Summer Course

Linear System Theory
Control
&
Matrix Computations

Monopoli September 8–12, 2008

Lecture 14: Dissipative systems

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Part I: General dissipative systems

Outline

The dissipation inequality

The dissipation equality

Dissipation inequality

Physical examples:

- **Resistive electrical circuits;**
- **Mechanical systems with friction;**
- **...**

Dissipation inequality

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Energy supplied to system $\rightsquigarrow$ **supply rate variable** F_Σ

Dissipation inequality

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- **Electrical circuits:** $V^\top I$ **with** V **(**I**) vector of voltages (currents)**

Dissipation inequality

Physical examples:

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- **Mechanical systems with friction;**
- **...**

Energy supplied to system $\rightsquigarrow$ **supply rate variable** F_Σ

- **Electrical circuits:** $V^\top I$ **with** V **(**I**) vector of voltages (currents)**
- **Mechanical systems:** $F^\top \frac{d}{dt}x$ **with** F **(**x**) vector of forces (displacements)**

Dissipation inequality

Energy supplied to system $\rightsquigarrow$ **supply rate variable** F_Σ

Energy stored in system $\rightsquigarrow$ **storage variable** F_S

Dissipation inequality

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- **Electrical circuits: $\frac{1}{2}C \cdot V^2$ for capacitor, $\frac{1}{2}L \cdot I^2$ for inductor**

Dissipation inequality

Energy supplied to system $\rightsquigarrow$ supply rate variable F_Σ

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- **Electrical circuits: $\frac{1}{2}C \cdot V^2$ for capacitor, $\frac{1}{2}L \cdot I^2$ for inductor**
- **Mechanical systems: $\frac{1}{2}K \cdot x^2$ for spring**

Dissipation inequality

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In a dissipative system,
energy cannot be stored faster than it is supplied

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Dissipation inequality

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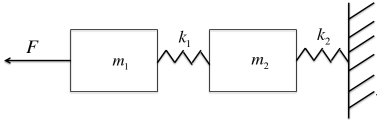
Dissipation equality

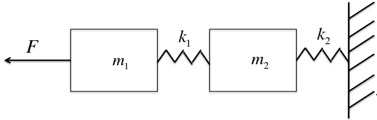
Lossless systems: $F_\Sigma = \frac{d}{dt} F_S$

Example : a mechanical system



$$
\begin{aligned}
m_1 \frac{d^2 w_1}{dt^2} + k_1 (w_1 - w_2) - F &= 0 \\
-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 &= 0
\end{aligned}
$$

Example : a mechanical system



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From physical considerations:

supply (power) $F \cdot \frac{dw_1}{dt}$

storage (total energy) $\frac{1}{2} m_1 \left(\frac{dw_1}{dt}\right)^2 + \frac{1}{2} m_2 \left(\frac{dw_2}{dt}\right)^2 + \frac{1}{2} k_1 (w_1 - w_2)^2 + \frac{1}{2} k_2 w_2^2$

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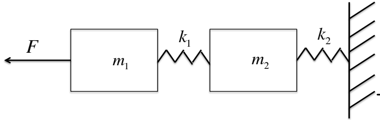

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Easy to see that

$$\frac{d}{dt}\left(F \frac{d}{dt} w_1\right) = \frac{1}{2} m_1 \left(\frac{dw_1}{dt}\right)^2 + \frac{1}{2} m_2 \left(\frac{dw_2}{dt}\right)^2 + \frac{1}{2} k_1 (w_1 - w_2)^2 + \frac{1}{2} k_2 w_2^2$$

Remarks

- **Supply, storage, dissipation for physical system example are functions of the system variables and their (first) derivatives.**

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- **¿Can we decide whether a system is dissipative by examining the supply rate?**

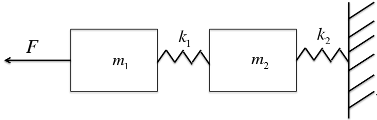
Remarks

- **Supply, storage, dissipation for physical system example are functions of the system variables and their (first) derivatives.**
- **Existential definition: "*if* $\exists$ *storage function...*"**
- **¿Can we decide whether a system is dissipative by examining the supply rate?**
- **What if linear time-invariant finite-dimensional systems, with quadratic supply rates?**

Remarks

- **Supply, storage, dissipation for physical system example are functions of the system variables and their (first) derivatives.**
- **Existential definition: "*if* $\exists$ *storage function...*"**
- **¿Can we decide whether a system is dissipative by examining the supply rate?**
- **What if linear time-invariant finite-dimensional systems, with quadratic supply rates?**
- **¡We need theoretical and algebraic tools!**

Mechanical system example revisited



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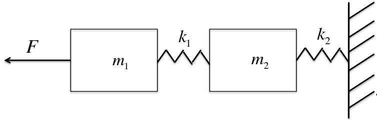

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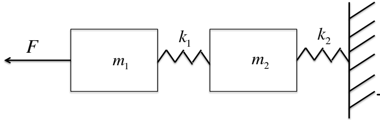
Only dynamics of w_1 of interest $\Longrightarrow$ eliminate w_2

