Summer Course

Linear System Theory
Control
&
Matrix Computations

Monopoli September 8–12, 2008

Lecture 2: Linear differential systems

Lecturer: Paolo Rapisarda

Part I: Representations

Outline

Kernel and image representations

The Smith form

Surjectivity/injectivity of polynomial differential operators

Inputs and outputs

Controllability

Observability

Definition

A linear differential system is a triple $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathcal{B})$ with $\mathcal{B}$

- **linear**

$$w_1, w_2 \in \mathcal{B} \text{ and } \alpha_1, \alpha_2 \in \mathbb{R} \Longrightarrow \alpha_1 w_1 + \alpha_2 w_2 \in \mathcal{B}$$

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- **time-invariant**

$$w \in \mathcal{B} \text{ and } \tau \in \mathbb{R} \Longrightarrow \sigma^{\tau} w \in \mathcal{B}$$

where $(\sigma^{\tau} w)(t) = w(t + \tau)$ for all $t \in \mathbb{R}$

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- **differential** i.e. $\mathcal{B}$ is the solution set of a system of differential equations.

$\mathcal{B}$ consists of the solutions
of a system of linear, constant-coefficient
differential equations.

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Linear differential behavior $\mathcal{B}$ with:

- $\mathtt{w}$ **variables** w_i, $i = 1, \ldots, \mathtt{w}$
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- **in** $\mathtt{g}$ **equations**

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- **$\mathtt{w}$ variables w_i, $i = 1, \ldots, \mathtt{w}$**
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represented as the solution space of

$$R\left(\frac{d}{dt}\right) w = 0$$

where

$$R(\xi) := R_0 + R_1\xi + \ldots + R_L\xi^L$$

Polynomial differential operators

Differential systems can be effectively represented by one-variable polynomial matrices

$$\mathcal{B} = \{ w \mid R\left(\frac{d}{dt}\right) w = 0 \} = \mathtt{ker}\; R\left(\frac{d}{dt}\right)$$

where

$$R\left(\frac{d}{dt}\right) : \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \longrightarrow \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{g}})$$

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Differential equations as differential operator equations

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In general, if $U \in \mathbb{R}^{p \times p}[\xi]$ is nonsingular, then U^{-1} is a matrix of *rational functions*.

$U \in \mathbb{R}^{p \times p}[\xi]$ is unimodular if $U^{-1} \in \mathbb{R}^{p \times p}[\xi]$.
Equivalent with det(U) $= c$, with $c \in \mathbb{R}$, $c \neq 0$.

The Smith form of a polynomial matrix

Let $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$. There exist unimodular matrices $U \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$ and $V \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ such that

$$URV = \begin{bmatrix} \mathrm{diag}(\delta_i)_{i=1,\ldots,\mathtt{r}} & 0_{\mathtt{r}\times(\mathtt{w}-\mathtt{r})} \\ 0_{(\mathtt{p}-\mathtt{r})\times\mathtt{r}} & 0_{(\mathtt{p}-\mathtt{r})\times(\mathtt{w}-\mathtt{r})} \end{bmatrix}$$

with δ_i monic, $i = 1,\ldots,\mathtt{r}$, and such that δ_i divides δ_{i+1}, $i = 1,\ldots,\mathtt{r}$.

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det(R) is the product of the diagonal elements of Δ.

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Scalar case: given arbitrary $g \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$, there exists w such that

$$p\left(\frac{d}{dt}\right) w = g$$

if and only if $p \neq 0$. Just integrate LHS!

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If $P \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ is *unimodular*, i.e. invertible in $\mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$, then $w := P\left(\frac{d}{dt}\right)^{-1} g$!

