Robust Control

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Outline

- Nonsingularity of Matrix Families
- LTI Robust Stability Analysis
- Time-Varying Uncertainties
- Robust Stability and Performance with Multipliers
- Controller Synthesis
- A Glimpse at Nonlinear Uncertainties and IQCs

Nonsingularity of Families of Matrices

The following linear algebra problem is fundamental in robust control:

Let us be given $M \in \mathbb{C}^{n\times m}$ and $\boldsymbol{V} \subset \mathbb{C}^{m\times n}$. Decide whether

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- **First goal:** Develop more refined computationally verifiable sufficient conditions that take the particular structure of $\boldsymbol{V}$ into account.

Key Idea

Observe that

$$\det(I - MV) \neq 0 \iff \text{image}\begin{pmatrix} I \\ M \end{pmatrix} \cap \text{image}\begin{pmatrix} V \\ I \end{pmatrix} = \{0\}.$$

For non-singularity we hence need to make sure that the graph of M and the inverse graph of V are separated (intersect at 0 only).

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Geometrically we guarantee separation if these graphs are located in the strictly negative/positive cone of some Hermitian $P \in \mathbb{C}^{(n+m)\times(n+m)}$ respectively:

- Positive cone of P: $\{x \in \mathbb{C}^{n+m} :\ x^*Px \geq 0\}$.
- Strictly negative cone of P: $\{x \in \mathbb{C}^{n+m} :\ x^*Px < 0\}$.

Multipliers

Suppose there exists a Hermitian **multiplier** $P \in \mathbb{C}^{(n+m)\times(n+m)}$ with

$$\begin{pmatrix} V \\ I \end{pmatrix}^* P \begin{pmatrix} V \\ I \end{pmatrix} \succcurlyeq 0 \ \text{ for all } \ V \in \boldsymbol{V} \tag{POS}$$

and at the same time

$$\begin{pmatrix} I \\ M \end{pmatrix}^* P \begin{pmatrix} I \\ M \end{pmatrix} \prec 0. \tag{NS-LMI}$$

Then $\det(I - MV) \neq 0$ for all $V \in \boldsymbol{V}$.

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Very easy to prove! Extensions? How to use in computations?

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For $z = Vw$ we infer $z \neq 0$ and

$$\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} I \\ M \end{pmatrix} z \;\text{ as well as }\; \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} V \\ I \end{pmatrix} w.$$

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Since $z \neq 0$ we obtain

$$0 > z^* \begin{pmatrix} I \\ M \end{pmatrix}^* P \begin{pmatrix} I \\ M \end{pmatrix} z = \begin{pmatrix} z \\ w \end{pmatrix}^* P \begin{pmatrix} z \\ w \end{pmatrix} = w^* \begin{pmatrix} V \\ I \end{pmatrix}^* P \begin{pmatrix} V \\ I \end{pmatrix} w \geq 0$$

which is a **contradiction**.

Numerical Implementation

Note that the set of P satisfying (POS) is **convex**. However this does not help much since this constraint involves infinitely many LMIs.

Key Idea for Computations: Relaxation

For practically relevant $\boldsymbol{V}$, try to identify some "nicely described" subclass of all P's which satisfy (POS).

Then search in this class of P's one which also satisfies (NS-LMI).

Recent years have witnessed a whole variety of possibilities along this line, using techniques from convex analysis and real algebraic geometry (Pólya's theorem, sum-of-squares). We only give three **examples**.

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Let us consider the set $\boldsymbol{V}$ of all V with

$$V = \mathrm{diag}(V_1, \ldots, V_p) \;\; \text{with} \;\; V_1, \ldots, V_p \in \mathbb{C}^{\bullet \times \bullet}, \;\; \sigma_{\mathsf{max}}(V) \leq 1.$$

These matrices admit a **diagonal structure with full complex blocks**.

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Repeated Diagonal Multipliers. (POS) is satisfied for all matrices

$$P = \begin{pmatrix} -\mathrm{diag}(q_1 I, \ldots, q_p I) & 0 \\ 0 & \mathrm{diag}(q_1 I, \ldots, q_p I) \end{pmatrix}$$

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Proof. Just note for $V \in \boldsymbol{V}$ and P as above that

$$\begin{pmatrix} V \\ I \end{pmatrix}^* P \begin{pmatrix} V \\ I \end{pmatrix} = \mathrm{diag}(q_k(I - V_k^* V_k)) \succcurlyeq 0.$$

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As we will see, another interesting set $\boldsymbol{V}$ consists of all

$$V = \mathrm{diag}(v_1 I, \ldots, v_p I) \ \text{ with } \ v_k \in \mathbb{R}, \quad |v_k| \leq 1, \quad k = 1, \ldots, p.$$

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with Hermitian $Q_1, \ldots, Q_p \succcurlyeq 0$ and skew-Hermitian $S_1, \ldots, S_p$.

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Proof. Just note for $V \in \boldsymbol{V}$ and P as above that

$$\begin{pmatrix} V \\ I \end{pmatrix}^* P \begin{pmatrix} V \\ I \end{pmatrix} = \operatorname{diag}(Q_k(1-v_k^2)+v_k(S_k+S_k^*)) = \operatorname{diag}(Q_k(1-v_k^2)) \succcurlyeq 0.$$

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$$\boldsymbol{V} = \text{convex hull}\{V_1, \ldots, V_N\}.$$

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Full Block Multipliers. (POS) is satisfied for all P with

$$\begin{pmatrix} I \\ 0 \end{pmatrix}^T P \begin{pmatrix} I \\ 0 \end{pmatrix} \preccurlyeq 0, \quad \begin{pmatrix} V_k \\ I \end{pmatrix}^T P \begin{pmatrix} V_k \\ I \end{pmatrix} \succcurlyeq 0, \quad k = 1, \ldots, N.$$

Set of multipliers described by **finitely many** LMI constraints.

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Proof. The first inequality implies that

$$V \to \begin{pmatrix} V \\ I \end{pmatrix}^T P \begin{pmatrix} V \\ I \end{pmatrix} \quad \text{is concave.}$$

Hence (POS) is valid iff the inequality holds at the generators $V_1, \ldots, V_N$.

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Consider the system

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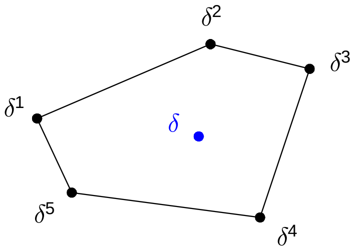
$\delta \in \boldsymbol{\delta} = \text{convex hull}\{\delta^1, \ldots, \delta^N\} \subset \mathbb{R}^p.$

Let $\delta = 0 \in \boldsymbol{\delta}$ be the nominal value and assume that $F(0)$ is Hurwitz.



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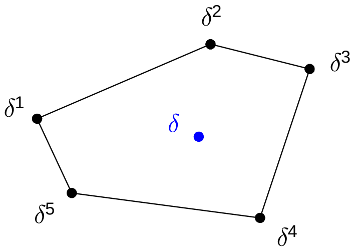


Can we decide whether all these systems are Hurwitz?