Mechanical system example revisited



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$$
\begin{aligned}
m_1 m_2 \frac{d^4}{dt^4} w_1 &+ (k_1 m_1 + k_2 m_1 + k_1 m_2) \frac{d^2}{dt^2} w_1 + k_1 k_2 w_1 \\
&= m_2 \frac{d^2}{dt^2} F + (k_1 + k_2) F
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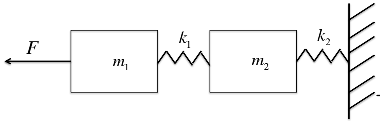
Mechanical system example revisited



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$$\begin{aligned} m_1 m_2 \frac{d^4}{dt^4} w_1 \quad &+ \quad (k_1 m_1 + k_2 m_1 + k_1 m_2) \frac{d^2}{dt^2} w_1 + k_1 k_2 w_1 \\ &= \quad m_2 \frac{d^2}{dt^2} F + (k_1 + k_2) F \end{aligned}$$

Higher-order equations. Physical insight bound to fail.

¿Stored energy, conservation laws, etc.?

Aim

An effective algebraic representation
of bilinear and quadratic functionals
of the system variables and their derivatives:

Operations/properties of functionals
$\Updownarrow$
algebraic operations/properties of representation

...a calculus of these functionals!

Recapitulation

- **Dissipation inequality and equality;**

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- **Dissipation function, storage function, supply rate;**

Recapitulation

- **Dissipation inequality and equality;**
- **Dissipation function, storage function, supply rate;**
- **Algebraic representation of systems needs algebraic representation of functionals.**

Part II: Bilinear- and quadratic differential forms

Outline

Definition

Two-variable polynomial matrices

The calculus of B/QDFs

Bilinear differential forms (BDFs)

$$\Phi := \left\{ \Phi_{k,\ell} \in \mathbb{R}^{w_1 \times w_2} \right\}_{k,\ell=0,\ldots,L}$$

$$L_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{w_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{w_2}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$$

$$L_\Phi(w_1, w_2) := \begin{bmatrix} w_1^\top & \frac{dw_1}{dt}^\top & \ldots \end{bmatrix} \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots \\ \vdots & \vdots & \cdots \\ \Phi_{k,0} & \Phi_{k,1} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} w_2 \\ \frac{dw_2}{dt} \\ \vdots \end{bmatrix}$$

$$= \sum_{k,\ell} \left(\frac{d^k}{dt^k} w_1 \right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dt^\ell} w_2 \right)$$

Quadratic differential forms (QDFs)

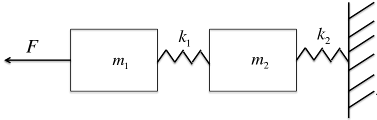
$$\Phi := \left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}\right\}_{k,\ell=0,\dots,L} \text{ symmetric, i.e. } \Phi_{k,\ell} = \Phi_{\ell,k}^\top$$

$$Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$$

$$Q_\Phi(w) := \begin{bmatrix} w^\top & \frac{dw}{dt}^\top & \dots \end{bmatrix} \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots \\ \vdots & \vdots & \cdots \\ \Phi_{k,0} & \Phi_{k,1} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} w \\ \frac{dw}{dt} \\ \vdots \end{bmatrix}$$

$$= \sum_{k,\ell=0}^{L} \left(\frac{d^k}{dt^k} w\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dt^\ell} w\right)$$

Example: total energy in mechanical system



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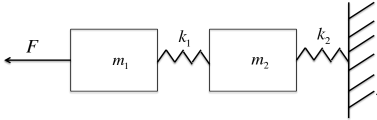

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Total energy is

$$\frac{1}{2} m_1 \left(\frac{d}{dt} w_1\right)^2 + \frac{1}{2} m_2 \left(\frac{d}{dt} w_2\right)^2 + \frac{1}{2} k_1 (w_1 - w_2)^2 + \frac{1}{2} k_2 w_2^2$$

$$= \begin{bmatrix} w_1 & w_2 & F & \frac{d}{dt} w_1 & \frac{d}{dt} w_2 & \frac{d}{dt} F \end{bmatrix} \begin{bmatrix} \frac{1}{2} k_1 & -\frac{1}{2} k_1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} k_1 & \frac{1}{2}(k_1 + k_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ F \\ \frac{d}{dt} w_1 \\ \frac{d}{dt} w_2 \\ \frac{d}{dt} F \end{bmatrix}$$

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Definition

Two-variable polynomial matrices

The calculus of B/QDFs

Two-variable polynomial matrices for BDFs

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$$L_\Phi(w_1, w_2) = \sum_{k,\ell=0}^{L} \left(\frac{d^k}{dt^k} w_1\right)^\top \Phi_{k,\ell} \, \frac{d^\ell}{dt^\ell} w_2$$

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$$\Phi(\zeta, \eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell} \, \zeta^k \, \eta^\ell$$

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2-variable polynomial matrix associated with L_Φ