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General case: use Smith form of $P = U \Delta V$

$$P\left(\frac{d}{dt}\right) w = U\left(\frac{d}{dt}\right) \Delta\left(\frac{d}{dt}\right) V\left(\frac{d}{dt}\right) w = g$$

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$$= U\left(\frac{d}{dt}\right) \Delta\left(\frac{d}{dt}\right) w' = g$$

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exists, with $g' := U\left(\frac{d}{dt}\right)^{-1} g$

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g' arbitrary, because $U\left(\frac{d}{dt}\right)$ bijective and g arbitrary

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¡$\mathtt{w}$ scalar problems $\delta_i\left(\frac{d}{dt}\right) w_i' = g_i'$!

Always solvable as long as $\delta_i \neq 0$...

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We proved

$P\left(\frac{d}{dt}\right) w = g$ solvable for all g

iff

P has full row rank as a polynomial matrix

Injectivity

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Scalar case: assuming w satisfies

$$p\left(\frac{d}{dt}\right) w = g,$$

such w is unique iff $p = 1$. Sufficiency is evident. Necessity holds since otherwise $\mathtt{ker}\left(p\left(\frac{d}{dt}\right)\right) \neq \{0\}$.

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General case: Use Smith form of $P = U \Delta V$ to write

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with $w' := V\left(\frac{d}{dt}\right) w$, $g' := U\left(\frac{d}{dt}\right)^{-1} g$

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Scalar equation $\delta_i\left(\frac{d}{dt}\right)w'_i = g'_i$ has only one solution iff $\delta_i = 1$

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w is the only solution to $P\left(\frac{d}{dt}\right) w = g$

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all nonzero invariant polynomials of P are unity

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Equivalently: $P(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$

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If all nonzero invariant polynomials of P are unity, then P admits a left inverse on $\mathfrak{C}^\infty(\mathbb{R})$:

$$P = U \begin{bmatrix} I_{\mathrm{m}} \\ 0 \end{bmatrix} V \Longrightarrow V^{-1} \begin{bmatrix} I_{\mathrm{m}} & 0 \end{bmatrix} U^{-1} \text{is left inverse}$$

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- **Injectivity: $P(\lambda)$ full column rank for all $\lambda \in \mathbb{C}$, as a matrix over $\mathbb{R}^{\bullet\times\bullet}$**

Outline

Free variables

Given $\mathcal{B} \in \mathfrak{L}^{\mathtt{w}}$ and $I := \{i_1, \ldots, i_{\mathtt{k}}\} \subseteq \{1, \ldots, \mathtt{w}\}$, let

$$\Pi_I \mathcal{B} := \{(w_{i_1}, \ldots, w_{i_{\mathtt{k}}}) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{k}}) \mid \exists\, w \in \mathcal{B} \text{ s.t. } w = (w_1, \ldots, w_{i_1}, \ldots, w_{i_{\mathtt{k}}}, \ldots, w_{\mathtt{w}}) \in \mathcal{B}\}$$

projection of $\mathcal{B}$ onto the variables w_{i_j}, $j = 1, \ldots, \mathtt{k}$

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projection of $\mathcal{B}$ onto the variables w_{i_j}, $j = 1, \ldots, \mathtt{k}$

Variables w_{i_j}, $j = 1, \ldots, \mathtt{k}$ are free if

$$\Pi_I \mathcal{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{k}})$$

Free variables

***Example*:**

$$p_1\left(\frac{d}{dt}\right)w_1 + p_2\left(\frac{d}{dt}\right)w_2 + p_3\left(\frac{d}{dt}\right)w_3 = 0$$

Assume $p_i \neq 0$, $i = 1, \ldots, 3$.

Let $I = \{1\}$; since $\begin{bmatrix} p_2(\xi) & p_3(\xi) \end{bmatrix}$ is full row rank, for every $w_1 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ there exist w_2, w_3 satisfying equation.

w_1 is free.

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w_1 is free.

w_1, w_2 (and w_2, w_3, and w_1, w_3) are also free.