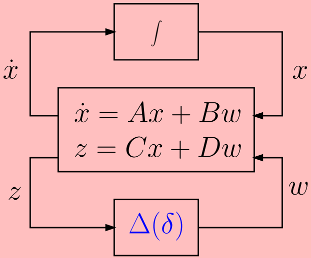
Linear Fractional Representation

Let $F(\delta)$ depend rationally on δ without having a pole in 0.

One can then write $\dot{x} = F(\delta)x$ as

$$\left.\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx + Dw \end{aligned}\right\} \quad w = \Delta(\delta)z$$

with $\Delta(\delta)$ being **linear** in $\delta \in \boldsymbol{\delta}$.



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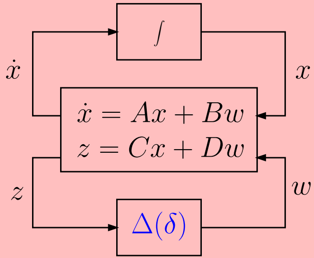

It is always possible to take $\Delta(\delta)$ with the structure

$$\Delta(\delta) = \begin{pmatrix} \delta_1 I_{\nu_1} & & 0 \\ & \ddots & \\ 0 & & \delta_p I_{\nu_p} \end{pmatrix} \quad \text{for some integers } \nu_1, \ldots, \nu_p \geq 0$$

where I_ν denotes the identity matrix of size ν.

Example I

$$\dot{x} = \begin{pmatrix} -1 & 2\delta_1 \\ -\frac{1}{2+\delta_1} & -4+3\delta_2 \end{pmatrix} x \quad \text{with} \quad |\delta_1| \leq r, \quad |\delta_2| \leq r, \quad r > 0.$$

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Rewrite as

$$\begin{pmatrix} 1 & 0 \\ 0 & 2+\delta_1 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 2\delta_1 \\ -1 & (-4+3\delta_2)(2+\delta_1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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or as

$$\begin{pmatrix} \dot{x}_1 \\ 2\dot{x}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w_1 = \begin{pmatrix} -1 & 0 \\ -1 & -4+3\delta_2 \end{pmatrix} \begin{pmatrix} x_1 \\ 2x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4+3\delta_2 \end{pmatrix} w_2$$

$$w_1 = \delta_1 z_1, \; z_1 = \dot{x}_2, \qquad w_2 = \delta_1 z_2, \; z_2 = x_2$$

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$$w_1 = \delta_1 z_1,\ z_1 = \dot{x}_2, \qquad w_2 = \delta_1 z_2,\ z_2 = x_2, \qquad w_3 = \delta_2 z_3,\ z_3 = 2x_2 + w_2$$

Example I

Hence $\dot{x} = F(\delta)x$ can be written as

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ -.5 & -4 \end{pmatrix} x + \begin{pmatrix} 0 & 2 & 0 \\ -.5 & -2 & 1.5 \end{pmatrix} w$$

$$z = \begin{pmatrix} -.5 & -4 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} x + \begin{pmatrix} -.5 & -2 & 1.5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} w, \quad w = \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \end{pmatrix} z.$$

Therefore we can choose

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|ccc} -1 & 0 & 0 & 2 & 0 \\ -.5 & -4 & -.5 & -2 & 1.5 \\ \hline -.5 & -4 & -.5 & -2 & 1.5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right), \quad \Delta(\delta) = \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \end{pmatrix}.$$

Linear Fractional Representation

Let $\dot{x} = F(\delta)x$ be represented as

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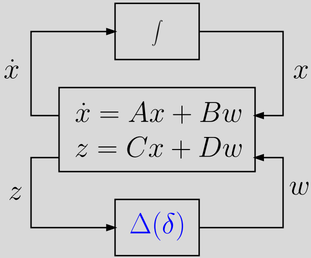

The given derivation shows that this actually means

$$F(\delta) = A + B\Delta(\delta)(I - D\Delta(\delta))^{-1}C =: \begin{pmatrix} A & B \\ C & D \end{pmatrix} \star \Delta(\delta).$$

This is called a **linear fractional representation** (LFR) of $F(\delta)$.
It is said to be **well-posed** if $I - D\Delta(\delta)$ is non-singular on $\boldsymbol{\delta}$.

Nonconservative Robust Stability Test

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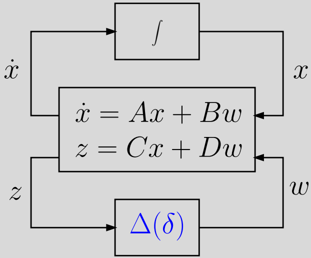

Define the transfer matrix $G(s) = D + C(sI - A)^{-1}B$.

The LFR is well-posed and $\dot{x} = F(\delta)x$ is Hurwitz for all $\delta \in \boldsymbol{\delta}$ iff

$$\det(I - G(i\omega)\Delta(\delta)) \neq 0 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}, \ \delta \in \boldsymbol{\delta}.$$

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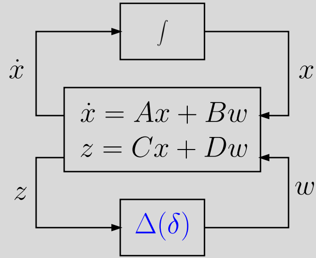

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Testing robust stability is reduced to a robust non-singularity condition for the matrices $G(i\omega)$, $\omega \in \mathbb{R} \cup \{\infty\}$, and the set $\boldsymbol{V} = \Delta(\boldsymbol{\delta})$.

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Sketch of Proof

Suppose the LFR is well-posed. Then $F(\delta)$ is Hurwitz for all $\delta \in \boldsymbol{\delta}$ iff

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$\iff$ (Schur determinant formula & well-posedness)

$$\det\begin{pmatrix} sI - A & -B\Delta(\delta) \\ -C & I - D\Delta(\delta) \end{pmatrix} \text{ for all } \mathrm{Re}(s) \geq 0,\ \delta \in \boldsymbol{\delta}$$

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Sketch of Proof

Suppose the LFR is well-posed. Then $F(\delta)$ is Hurwitz for all $\delta \in \boldsymbol{\delta}$ iff

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Now just observe that the latter condition is well-posedness for $\omega = \infty$.

Computational Procedure

- Choose a list frequencies $\omega_1, \ldots, \omega_m \in \mathbb{R} \cup \{\infty\}$.

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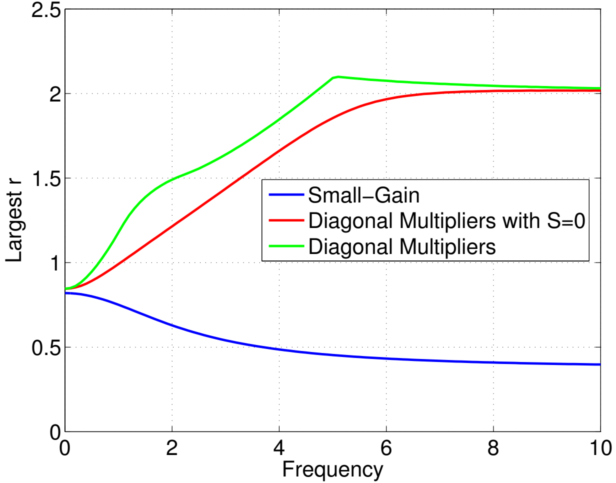
Note that there is a risk of missing crucial frequencies! Can be handled!

