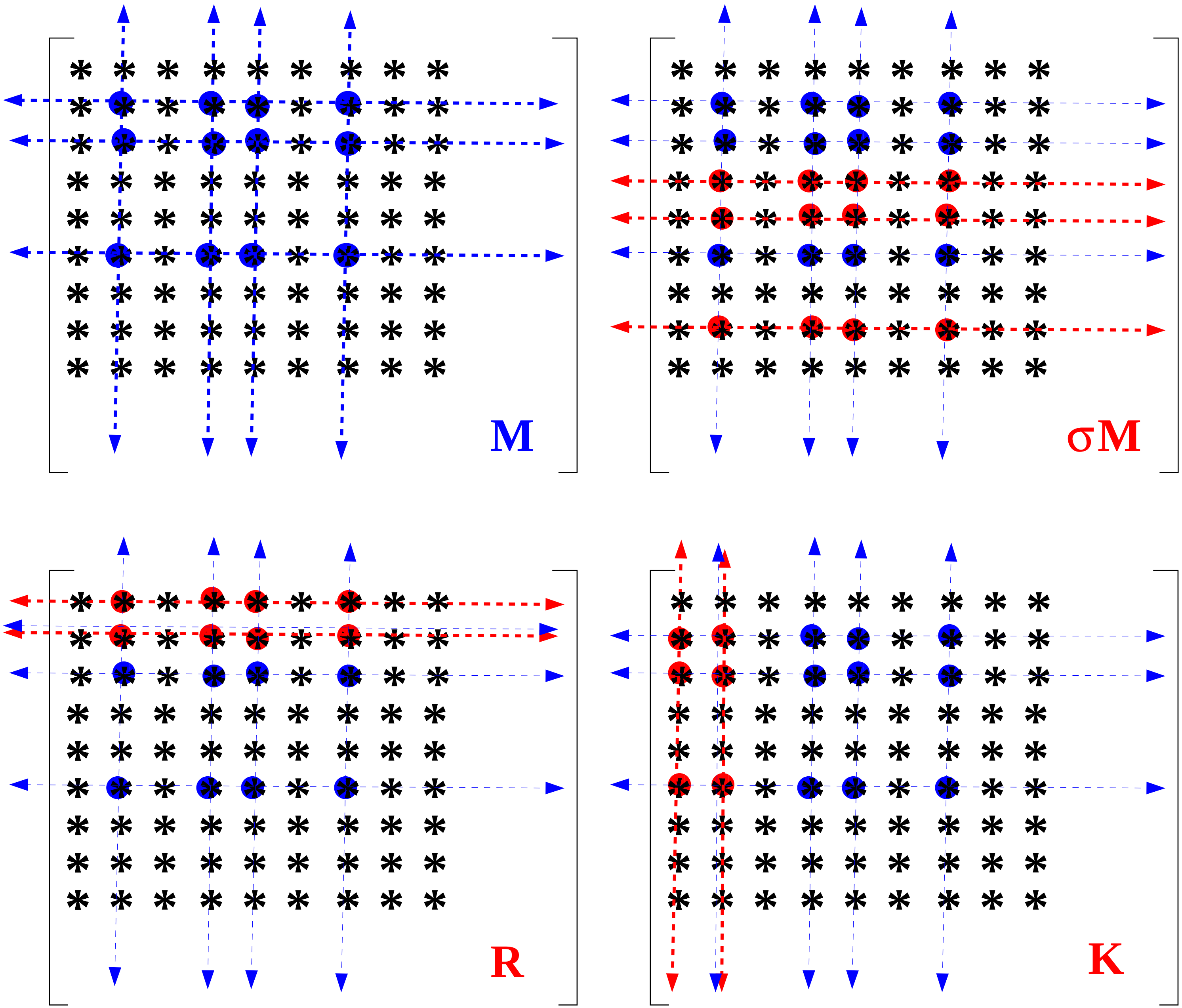
Equivalently, by the Hankel structure, the sub-matrix of $\mathscr{H}_H$ **formed by the elements in rows** $\mathtt{r}_1, \mathtt{r}_2, \ldots, \mathtt{r}_{\mathtt{n}'}$ **and columns** $\mathtt{k}_1 + \mathtt{m}, \mathtt{k}_2 + \mathtt{m}, \ldots, \mathtt{k}_{\mathtt{n}''} + \mathtt{m}$.