Maximally free sets

Let $I = \{i_1, \ldots, i_{\mathtt{k}}\} \subseteq \{1, \ldots, \mathtt{w}\}$. The variables $w_{i_1}, \ldots, w_{i_{\mathtt{k}}}$ form a maximally free set if

- **they are free; and**
- **for every $I' = \{i'_1, \ldots, i'_{\mathtt{k}}\} \underset{\neq}{\subset} \{1, \ldots, \mathtt{w}\}$ such that $I \underset{\neq}{\subset} I'$ it holds**

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Maximally free: it's free, and any added variable is not

Maximally free sets

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w_1 (and w_2, and w_3) is free, but not maximally so.

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w_1 (and w_2, and w_3) is free, but not maximally so.

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Note nonunicity of maximally free sets!

Inputs and outputs

Let $\mathcal{B} \in \mathfrak{L}^{\mathtt{w}}$. Assume (if necessary, after permutation of the variables) w partitioned as

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

with w_1 a set of maximally free variables.

Then w_1 are inputs and w_2 outputs.

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***Example*: for $p_1\left(\frac{d}{dt}\right)w_1 + p_2\left(\frac{d}{dt}\right)w_2 + p_3\left(\frac{d}{dt}\right)w_3 = 0$ and assuming $p_i \neq 0$ for $i = 1, \ldots, 3$, we can choose**

- **w_1, w_2 or**
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- **w_1, w_3**

as inputs.

Remarks

- **Nonunicity an issue? What about (linear) resistors**

$$\mathcal{B} = \{(V, I) \mid V = R \cdot I\}?$$

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Input-output representations

$$\mathcal{B} = \{(u, y) \mid P\left(\frac{d}{dt}\right) y = Q\left(\frac{d}{dt}\right) u\}$$

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u maximally free: add one component of y to those of u, resulting set satisfies differential equation $\Longrightarrow$ it is not free.

Input-output representations

Let $\mathcal{B} \in \mathfrak{L}^{\mathtt{w}}$. There exists (possibly after permuting components) a partition of $w = (u, y)$ and $P \in \mathbb{R}^{\mathtt{y} \times \mathtt{y}}[\xi]$ nonsingular, $Q \in \mathbb{R}^{\mathtt{y} \times \mathtt{u}}[\xi]$ such that

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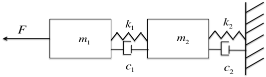
***Proof*: Assume w.l.o.g. that $\mathcal{B} = \mathtt{ker}\ R\left(\frac{d}{dt}\right)$ with R of full row rank $\mathtt{p}$.**

Since R of full row rank, there exists a nonsingular submatrix P.

For $P^{-1}Q$ proper, select P to be a maximal determinantal degree (nonsingular) submatrix of R.

Example



$$m_1 \frac{d^2 w_1}{dt^2} + c_1 \left(\frac{d}{dt} w_1 - \frac{d}{dt} w_2 \right) + k_1 (w_1 - w_2) - F = 0$$

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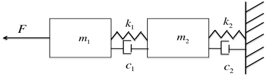

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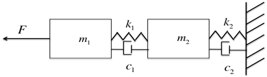

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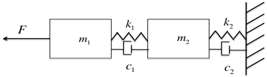

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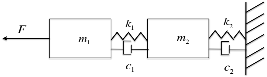

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w_1 and w_2 outputs, F input; $P^{-1}Q$ strictly proper

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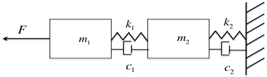

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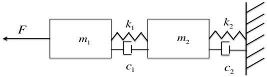

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No inputs: autonomous systems

$\mathcal{B}$ is called autonomous if

$$w_1, w_2 \in \mathcal{B} \quad \text{and} \quad w_1 \big|_{(-\infty,0]} = w_2 \big|_{(-\infty,0]}$$
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Equivalent with

- $\mathtt{m}(\mathcal{B}) = 0$ **(no inputs);**
- **there exists** $R \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ **nonsingular such that** $\mathcal{B} = \mathtt{ker}\; R\left(\frac{d}{dt}\right)$