Example I

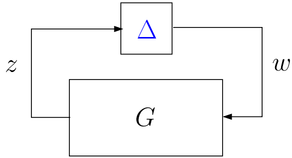
Determine largest r such that the non-singularity test is successful.



Observation: The larger the class of multipliers the better the test!

Summary and Comments

- Sketched key ideas to obtain linear fractional representation



which forms the basis for advanced robustness analysis.

- Have reduced robust stability to non-singularity test.
- Developed multiplier relaxation schemes to verify robust stability.
- Extends to stable **structured dynamic** uncertainties Δ that satisfy

$$\Delta(i\omega) \in \boldsymbol{V} \quad \text{for all} \quad \omega \in \mathbb{R} \cup \{\infty\}.$$

This touches the so-called **structured singular value** theory.

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Robust Stability: Time-Varying Uncertainties

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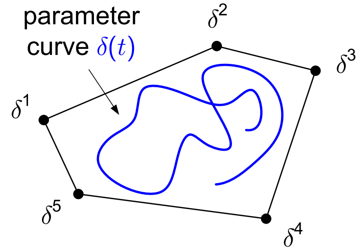

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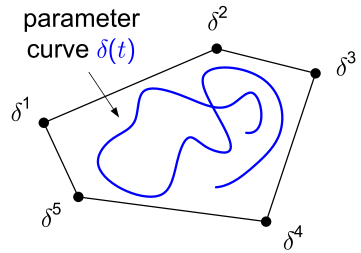

Quadratic Stability

All systems are exponentially stable if there exists some X with

$$X \succ 0, \quad F(\delta)^T X + X F(\delta) \prec 0 \text{ for all } \delta \in \boldsymbol{\delta}.$$

How to Check?

Consider the **robust LMI**

$$\begin{pmatrix} I \\ F(\delta) \end{pmatrix}^T \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} I \\ F(\delta) \end{pmatrix} = F(\delta)^T X + X F(\delta) \prec 0 \quad \forall\, \delta \in \boldsymbol{\delta}.$$

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LFR is well-posed and the robust LMI holds **iff** there exists a P with

$$\begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} \succcurlyeq 0 \text{ for all } \delta \in \boldsymbol{\delta} \qquad \text{(POS)}$$

that also satisfies

$$\begin{pmatrix} A^T X + XA & XB \\ B^T X & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^T P \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0. \qquad \text{(QS-LMI)}$$

Our numerical procedure applies for checking sufficient conditions!

Sketch of Algebraic Proof of "if"

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Abbreviate $H = (I - D\Delta(\delta))^{-1}C$ to infer from (QS-LMI) that

$$0 \succ \begin{pmatrix} I \\ \Delta(\delta)H \end{pmatrix}^T \text{lhs of LMI} \begin{pmatrix} I \\ \Delta(\delta)H \end{pmatrix} =$$

$$= F(\delta)^T X + X F(\delta) \quad + \quad \underbrace{H^T \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} H}_{\succcurlyeq 0 \text{ due to (POS)}}$$

where $=$ follows by simple computation.

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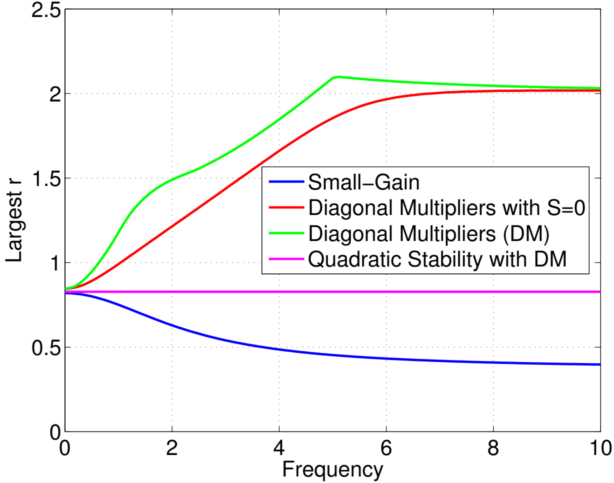
where $=$ follows by simple computation. Hence $F(\delta)^T X + X F(\delta) \prec 0$.

Example I

Determine largest r such that robust quadratic stability can be verified.



Observation: Small gap between time-invariant/time-varying case!

Example II

Test quadratic stability for polynomial parameter dependence:

$$\dot{x} = \begin{pmatrix} -1.25 & 1 - \delta_1 {\delta_2}^2 \\ 1 - \delta_1 \delta_2 & -1 \end{pmatrix} x, \quad \delta_1 \in [-1, 1], \quad \delta_2 \in [-1, 0].$$

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This family can be covered by the following uncertain system with affine parameter dependence:

$$\dot{x} = \begin{pmatrix} -1.25 & 1 - x \\ 1 - y & -1 \end{pmatrix} x, \quad x \in [-r, r], \quad y \in [-r, r], \quad r = 1.$$

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- Polytopic technique from Lecture 6 successful only for $r \approx 0.11$.
- The multipliers from slide 9 allow to guarantee quadratic stability for original uncertain system with polynomial parameter dependence!

Example II: What's going on?

Blue Region

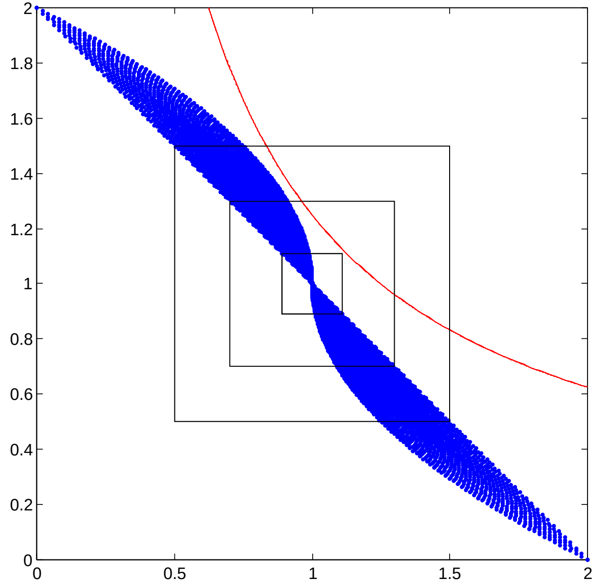
Set of parameter-dependent elements of original system

Red Line

Boundary of Hurwitz region for original system

Black Boxes

Set of parameter-dependent elements of affine covering



Comments

- A trajectory-based proof for robust stability will be given below.
- With affine $Q_0(v)$, $R_0(v)$, $S_0(v)$ in the decision variable v, the same technique (proof) applies to finding v which robustly satisfies

$$\begin{pmatrix} I \\ F(\delta) \end{pmatrix}^T \begin{pmatrix} Q_0(v) & S_0(v) \\ S_0(v)^T & R_0(v) \end{pmatrix} \begin{pmatrix} I \\ F(\delta) \end{pmatrix} \prec 0, \quad R_0(v) \succcurlyeq 0.$$

  Examples: Discrete-time stability, eigenvalue-location in LMI region.