Let $R \in \mathbb{R}^{\mathtt{m} \times \mathtt{n}''}$ **be the sub-matrix of** $\mathscr{H}_H$ **formed by the elements in the first** $\mathtt{p}$ **rows and columns** $\mathtt{k}_1, \mathtt{k}_2, \ldots, \mathtt{k}_{\mathtt{n}''}$**.**

Let $K \in \mathbb{R}^{\mathtt{n}' \times \mathtt{p}}$ **be the sub-matrix of** $\mathscr{H}_H$ **formed by the elements in the rows** $\mathtt{r}_1, \mathtt{r}_2, \ldots, \mathtt{r}_{\mathtt{n}'}$ **and the first** $\mathtt{m}$ **columns.**

$M, \sigma M, R, K$



A general realization algorithm

Step 3: **Find** $P \in \mathbb{R}^{\mathtt{n}' \times \mathtt{n}_{\min}(H)}$ **and** $Q \in \mathbb{R}^{\mathtt{n}_{\min}(H) \times \mathtt{n}''}$ **such that**

$$PMQ = I_{\mathtt{n}_{\min}(H)}$$

A general realization algorithm

<u>Step 3</u>: Find $P \in \mathbb{R}^{\mathtt{n}' \times \mathtt{n}_{\min}(H)}$ **and** $Q \in \mathbb{R}^{\mathtt{n}_{\min}(H) \times \mathtt{n}''}$ **such that**

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<u>Step 4</u>: A minimal state representation $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **of** H **is obtained as follows:**

$$\begin{aligned} A &= P\sigma MQ \\ B &= PR \\ C &= KQ \\ D &= H(0) \end{aligned}$$

Proof of the general algorithm

Realization algorithms

Important special cases of this general algorithm:

1. The Ho-Kalman algorithm: of historical interest.

2. Silverman's algorithm

M **: a non-singular maximal rank (hence** $\mathrm{n}_{\mathrm{min}(H)} \times \mathrm{n}_{\mathrm{min}(H)}$**) submatrix of** H**.** $\rightsquigarrow$ **minimal realizations:**

$$\begin{aligned} A &= \sigma M M^{-1} \\ B &= R \\ C &= K M^{-1} \\ D &= H(0) \end{aligned}$$

$$\begin{aligned} A &= M^{-1} \sigma M \\ B &= M^{-1} R \\ C &= K \\ D &= H(0) \end{aligned}$$

Leonard Silverman
born 1939

Very efficient!

SVD based realization algorithms

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SVD based realization algorithms

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Step 3': Determine an SVD of $M \rightsquigarrow M = U\Sigma V^\top$.

$$\Sigma = \begin{bmatrix} \Sigma_{\text{reduced}} & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_{\text{reduced}} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ldots & 0 \\ 0 & 0 & \cdots & \sigma_{\mathbf{n}_{\min}(H)} \end{bmatrix}$$

SVD based realization algorithms

Step 4':

$$A = \begin{bmatrix} \sqrt{\Sigma_{\text{reduced}}^{-1}} & 0 \end{bmatrix} U^\top \sigma M V \begin{bmatrix} \sqrt{\Sigma_{\text{reduced}}^{-1}} \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \sqrt{\Sigma_{\text{reduced}}^{-1}} & 0 \end{bmatrix} U^\top R$$

$$C = K \begin{bmatrix} \sqrt{\Sigma_{\text{reduced}}^{-1}} \\ 0 \end{bmatrix}$$

$$D = H(0)$$

Balanced realization algorithm

Our general algorithm also holds when M is an infinite sub-matrix of $\mathscr{H}_H$ (or when $M = \mathscr{H}_H$). However, convergence issues arise then when multiplying infinite matrices.

In the SVD case, for instance, we need to make some assumptions that guarantee the existence of an SVD.

Balanced realization algorithm

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In the SVD case, for instance, we need to make some assumptions that guarantee the existence of an SVD.

Assume

$$\sum_{t=1}^{\infty} ||H(t)|| < \infty$$

This condition corresponds to input/output stability of the convolution systems. It implies the existence of an SVD of $\mathscr{H}_H$.