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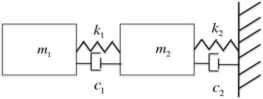

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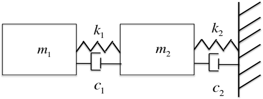

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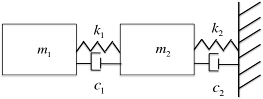
Classical mechanics: motion depends only on ‘initial conditions’

Example



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R **nonsingular** $\rightsquigarrow$ **autonomous system**

On autonomous system trajectories

Scalar case:

$$p\left(\frac{d}{dt}\right)w = 0 \Longleftrightarrow w(t) = \sum_{i=1}^{n}\sum_{j=0}^{n_i} \alpha_{ij} t^j e^{\lambda_i t}$$

where

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λ_i **are called characteristic frequencies of** p**.**

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For $\mathtt{w} > 1$, resort to Smith form $R = U\Delta V$:

$$R\left(\frac{d}{dt}\right) w = 0 \Longleftrightarrow \Delta\left(\frac{d}{dt}\right) \underbrace{V\left(\frac{d}{dt}\right) w}_{=: w'} = 0$$

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Assume for simplicity all roots of det(R) are simple:

$$w = V\left(\frac{d}{dt}\right)^{-1} w' \iff w(t) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t}$$

with $\alpha_i \in \mathbb{C}^{\mathtt{w}}$ such that $R(\lambda_i)\alpha_i = 0$, $i = 1, \ldots, n$.

Remarks

- **Linear combinations of polynomial exponential vector trajectories**

$$\sum_{i=1}^{n} \sum_{j=0}^{n_i} \alpha_{ij} t^j e^{\lambda_i t}$$

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- **Characteristic frequencies** λ_i **are roots of** $\det(R)$.
- $\mathcal{B}$ **is finite-dimensional subspace of** $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.
- **If real part of** λ_i **is negative,** $i = 1, \ldots, n$**, then** $\mathcal{B}$ **is asymptotically stable:** $\lim_{t\to\infty} w(t) = 0 \; \forall \; w \in \mathcal{B}$.

Outline

Kernel and image representations

The Smith form

Surjectivity/injectivity of polynomial differential operators

Inputs and outputs

Controllability

Observability

Controllability

$\mathcal{B}$ **controllable** if for all $w_1, w_2 \in \mathcal{B}$ there exists $w \in \mathcal{B}$ and $T \geq 0$ such that

$$w(t) = \begin{cases} w_1(t) & \text{for} \quad t < 0 \\ w_2(t) & \text{for} \quad t \geq T \end{cases}$$

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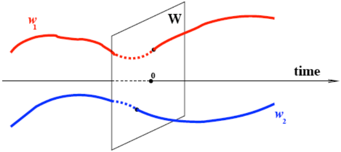

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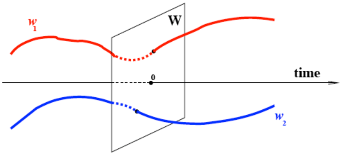

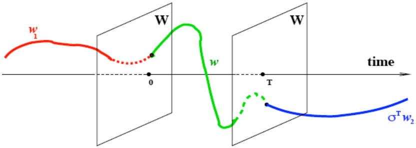

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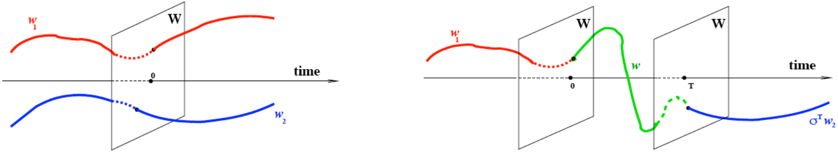

Past of any trajectory can be "patched up" with future of any trajectory

Examples

$$r\left(\frac{d}{dt}\right)w = 0$$

where $0 \neq r \in \mathbb{R}[\xi]$ has degree n.