- The result is a concrete version of the so-called full block S-procedure. It serves to handle general robust LMI problems in which the uncertain parameters enter in a rational fashion.

  C.W. Scherer, LMI Relaxations in Robust Control, Eur. J. Cont. 12 (2006) 3-29.

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Robust Energy-Gain Performance

Consider the uncertain input-output system described as

$$\begin{aligned} \dot{x}(t) &= F(\delta(t))x(t) + G(\delta(t))d(t) \\ e(t) &= H(\delta(t))x(t) + J(\delta(t))d(t) \end{aligned}$$

with continuous parameter-curves $\delta(.)$ that satisfy

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Robust Energy-Gain Performance of level γ

For all parameter-curves, $\dot{x}(t) = F(\delta(t))x(t)$ is exponentially stable

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Robust Energy-Gain Performance of level γ

For all parameter-curves, $\dot{x}(t) = F(\delta(t))x(t)$ is exponentially stable and the system's energy-gain is bounded by γ:

$$\int_0^\infty e(t)^T e(t)\, dt \leq \gamma^2 \int_0^\infty d(t)^T d(t)\, dt \text{ for } d \in \mathcal{L}_2(\mathbb{R}_+, \mathbb{R}^{n_d}),\ x(0) = 0.$$

Robust $\mathcal{L}_2$-Gain Performance

Obtain LFR of matrices describing system:

$$\begin{pmatrix} F(\delta) & G(\delta) \\ H(\delta) & J(\delta) \end{pmatrix} = \left(\begin{array}{cc|c} A & B_1 & B_2 \\ C_1 & D_1 & D_{12} \\ \hline C_2 & D_{21} & D_2 \end{array}\right) \star \Delta(\delta).$$

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If LFR well-posed, we have the following alternative system description:

$$\begin{pmatrix} x(t) \\ \dot{x}(t) \\ \hline d(t) \\ e(t) \\ \hline w(t) \\ z(t) \end{pmatrix} = \left(\begin{array}{ccc} I & 0 & 0 \\ A & B_1 & B_2 \\ \hline 0 & I & 0 \\ C_1 & D_1 & D_{12} \\ \hline 0 & 0 & I \\ C_2 & D_{21} & D_2 \end{array}\right) \begin{pmatrix} x(t) \\ d(t) \\ w(t) \end{pmatrix}, \quad \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \Delta(\delta(t)) \\ I \end{pmatrix} z(t).$$

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Robust $\mathcal{L}_2$-Gain Performance

The LFR is well-posed and the system satisfies robust quadratic performance if there exist $P = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}$ and $X \succ 0$ with

$$\begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} \succcurlyeq 0 \quad \text{for all} \quad \delta \in \boldsymbol{\delta} \quad \text{and} \qquad \text{(POS)}$$

$$\left(\begin{array}{ccc} I & 0 & 0 \\ A & B_1 & B_2 \\ \hline 0 & I & 0 \\ C_1 & D_1 & D_{12} \\ \hline 0 & 0 & I \\ C_2 & D_{21} & D_2 \end{array}\right)^T \left(\begin{array}{cc|cc|cc} 0 & X & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q & S \\ 0 & 0 & 0 & 0 & S^T & R \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ A & B_1 & B_2 \\ \hline 0 & I & 0 \\ C_1 & D_1 & D_{12} \\ \hline 0 & 0 & I \\ C_2 & D_{21} & D_2 \end{array}\right) \prec 0. \quad \text{(RP)}$$

Proof of Well-Posdeness

Considering the right-lower block of (RP) reveals that

$$D_{12}^T D_{12} + \begin{pmatrix} I \\ D_2 \end{pmatrix}^T P \begin{pmatrix} I \\ D_2 \end{pmatrix} =$$

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This implies $\begin{pmatrix} I \\ D_2 \end{pmatrix}^T P \begin{pmatrix} I \\ D_2 \end{pmatrix} \prec 0$ and with (POS) well-posedness.

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Sketch of Trajectory-Based Proof

Due to (RP) there exists some $\epsilon > 0$ such that

$$\begin{pmatrix} I & 0 & 0 \\ A & B_1 & B_2 \\ \hline 0 & I & 0 \\ C_1 & D_1 & D_{12} \\ \hline 0 & 0 & I \\ C_2 & D_{21} & D_2 \end{pmatrix}^T \left(\begin{array}{cc|cc|cc} 0 & X & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q & S \\ 0 & 0 & 0 & 0 & S^T & R \end{array}\right) \begin{pmatrix} I & 0 & 0 \\ A & B_1 & B_2 \\ \hline 0 & I & 0 \\ C_1 & D_1 & D_{12} \\ \hline 0 & 0 & I \\ C_2 & D_{21} & D_2 \end{pmatrix} + \begin{pmatrix} \epsilon X & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \prec 0.$$

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Choose parameter trajectory $\delta(t) \in \boldsymbol{\delta}$ and $d \in \mathscr{L}_2(\mathbb{R}_+, \mathbb{R}^{n_d})$, and let $x(.)$ and $e(.)$ be some corresponding state- and output trajectories for any initial condition $x(0)$.

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Sketch of Trajectory-Based Proof

Due to (RP) there exists some $\epsilon > 0$ such that

$$\begin{pmatrix} I & 0 & 0 \\ A & B_1 & B_2 \\ \hline 0 & I & 0 \\ C_1 & D_1 & D_{12} \\ \hline 0 & 0 & I \\ C_2 & D_{21} & D_2 \end{pmatrix}^T \left(\begin{array}{cc|cc|cc} 0 & X & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q & S \\ 0 & 0 & 0 & 0 & S^T & R \end{array}\right) \begin{pmatrix} I & 0 & 0 \\ A & B_1 & B_2 \\ \hline 0 & I & 0 \\ C_1 & D_1 & D_{12} \\ \hline 0 & 0 & I \\ C_2 & D_{21} & D_2 \end{pmatrix} + \begin{pmatrix} \epsilon X & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \prec 0.$$

Choose parameter trajectory $\delta(t) \in \boldsymbol{\delta}$ and $d \in \mathscr{L}_2(\mathbb{R}_+, \mathbb{R}^{n_d})$, and let $x(.)$ and $e(.)$ be some corresponding state- and output trajectories for any initial condition $x(0)$. Due to well-posedness, these trajectories satisfy the relations on slide 28 for suitable $w(.)$, $z(.)$.

Now right-multiply $\mathrm{col}(x(t), d(t), w(t))$ and left-multiply its transpose.

Sketch of Trajectory-Based Proof

We obtain

$$\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}^T \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} + \epsilon x(t)^T X x(t) + \\ + \begin{pmatrix} d(t) \\ e(t) \end{pmatrix}^T \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} d(t) \\ e(t) \end{pmatrix} + \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T P \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} \leq 0.$$

Sketch of Trajectory-Based Proof

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As a key feature observe that $w(t) = \Delta(\delta(t))z(t)$ and hence with (POS):

$$\begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T P \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} = z(t)^T \begin{pmatrix} \Delta(\delta(t)) \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta(\delta(t)) \\ I \end{pmatrix} z(t) \succcurlyeq 0.$$

Sketch of Trajectory-Based Proof

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$$+ \begin{pmatrix} d(t) \\ e(t) \end{pmatrix}^T \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} d(t) \\ e(t) \end{pmatrix} + \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T P \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} \leq 0.$$

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Even after canceling this term, the above inequality hence stays valid.