4. An algorithm based on the SVD of $\mathscr{H}_H$:

Step 3”: Determine a ‘partial’ SVD

$$\mathscr{H}_H = U \Sigma_{\text{reduced}} V^\top, \qquad \Sigma_{\text{reduced}} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ldots & 0 \\ 0 & 0 & \cdots & \sigma_{\mathtt{n}_{\min}(H)} \end{bmatrix}.$$

U, V **have an ∞ number of rows, and columns $\in \ell_2$.**

Balanced realization algorithm

Step 4”:

$$
\begin{aligned}
A &= \sqrt{\Sigma_{\text{reduced}}^{-1}}\, U^\top \mathscr{H}_{\sigma\mathscr{H}} V \sqrt{\Sigma_{\text{reduced}}^{-1}} \\
B &= \sqrt{\Sigma_{\text{reduced}}^{-1}}\, U^\top \mathscr{H}_H^{\infty,1} \\
C &= \mathscr{H}_H^{1,\infty} V \sqrt{\Sigma_{\text{reduced}}^{-1}} \\
D &= H(0)
\end{aligned}
$$

This leads to a system $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **with nice properties: it is balanced (see lecture 8).**

Recapitulation

- **The Hankel matrix $\mathscr{H}_H$ plays the central role in the realization of a discrete time convolution.**
- **The rank of $\mathscr{H}_H$ is equal to the dimension of a minimal realization.**
- **A realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ can be computed from a maximal rank submatrix of $\mathscr{H}_H$**
- **SVD based algorithms form an important special case.**

System norms

Function spaces

$|\cdot|$ **denotes the (Euclidean) norm on** $\mathbb{R}^{\mathtt{n}}$**.**

1. Time-domain signal norms.

- $\mathscr{L}_p(\mathbb{R},\mathbb{R}^{\mathtt{n}}) = \{f:\mathbb{R}\to\mathbb{R}^{\mathtt{n}} \mid \|f\|_{\mathscr{L}_p} := \left(\int_{-\infty}^{+\infty}|f(t)|^p\,dt\right)^{\frac{1}{p}} < \infty\}$ **for** $1\leq p<\infty$
- $\mathscr{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{n}}) = \{f:\mathbb{R}\to\mathbb{R}^{\mathtt{n}} \mid \|f\|_{\mathscr{L}_2} := \sqrt{\int_{-\infty}^{+\infty} f^\top(t)f(t)\,dt} < \infty\}$
- $\mathscr{L}_\infty(\mathbb{R},\mathbb{R}^{\mathtt{n}}) = \{f:\mathbb{R}\to\mathbb{R}^{\mathtt{n}} \mid \|f\|_{\mathscr{L}_\infty} :=$ **essential sup**$(|f|) < \infty\}$

with suitable modifications for other domains (e.g. $\mathbb{R}_+$**) and co-domains (e.g. complex- or matrix-valued functions).**

If $\hat{f}\in\mathscr{L}_2(\mathbb{R},\mathbb{C}^{\mathtt{n}})$ **is the Fourier transform of** $f\in\mathscr{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{n}})$**, then**

- $\|f\|_{\mathscr{L}_2} = \frac{1}{\sqrt{2\pi}}\|\hat{f}\|_{\mathscr{L}_2}$

Four system norms

2. Frequency-domain norms. **We only consider rational functions. Let G be a matrix of rational functions.**

- $\|G\|_{\mathscr{H}_\infty} < \infty$ **iff** G **proper, no poles in** $\mathbb{C}_+ := \{s \in \mathbb{C} | \mathbb{R}\text{eal}(s) \geq 0\}$.

$$\|G\|_{\mathscr{H}_\infty} := \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(i\omega)) \quad \sigma_{\max} := \text{ maximum SV}$$

- $\|G\|_{\mathscr{H}_2} < \infty$ **iff** G **is strictly proper and has no poles in** $\mathbb{C}_+$.

$$\|G\|_{\mathscr{H}_2} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathtt{trace}(G^\top(-i\omega)G(i\omega))\, d\omega}$$

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Relation with the impulse response

The $\mathscr{H}_\infty, \mathscr{H}_2, \mathscr{L}_\infty, \mathscr{L}_2$ **norms of a system** $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **are defined in terms of its TF** G**. In terms of the IR, the** $_2$ **norms are infinite if** $D \neq 0$**. If** $D = 0$**, we have**

$$\|W\|_{\mathscr{L}_2} = \|G\|_{\mathscr{H}_2}$$

However, one should be very careful in applying these norms for uncontrollable systems, since they ignore the uncontrollable part of a system!