System autonomous: every solution uniquely determined by ‘initial conditions’ $\frac{d^i w}{dt^i}(t)$, $i = 0, \ldots, n-1$, so no patching possible among different trajectories.

Past of trajectory uniquely determines its future.

Examples

Classical state-space system

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

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If x minimal, then $\mathcal{B}$ controllable $\Longrightarrow$ $\mathcal{B}_s$ controllable.

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“State point-controllability": for all $x_1, x_2 \in \mathbb{R}^{\mathtt{n}}$ $\exists$ $x \in \mathcal{B}_x$ and $T \geq 0$ s.t. $x(0) = x_0$ and $x(T) = x_1$.

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If x minimal, then $\mathcal{B}$ controllable iff $\mathcal{B}_s$ controllable $\iff$ $\mathcal{B}_s$ state point-controllable.

Algebraic characterization of controllability

$\mathcal{B} = \mathtt{ker}\ R\left(\frac{d}{dt}\right)$ **is controllable**

iff

$\mathbf{rank}(R(\lambda))$ **is constant for all** $\lambda \in \mathbb{C}$

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Proof: **Compute Smith form**

$$R = U \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} V \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$$

$U\left(\frac{d}{dt}\right)$, $V\left(\frac{d}{dt}\right)$ **bijective** $\Longrightarrow$ $\texttt{ker}\, R\left(\frac{d}{dt}\right)$ **controllable iff** $\texttt{ker}\, \Delta\left(\frac{d}{dt}\right)$ **is.**

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Change variables $w \rightsquigarrow w' := V\left(\frac{d}{dt}\right) w$**, define** $\mathcal{B}' := V\left(\frac{d}{dt}\right)\mathcal{B} = \mathtt{ker}\ \Delta\left(\frac{d}{dt}\right)$.

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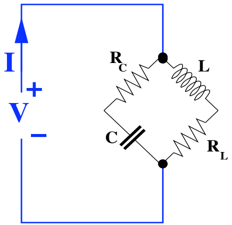
Proof: **Last** $\mathtt{p} - \mathrm{rank}(R)$ **trajectories of** $\mathcal{B}' = \mathtt{ker}\, \Delta\left(\frac{d}{dt}\right)$ **are free.**

First $\mathrm{rank}(R)$ **ones patchable if and only if** $\delta_i = 1$**.**

Example



Case 1: $CR_C \neq \frac{L}{R_L}$

$$\left(\frac{R_C}{R_L} + \left(1 + \frac{R_C}{R_L}\right) CR_C \frac{d}{dt} + CR_C \frac{L}{R_L} \frac{d^2}{dt^2}\right) V_{\text{externalport}}$$

$$= \left(1 + CR_C \frac{d}{dt}\right)\left(1 + \frac{L}{R_L}\frac{d}{dt}\right) R_C I_{\text{externalport}}$$

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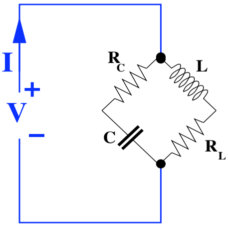

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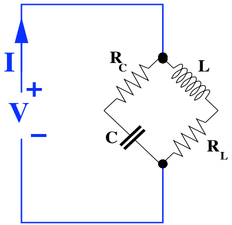

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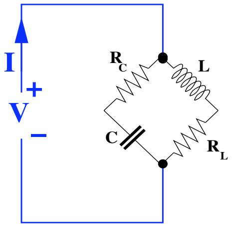
Are there common roots among the two polynomials?

Example



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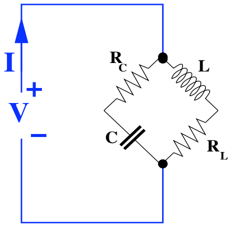
Are there common roots among the two polynomials?