Sketch of Trajectory-Based Proof

With the product-rule we arrive at

$$\frac{d}{dt}x(t)^T X x(t) + \epsilon x(t)^T X x(t) + e(t)^T e(t) - \gamma^2 d(t)^T d(t) \leq 0.$$

Sketch of Trajectory-Based Proof

With the product-rule we arrive at

$$\frac{d}{dt}x(t)^T X x(t) + \epsilon x(t)^T X x(t) + e(t)^T e(t) - \gamma^2 d(t)^T d(t) \leq 0.$$

- If $d(.) = 0$ we infer $\frac{d}{dt}x(t)^T X x(t) + \epsilon x(t)^T X x(t) \leq 0$. Exploit $X \succ 0$ to obtain uniform exponential stability as in Lecture 6.

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- If $x(0) = 0$ drop the term $\epsilon x(t)^T X x(t)$ and observe that the inequality stays true. We then infer by integration on $[0, T]$ that
$$x(T)^T X x(T) + \int_0^T e(t)^T e(t) - \gamma^2 d(t)^T d(t)\, dt \leq 0.$$

Sketch of Trajectory-Based Proof

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Since $X \succ 0$, we can drop term $x(T)^T X x(T)$ without violating the inequality.

Sketch of Trajectory-Based Proof

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- If $d(.) = 0$ we infer $\frac{d}{dt}x(t)^T X x(t) + \epsilon x(t)^T X x(t) \leq 0$. Exploit $X \succ 0$ to obtain uniform exponential stability as in Lecture 6.

- If $x(0) = 0$ drop the term $\epsilon x(t)^T X x(t)$ and observe that the inequality stays true. We then infer by integration on $[0, T]$ that

$$x(T)^T X x(T) + \int_0^T e(t)^T e(t) - \gamma^2 d(t)^T d(t)\, dt \leq 0.$$

  Since $X \succ 0$, we can drop term $x(T)^T X x(T)$ without violating the inequality. Then $T \to \infty$ finally leads to $\|e\|_{\mathscr{L}_2(\mathbb{R}_+, \mathbb{R}^{n_e})} \leq \gamma \|d\|_{\mathscr{L}_2(\mathbb{R}_+, \mathbb{R}^{n_d})}$.

How to Apply?

Test feasibility of LMIs

$$\begin{pmatrix} I \\ 0 \end{pmatrix}^T P \begin{pmatrix} I \\ 0 \end{pmatrix} \prec 0, \quad \begin{pmatrix} \Delta(\delta^k) \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta(\delta^k) \\ I \end{pmatrix} \succ 0, \quad k = 1, \ldots, N,$$

$$X \succ 0, \quad \left(\begin{array}{ccc} I & 0 & 0 \\ A & B_1 & B_2 \\ \hline 0 & I & 0 \\ C_1 & D_1 & D_{12} \\ \hline 0 & 0 & I \\ C_2 & D_{21} & D_2 \end{array}\right)^T \left(\begin{array}{cc|cc|cc} 0 & X & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q & S \\ 0 & 0 & 0 & 0 & S^T & R \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ A & B_1 & B_2 \\ \hline 0 & I & 0 \\ C_1 & D_1 & D_{12} \\ \hline 0 & 0 & I \\ C_2 & D_{21} & D_2 \end{array}\right) \prec 0.$$

Feasibility guarantees a robust energy-gain level of γ.

Minimize γ^2 to determine best possible bound ... with this technique.

Summary and Comments

- Obtained non-trivial robust stability and robust performance tests which are based on multiplier relaxations.

- Observed trade-off between conservatism and "size" of multiplier set (computational complexity).

- Substantially more instances of the same theme are known.

  Examples: Uncertainty phase information in μ-theory
  Parameter-dependent Lyapunov functions
  Semi-algebraic uncertainty sets

  C.W. Scherer, LMI Relaxations in Robust Control, Eur. J. Cont. 12 (2006) 3-29.

- Finally: Hints on synthesis. Larger classes of uncertainties.

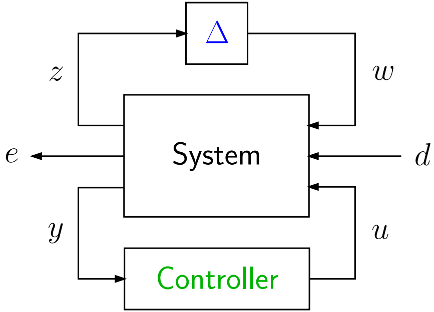
Outline

- Nonsingularity of Matrix Families
- LTI Robust Stability Analysis
- Time-Varying Uncertainties
- Robust Stability and Performance with Multipliers
- Controller Synthesis
- A Glimpse at Nonlinear Uncertainties and IQCs

Configuration for Robust Controller Synthesis

Design controller guaranteeing

- robust stability
- desired robust performance specification on $d \to e$.



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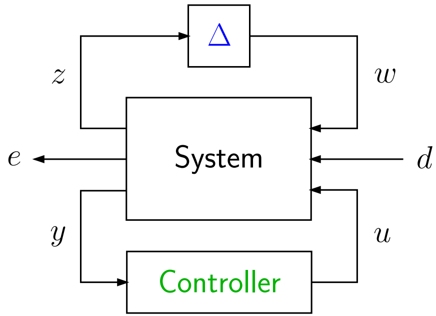

Consider following approach:

- Use robust performance characterization with multipliers
- Try to satisfy the multiplier characterization with suitable controller

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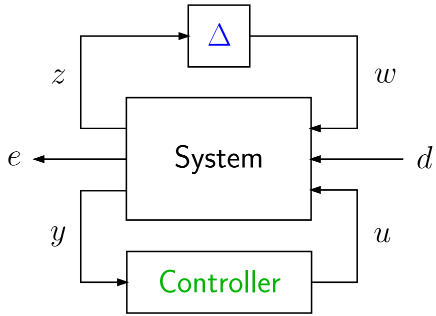


Consider following approach:

- Use robust performance characterization with multipliers
- Try to satisfy the multiplier characterization with suitable controller

For notational simplicity: Concentrate on robust quadratic stability with full-block multiplier relaxation.

Carsten Scherer

System Descriptions

Uncontrolled LTI part:

$$\begin{aligned}\dot{x} &= Ax + B_1 w + Bu\\ z &= C_1 x + D_1 w + Eu\\ y &= Cx + Fw\end{aligned}$$

w: uncertainty input

z: uncertainty output

u: control input

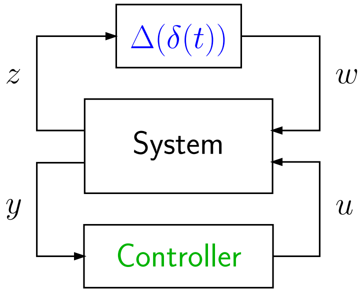
y: measured output

Controller:

$$\begin{aligned}\dot{x}_c &= A_K x_c + B_K y\\ u &= C_K x_c + D_K y\end{aligned}$$

Controlled LTI part:

$$\begin{aligned}\dot{\xi} &= \mathcal{A}\xi + \mathcal{B}w\\ z &= \mathcal{C}\xi + \mathcal{D}w\end{aligned}$$



Uncertainty: $w(t) = \Delta(\delta(t))z(t)$.