Input/output stability

$\mathscr{L}_p$ stability

Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$**. The output** $y : \mathbb{R}_+ \to \mathbb{R}^{\mathtt{p}}$ **corresponding to the input** $u : \mathbb{R}_+ \to \mathbb{R}^{\mathtt{m}}$ **with** $x(0) = 0$ **is**

$$y(t) = Du(t) + \int_0^\infty Ce^{At'} Bu(t - t')\, dt'.$$

The system is said to be **$\mathscr{L}_p$-input/output stable** **if** $u \in \mathscr{L}_p(\mathbb{R}_+, \mathbb{R}^{\mathtt{m}})$ **implies that the corresponding output** $y \in \mathscr{L}_p(\mathbb{R}_+, \mathbb{R}^{\mathtt{p}})$**.**

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$$\sup_{0 \neq u \in \mathscr{L}_p(\mathbb{R}_+, \mathbb{R}^{\mathtt{m}})} \frac{||y||_{\mathscr{L}_p(\mathbb{R}_+, \mathbb{R}^{\mathtt{p}})}}{||u||_{\mathscr{L}_p(\mathbb{R}_+, \mathbb{R}^{\mathtt{m}})}}$$

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The $\mathscr{L}_p$-input/output gain is bounded by $|D| + \|W\|_{\mathscr{L}_1}$
For $p = 1, \infty$ **this bound is sharp (for suitable choice of the norm on** $\mathbb{R}^{\mathtt{m}}$ **and** $\mathbb{R}^{\mathtt{p}}$**).**

$\mathscr{L}_2$-gain

The $\mathscr{L}_2(\mathbb{R}_+, \mathbb{R}^{\mathtt{m}})$ **to** $\mathscr{L}_2(\mathbb{R}_+, \mathbb{R}^{\mathtt{p}})$ **induced norm is**

$$\|G\|_{\mathscr{H}_\infty} = \sup_{0 \neq u \in \mathscr{L}_2(\mathbb{R}_+, \mathbb{R}^{\mathtt{m}})} \frac{\|y\|_{\mathscr{L}_2(\mathbb{R}_+, \mathbb{R}^{\mathtt{p}})}}{\|u\|_{\mathscr{L}_2(\mathbb{R}_+, \mathbb{R}^{\mathtt{m}})}}$$

with G **the TF. Of course,** $\|G\|_{\mathscr{H}_\infty} \leq |D| + ||W||_{\mathscr{L}_1}$**, and usually this inequality is strict.**

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with G the TF. Of course, $\|G\|_{\mathscr{H}_\infty} \leq |D| + \|W\|_{\mathscr{L}_1}$, and usually this inequality is strict.

If $D = 0$, and $\mathtt{m} = \mathtt{p} = 1$, then the $\mathscr{L}_2(\mathbb{R}_+,\mathbb{R}^{\mathtt{m}})$ to $\mathscr{L}_\infty(\mathbb{R}_+,\mathbb{R}^{\mathtt{p}})$ (and the $\mathscr{L}_\infty(\mathbb{R}_+,\mathbb{R}^{\mathtt{m}})$ to $\mathscr{L}_2(\mathbb{R}_+,\mathbb{R}^{\mathtt{p}})$) induced norm is

$$\|G\|_{\mathscr{H}_2} = \|W\|_{\mathscr{L}_2} = \sup_{0\neq u\in\mathscr{L}_2(\mathbb{R}_+,\mathbb{R})} \frac{\|y\|_{\mathscr{L}_\infty(\mathbb{R}_+,\mathbb{R}^{\mathtt{p}})}}{\|u\|_{\mathscr{L}_2(\mathbb{R}_+,\mathbb{R}^{\mathtt{m}})}}$$

In the multivariable case, there are stochastic interpretations of the $\mathscr{H}_2$-norm, and *at hoc* deterministic interpretations, but no induced norm interpretation.

Input/output and state stability

Theorem

The following are equivalent for $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$**, assumed controllable and observable:**

- **it is state stable**

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- **it is** $\mathscr{L}_p$**-input/output stable**
- $||W||_{\mathscr{L}_1} < \infty$
- $\|G\|_{\mathscr{H}_\infty} < \infty$
- **(assuming** $D = 0$**)** $\|G\|_{\mathscr{H}_2} = \|W\|_{\mathscr{L}_2} < \infty$

Proof of the input/output stability theorem

Recapitulation

- **The $\mathscr{H}_\infty$-norm of the transfer function is the $\mathscr{L}_2$ induced norm.**
- **Boundedness of the $\mathscr{H}_\infty$-norm and of the $\mathscr{H}_2$-norm (assuming $D = 0$) are equivalent to state stability or input/output stability of** $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$**, assumed state controllable and observable.**

Summary of Lecture 4

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End of lecture 4