No $\Longrightarrow$ system is controllable

Example



Case 2: $CR_C = \frac{L}{R_L}$

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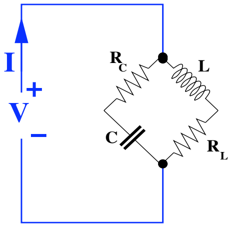

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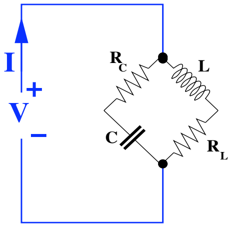

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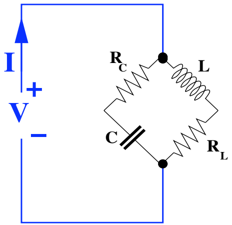

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If $R_C = R_L$ yes $\Longrightarrow$ system is not controllable

Remarks

- $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)$, with $R \in \mathbb{R}^{w \times w}[\xi]$ nonsingular, is controllable $\iff$ R is unimodular $\iff$ $\mathcal{B} = \{0\}$

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- Rank constancy test generalization of 'Hautus test' for state-space systems.
- Trajectory-, not representation-based definition as in state-space framework.

Decomposition of behaviors

Let $\mathcal{B} = \texttt{ker}\, R\left(\frac{d}{dt}\right)$, with $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ full row rank.
There exist $\mathcal{B}_{aut} \subseteq \mathcal{B}$ and $\mathcal{B}_{contr} \subseteq \mathcal{B}$ such that

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Proof: **Write Smith form of $R = U \begin{bmatrix} D & 0_{\mathtt{p}\times(\mathtt{w}-\mathtt{p})} \end{bmatrix} V$,**
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with $w_1' \in \mathtt{ker}\, D\left(\frac{d}{dt}\right)$, w_2' free.

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If $D = I_p \Longrightarrow$ take $\mathcal{B}'_{\mathrm{contr}} = \mathcal{B}'$, $\mathcal{B}'_{\mathrm{aut}} = \{0\}$.

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If $D \neq I_{\mathtt{p}}$, define

$$\mathcal{B}'_{\text{contr}} = \left\{ \begin{bmatrix} w_1' \\ 0 \end{bmatrix} \mid w_1' \in \mathtt{ker}\, D\left(\frac{d}{dt}\right) \right\}$$

$$\mathcal{B}'_{\text{aut}} = \left\{ \begin{bmatrix} 0 \\ w_2' \end{bmatrix} \mid w_2' \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}-\mathtt{p}}) \right\}.$$

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Then transform back to w variables.

Image representations and controllability

There exists $M \in \mathbb{R}^{\mathtt{w}\times\bullet}[\xi]$ such that $\mathcal{B} = \mathrm{im}\, M\left(\frac{d}{dt}\right)$

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***Only if*: Full behavior is controllable, since has kernel representation induced by**

$$\begin{bmatrix} I_{\mathtt{w}} & -M(\xi) \end{bmatrix}$$

with constant rank over $\mathbb{C}$.

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***If*: Take R for minimal kernel representation of $\mathcal{B}$. Apply constancy of rank to conclude Smith form of R is $R = U\begin{bmatrix} I_{\mathtt{p}} & 0_{\mathtt{p}\times\mathtt{m}} \end{bmatrix} V$.**

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Now $U\left(\frac{d}{dt}\right) \begin{bmatrix} I_{\mathtt{p}} & 0_{\mathtt{p}\times\mathtt{m}} \end{bmatrix} \underbrace{V\left(\frac{d}{dt}\right) w}_{=:w'} = 0$ if and only if

$\begin{bmatrix} I_{\mathtt{p}} & 0_{\mathtt{p}\times\mathtt{m}} \end{bmatrix} w' = 0$ if and only if

$$w' = \begin{bmatrix} 0_{\mathtt{p}} \\ I_{\mathtt{m}} \end{bmatrix} \ell$$

with $\ell \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$ free.