Robust Stability Analysis Inequalities

Assume $\delta(t) \in \boldsymbol{\delta} = \text{co}\{\delta^1, \ldots, \delta^N\}$.

Robust stability guaranteed if exist $\mathcal{X}$ and Q, R, S with

$$\begin{pmatrix} I \\ 0 \end{pmatrix}^T P \begin{pmatrix} I \\ 0 \end{pmatrix} \prec 0, \quad \begin{pmatrix} \Delta(\delta^k) \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta(\delta^k) \\ I \end{pmatrix} \succ 0, \;\; k = 1, \ldots, N,$$

$$\mathcal{X} \succ 0, \quad \left(\begin{array}{cc} I & 0 \\ \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \hline 0 & I \\ \mathcal{C} & \mathcal{D} \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^T & R \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \hline 0 & I \\ \mathcal{C} & \mathcal{D} \end{array}\right) \prec 0.$$

Apply standard procedure to step from analysis to synthesis.

Robust Synthesis Inequalities

Exists controller guaranteeing robust stability if exist v, Q, R, S:

$$\begin{pmatrix} I \\ 0 \end{pmatrix}^T P \begin{pmatrix} I \\ 0 \end{pmatrix} \prec 0, \quad \begin{pmatrix} \Delta(\delta^k) \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta(\delta^k) \\ I \end{pmatrix} \succ 0, \quad k = 1, \ldots, N,$$

$$\boldsymbol{X}(v) \succ 0, \quad \left(\begin{array}{cc} I & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \hline 0 & I \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^T & R \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \hline 0 & I \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{array}\right) \prec 0.$$

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Unfortunately **not convex** in all variables v and Q, R, S!

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Usual heuristic remedy: Controller multiplier iteration.

Dualization Lemma

For real matrices $P = P^T$ and W of compatible size, the conditions

$$\begin{pmatrix} 0 \\ I \end{pmatrix}^T P \begin{pmatrix} 0 \\ I \end{pmatrix} \succ 0 \;\; \text{and} \;\; \begin{pmatrix} I \\ W \end{pmatrix}^T P \begin{pmatrix} I \\ W \end{pmatrix} \prec 0$$

are equivalent to

$$\begin{pmatrix} I \\ 0 \end{pmatrix}^T P^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} \prec 0 \;\; \text{and} \;\; \begin{pmatrix} W^T \\ -I \end{pmatrix}^T P^{-1} \begin{pmatrix} W^T \\ -I \end{pmatrix} \succ 0.$$

Note that $\mathrm{im} \begin{pmatrix} I \\ W \end{pmatrix}^\perp$ equals $\mathrm{im} \begin{pmatrix} W^T \\ -I \end{pmatrix}$.

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In general: Let $P = P^T$ be nonsingular with k negative eigenvalues.
If the subspace $\mathcal{S}$ with dimension k is P-negative then $\mathcal{S}^\perp$ is P-positive.

Dual Robust Synthesis Inequalities

Exists controller guaranteeing robust stability if exist v, $\tilde{Q}$, $\tilde{R}$, $\tilde{S}$:

$$\begin{pmatrix} 0 \\ I \end{pmatrix}^T \tilde{P} \begin{pmatrix} 0 \\ I \end{pmatrix} \succ 0, \quad \begin{pmatrix} -I \\ \Delta(\delta^k) \end{pmatrix}^T \tilde{P} \begin{pmatrix} -I \\ \Delta(\delta^k) \end{pmatrix} \prec 0, \quad k = 1, \ldots, N$$

$$\boldsymbol{X}(v) \succ 0, \begin{pmatrix} \boldsymbol{A}(v)^T & \boldsymbol{C}(v)^T \\ -I & 0 \\ \hline \boldsymbol{B}(v)^T & \boldsymbol{D}(v)^T \\ 0 & -I \end{pmatrix}^T \left(\begin{array}{cc|cc} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} \\ 0 & 0 & \tilde{S}^T & \tilde{R} \end{array}\right) \begin{pmatrix} \boldsymbol{A}(v)^T & \boldsymbol{C}(v)^T \\ -I & 0 \\ \hline \boldsymbol{B}(v)^T & \boldsymbol{D}(v)^T \\ 0 & -I \end{pmatrix} \succ 0.$$

Note that we use the partition $\tilde{P} = \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix}$.

No progress in general.

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No progress in general. However **it helps** for state-feedback synthesis!

Lucky Case: Static State-Feedback Synthesis

Recall block substitution:

$$\begin{pmatrix} \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix} = \begin{pmatrix} AY + BM & B_1 \\ C_1Y + EM & D_1 \end{pmatrix}, \quad \boldsymbol{X}(v) = Y.$$

Last column does not depend on v ...

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... dual inequalities are **affine** in all variables ...

... robust state-feedback synthesis possible with LMI's!

Extends to robust performance specification in straightforward fashion!

Dual Robust Synthesis Inequalities: State-feedback

Exists state-feedback controller guaranteeing robust stability if there exist Y, M, $\tilde{Q}$, $\tilde{R}$, $\tilde{S}$ satisfying

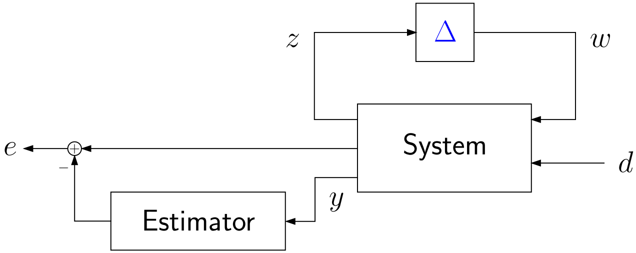
$$\begin{pmatrix} 0 \\ I \end{pmatrix}^T \tilde{P} \begin{pmatrix} 0 \\ I \end{pmatrix} \succ 0, \quad \begin{pmatrix} -I \\ \Delta(\delta^k) \end{pmatrix}^T \tilde{P} \begin{pmatrix} -I \\ \Delta(\delta^k) \end{pmatrix} \prec 0, \quad k = 1, \ldots, N$$

$$Y \succ 0, \quad \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}^T \left(\begin{array}{cc|cc} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} \\ 0 & 0 & \tilde{S}^T & \tilde{R} \end{array}\right) \left(\begin{array}{cc} (AY + BM)^T & (C_1 Y + EM)^T \\ -I & 0 \\ \hline B_1^T & D_1^T \\ 0 & -I \end{array}\right) \succ 0.$$

Is indeed - obviously - an LMI problem!