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Consequently,

$$w' = V\left(\frac{d}{dt}\right) w = \begin{bmatrix} 0_{\mathtt{p}} \\ I_{\mathtt{m}} \end{bmatrix} \ell$$

from which

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Note also that M can be chosen with $\mathtt{m}(B)$ columns.

Outline

Kernel and image representations

The Smith form

Surjectivity/injectivity of polynomial differential operators

Inputs and outputs

Controllability

Observability

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w_1
observed variables

w_2
to-be-deduced variables

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¿Can w_2 be determine knowing w_1 and the system dynamics?

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$\mathcal{B} \in \mathfrak{L}^{\mathtt{w}}$, $w = (w_1, w_2)$. w_2 is **observable** from w_1 if

$$(w_1, w_2'), (w_1, w_2'') \in \mathcal{B} \Longrightarrow w_2' = w_2''$$

Algebraic characterization of observability

Assume $\mathcal{B}$ represented in kernel form as

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It has iff $R_2\left(\frac{d}{dt}\right)$ injective iff $R_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$

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if and only if

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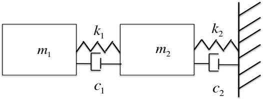
Example



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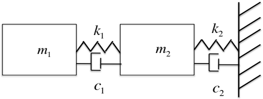

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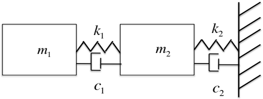

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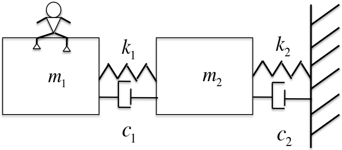
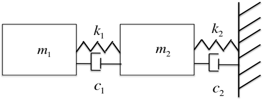
¿Can one determine w_2
from knowledge of w_1 and the system dynamics?



Example

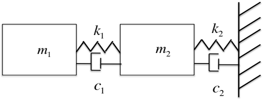


$$
\begin{aligned}
m_1 \frac{d^2 w_1}{dt^2} + c_1 \left(\frac{d}{dt} w_1 - \frac{d}{dt} w_2 \right) + k_1 (w_1 - w_2) &= 0 \\
-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + c_2 \frac{d}{dt} w_2 + c_1 \left(\frac{d}{dt} w_2 - \frac{d}{dt} w_1 \right) + (k_1 + k_2) w_2 &= 0
\end{aligned}
$$

¿Is w_2 observable from w_1?

$$
\begin{bmatrix} m_1 \frac{d^2}{dt^2} + c_1 \frac{d}{dt} + k_1 \\ -c_1 \frac{d}{dt} - k_1 \end{bmatrix} w_1 = \begin{bmatrix} c_1 \frac{d}{dt} + k_1 \\ -m_2 \frac{d^2}{dt^2} - (c_2 + c_1) \frac{d}{dt} - (k_1 + k_2) \end{bmatrix} w_2
$$

Example



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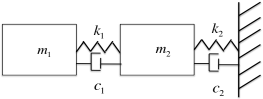
$$

Is polynomial differential operator on RHS injective?

Example



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$$

Is polynomial differential operator on RHS injective?

$$
\begin{bmatrix} c_1 \lambda + k_1 \\ -m_2 \lambda^2 - (c_2 + c_1) \lambda - (k_1 + k_2) \end{bmatrix}
$$

has full column rank $\forall \lambda \in \mathbb{C}$ ($\Longleftrightarrow$ observability) iff

$$
-m_2 k_1^2 + c_1 c_2 k_1 - k_2 c_2^2 \neq 0
$$

Remarks

- **Rank constancy test generalization of ‘Hautus test’ for state-space systems.**

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- **Trajectory-, not representation-based definition as in state-space framework.**

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Summary

- **Polynomial differential operators and their properties are key;**
- **Inputs: free variables;**
- **Autonomous systems;**
- **Controllability and observability;**
- **Algebraic characterizations;**
- **Image representations.**