Lucky Case: Robust Estimator Synthesis

Configuration for robust estimator synthesis:



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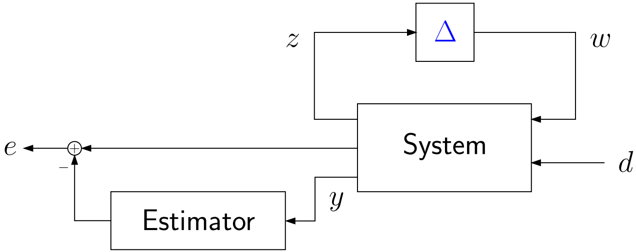

The open-loop system with performance channel reads as

$$\begin{pmatrix} \dot{x} \\ \hline z \\ e \\ y \end{pmatrix} = \left(\begin{array}{c|ccc} A & B_1 & B_2 & 0 \\ \hline C_1 & D_1 & D_{12} & 0 \\ C_2 & D_{21} & D_2 & -I \\ C & F_1 & F_2 & 0 \end{array}\right) \begin{pmatrix} x \\ \hline w \\ d \\ u \end{pmatrix}.$$

Lucky Case: Robust Estimator Synthesis

General variable substitution simplifies to

$$\boldsymbol{X}(v) = \begin{pmatrix} Y & I \\ I & X \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{A}(v) & \boldsymbol{B_1}(v) & \boldsymbol{B_2}(v) \\ \boldsymbol{C_1}(v) & \boldsymbol{D_1}(v) & \boldsymbol{D_{12}}(v) \\ \boldsymbol{C_2}(v) & \boldsymbol{D_{21}}(v) & \boldsymbol{D_2}(v) \end{pmatrix} =$$

$$= \left(\begin{array}{cc|cc} AY & A & B_1 & B_2 \\ K & XA + LC & XB_1 + LF_1 & XB_2 + LF_2 \\ \hline C_1Y & C_1 & D_1 & D_{12} \\ C_2Y - M & C_2 - NC & D_{21} - NF_1 & D_2 - NF_2 \end{array}\right)$$

Lucky Case: Robust Estimator Synthesis

Robust $\mathcal{L}_2$-gain estimator synthesis: Multiplier constraints and LMIs

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0$$

$$* \left(\begin{array}{cccc|cc|cc} 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & 0 & 0 & S^T & R & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right) \left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ AY & A & B_1 & B_2 \\ K & XA + LC & XB_1 + LF_1 & XB_2 + LF_2 \\ \hline 0 & 0 & I & 0 \\ C_1 Y & C_1 & D_1 & D_{12} \\ \hline 0 & 0 & 0 & I \\ C_2 Y - M & C_2 - NC & D_{21} - NF_1 & D_2 - NF_2 \end{array}\right) \prec 0$$

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Non-convex.

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Non-convex. Congruence trafos with $\mathrm{diag}(Y^{-1}, I)$, $\mathrm{diag}(Y^{-1}, I, I, I)$ ...

Lucky Case: Robust Estimator Synthesis

... leads to

$$\begin{pmatrix} Y^{-1} & Y^{-1} \\ Y^{-1} & X \end{pmatrix} \succ 0$$

$$* \left(\begin{array}{cccc|cc|cc} 0&0&I&0&0&0&0&0 \\ 0&0&0&I&0&0&0&0 \\ I&0&0&0&0&0&0&0 \\ 0&I&0&0&0&0&0&0 \\ \hline 0&0&0&0&Q&S&0&0 \\ 0&0&0&0&S^T&R&0&0 \\ \hline 0&0&0&0&0&0&-\gamma^2 I&0 \\ 0&0&0&0&0&0&0&I \end{array}\right) \left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ Y^{-1}A & Y^{-1}A & Y^{-1}B_1 & Y^{-1}B_2 \\ KY^{-1} & XA+LC & XB_1+LF_1 & XB_2+LF_2 \\ \hline 0 & 0 & I & 0 \\ C_1 & C_1 & D_1 & D_{12} \\ \hline 0 & 0 & 0 & I \\ C_2-MY^{-1} & C_2-NC & D_{21}-NF_1 & D_2-NF_2 \end{array}\right) \prec 0$$

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... leads to

$$\begin{pmatrix} Y^{-1} & Y^{-1} \\ Y^{-1} & X \end{pmatrix} \succ 0$$

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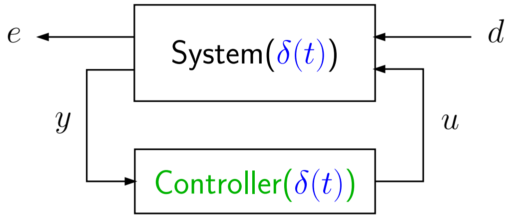
which **is convex** in new variables $\hat{Y} = Y^{-1}$, $\hat{K} = KY^{-1}$, $\hat{M} = MY^{-1}$!

Summary and Comments

- Identified trouble in output-feedback synthesis.
- Discussed lucky cases for robust synthesis by LMIs.
- **Gain-scheduling synthesis:** The controller is allowed to adapt itself according to on-line measurement parameters:



Output-feedback synthesis **can be transformed into LMIs**.

Outline

- Nonsingularity of Matrix Families
- LTI Robust Stability Analysis
- Time-Varying Uncertainties
- Robust Stability and Performance with Multipliers
- Controller Synthesis
- A Glimpse at Nonlinear Uncertainties and IQCs

Nonlinear Uncertainties

Consider the system

$$\begin{pmatrix} \dot{x}(t) \\ e(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} A & B_1 & B_2 \\ C_1 & D_1 & D_{12} \\ C_2 & D_{21} & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ d(t) \\ w(t) \end{pmatrix}, \quad w(t) = \Delta(z(t)).$$

which involves the (smooth) **nonlinear uncertainty** $\Delta : \mathbb{R}^{n_z} \to \mathbb{R}^{n_w}$.

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The statement on slide 29 persists to hold if replacing (POS) by

$$\begin{pmatrix} \Delta(z) \\ z \end{pmatrix}^T P \begin{pmatrix} \Delta(z) \\ z \end{pmatrix} \geq 0 \text{ for all vectors } z \in \mathbb{R}^{n_z}.$$

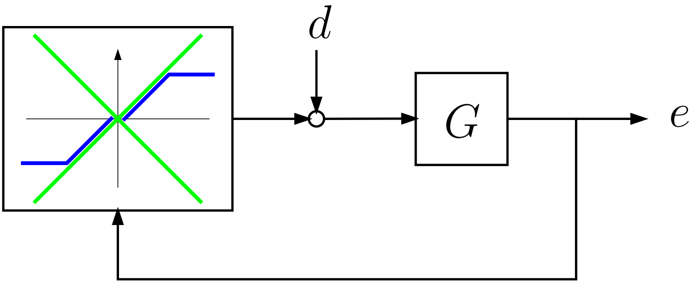
Proof. Literally as before!

IQCs: Example

For transfer function

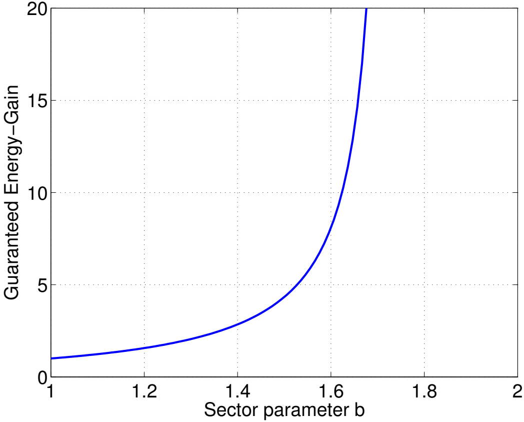
$$G(s) = \frac{-1}{2(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}$$

consider the following interconnection with saturation nonlinearity:



Compute a good bound on the energy-gain of $d \to e$.

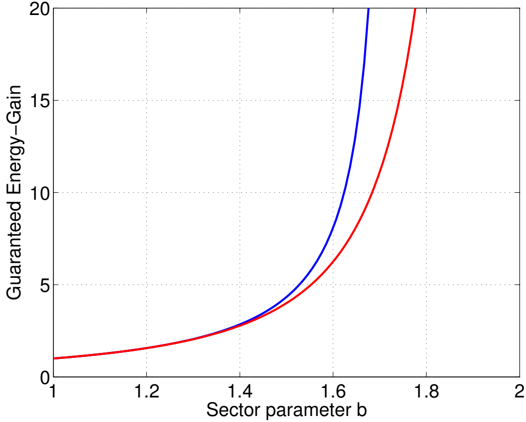
IQCs: Example



Saturation nonlinearity with gain b satisfies

$$|\Delta(z)| \leq b\,|z| \quad \text{or} \quad \begin{pmatrix} \Delta(z) \\ z \end{pmatrix}' \begin{pmatrix} -1 & 0 \\ 0 & b^2 \end{pmatrix} \begin{pmatrix} \Delta(z) \\ z \end{pmatrix} \geq 0$$

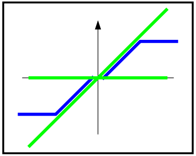
IQCs: Example



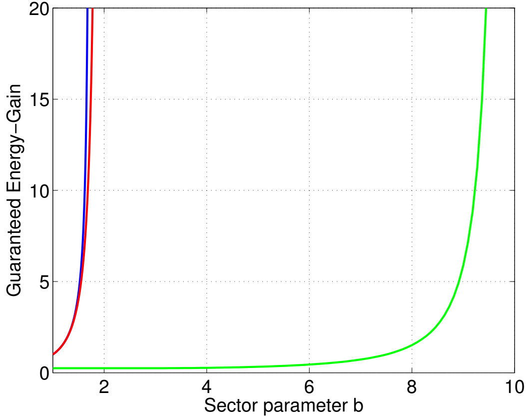
Introduce multiplier to reduce conservatism:

$$\begin{pmatrix} \Delta(z) \\ z \end{pmatrix}' \begin{pmatrix} -\tau & 0 \\ 0 & \tau b^2 \end{pmatrix} \begin{pmatrix} \Delta(z) \\ z \end{pmatrix} \geq 0 \quad \text{for all} \quad \tau \geq 0$$

IQCs: Example



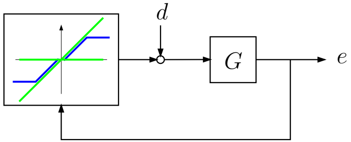
$\Delta(z)^2 \leq b\, z \Delta(z)$



Refined information about saturation:

$$\begin{pmatrix} \Delta(z) \\ z \end{pmatrix}' \begin{pmatrix} -2\tau & b\tau \\ b\tau & 0 \end{pmatrix} \begin{pmatrix} \Delta(z) \\ z \end{pmatrix} \geq 0 \quad \text{for all} \quad \tau \geq 0$$

Integral Quadratic Constraints

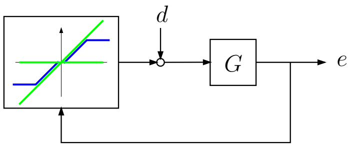


For any $\tau = (\tau_1, \tau_2)$ (elementwise) define the **dynamic** multiplier

$$\Pi_\tau(s) = \tau_1 \begin{pmatrix} -2 & b \\ b & 0 \end{pmatrix} + \tau_2 \begin{pmatrix} 0 & \frac{s}{s+100} \\ \frac{-s}{-s+100} & 0 \end{pmatrix}.$$

Carsten Scherer

Integral Quadratic Constraints



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Saturation satisfies **I**ntegral **Q**uadratic **C**onstraint (IQC)

$$\int_{-\infty}^{\infty} \begin{pmatrix} \lambda \widehat{\Delta(z)}(i\omega) \\ \hat{z}(i\omega) \end{pmatrix}^* \Pi_\tau(i\omega) \begin{pmatrix} \lambda \widehat{\Delta(z)}(i\omega) \\ \hat{z}(i\omega) \end{pmatrix} d\omega \geq 0$$

for all $z \in \mathscr{L}_2(\mathbb{R}_+, \mathbb{R})$, $\lambda \in [0,1]$, $\tau \geq 0$ (elementwise).

Dynamic (frequency-dependent) multipliers!

Integral Quadratic Constraints

Suppose that

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \text{ is the transfer matrix of } \begin{pmatrix} d \\ w \end{pmatrix} \to \begin{pmatrix} e \\ z \end{pmatrix}.$$

Integral Quadratic Constraints

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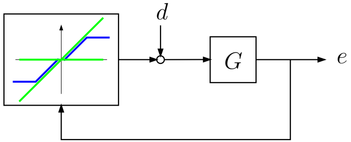
Robust stability and energy-gain performance of level γ is guaranteed if there exists $\tau \geq 0$ for which the following FDI holds:

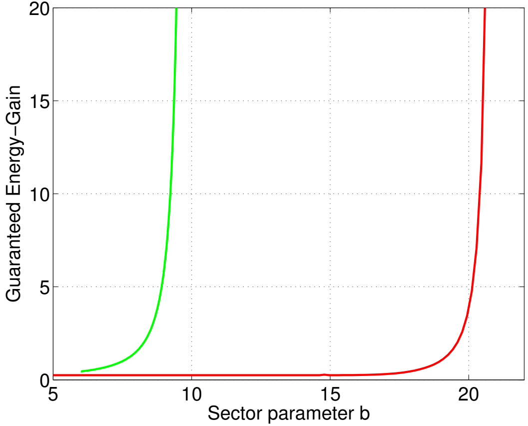
$$\begin{pmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{pmatrix}^* \left(\begin{array}{c|c} \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & I \end{pmatrix} & 0 \\ \hline 0 & \Pi_\tau(i\omega) \end{array} \right) \begin{pmatrix} I & 0 \\ T_{11}(i\omega) & T_{12}(i\omega) \\ \hline 0 & I \\ T_{21}(i\omega) & T_{22}(i\omega) \end{pmatrix} \prec 0 \;\; \forall \omega \in \mathbb{R} \cup \{\infty\}.$$

Computation: Application of KYP lemma leads to LMI feasibility test.

A. Megretski, A. Rantzer, System analysis via Integral Quadratic Constraints, IEEE Trans. Autom. Contr. 42 (1997) 819-830.

IQCs: Example





Dynamics are highly beneficial!

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