Two-variable polynomial matrices for QDFs

$$\left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}\right\}_{k,\ell=0,\ldots,L} \text{ symmetric } (\Phi_{k,\ell} = \Phi_{\ell,k}^{\top})$$

$$Q_{\Phi}(w) = \sum_{k,\ell=0}^{L} \left(\frac{d^k}{dt^k} w\right)^{\top} \Phi_{k,\ell} \frac{d^{\ell}}{dt^{\ell}} w$$

$$\Phi(\zeta,\eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell}\, \zeta^k\, \eta^{\ell}$$

symmetric: $\Phi(\zeta,\eta) = \Phi(\eta,\zeta)^{\top}$

Example: total energy in mechanical system

$$Q_E(w_1, w_2, F) =$$

$$\begin{bmatrix} w_1 & w_2 & F & \frac{d}{dt}w_1 & \frac{d}{dt}w_2 & \frac{d}{dt}F \end{bmatrix} \begin{bmatrix} \frac{1}{2}k_1 & -\frac{1}{2}k_1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}k_1 & \frac{1}{2}(k_1+k_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ F \\ \frac{d}{dt}w_1 \\ \frac{d}{dt}w_2 \\ \frac{d}{dt}F \end{bmatrix}$$

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$$E(\zeta, \eta) = \begin{bmatrix} \frac{1}{2}k_1 & -\frac{1}{2}k_1 & 0 \\ -\frac{1}{2}k_1 & \frac{1}{2}(k_1+k_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\zeta\eta & 0 & 0 \\ 0 & \frac{1}{2}\zeta\eta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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Example: total energy in mechanical system

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Outline

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The calculus of B/QDFs

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Using powers of ζ and η as placeholders,

B/QDF $\leftrightsquigarrow$ two-variable polynomial matrix

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Using powers of ζ and η as placeholders,

B/QDF $\leftrightsquigarrow$ two-variable polynomial matrix

Operations and properties of B/QDF	$\leftrightsquigarrow$	**algebraic operations/properties on two-variable matrix**

Differentiation

$\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. $\overset{\bullet}{\Phi}$ **derivative** of Q_Φ:

$$Q_{\overset{\bullet}{\Phi}} : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$$

$$Q_{\overset{\bullet}{\Phi}}(w) := \frac{d}{dt}(Q_\Phi(w))$$

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$$\overset{\bullet}{\Phi}(\zeta,\eta) = (\zeta+\eta)\Phi(\zeta,\eta)$$

Two-variable version of Leibniz's rule

Integration

$\mathfrak{D}(\mathbb{R}, \mathbb{R}^\bullet)$ $\mathfrak{C}^\infty$**-compact-support trajectories**

$$L_\Phi : \mathfrak{D}(\mathbb{R}, \mathbb{R}^{w_1}) \times \mathfrak{D}(\mathbb{R}, \mathbb{R}^{w_2}) \longrightarrow \mathfrak{D}(\mathbb{R}, \mathbb{R})$$

$$\int L_\Phi : \mathfrak{D}(\mathbb{R}, \mathbb{R}^{w_1}) \times \mathfrak{D}(\mathbb{R}, \mathbb{R}^{w_2}) \longrightarrow \mathbb{R}$$
$$\int L_\Phi(w_1, w_2) := \int_{-\infty}^{+\infty} L_\Phi(w_1, w_2)dt$$

Analogous for QDFs

Part III: LTI dissipative differential systems

Outline

Characterizations of dissipativity

Dissipation and storage in an algebraic setting

Setting the stage

LTI systems $\rightsquigarrow$ **supply, dissipation, storage are quadratic functionals of the system variables and their derivatives**

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LTI systems $\rightsquigarrow$ **supply, dissipation, storage are quadratic functionals of the system variables and their derivatives**

Dissipation equality:

$$Q_\Phi(w) = Q_\Delta(w) + \frac{d}{dt} Q_\Psi(w)$$

where $w \in \mathcal{B}$

Setting the stage

Controllable system

$$w = M(\tfrac{d}{dt})\ell \rightsquigarrow M(\xi)$$

Power ('supply rate')

$$Q_\Phi \rightsquigarrow \Phi(\zeta, \eta)$$

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$$Q_\Phi(w) = Q_\Phi(M(\tfrac{d}{dt})\ell)$$

$$\Phi'(\zeta, \eta) := M(\zeta)^\top \Phi(\zeta, \eta) M(\eta)$$

$Q_{\Phi'}$ acts on free variable ℓ, i.e. $\mathfrak{C}^\infty$

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When is a system dissipative?

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When is a system dissipative?

Dissipation equality:

$$Q_\Phi = \frac{d}{dt} Q_\Psi + Q_\Delta$$

$$\int_{-\infty}^{+\infty} Q_\Phi(w)\,dt \geq 0$$

for all compact-support trajectories $w \in \mathcal{B}$

When is a system dissipative?

$$\int_{-\infty}^{+\infty} Q_\Phi(w)\,dt \geq 0$$

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If $w = M(\frac{d}{dt})\ell$, equivalent to

$$Q_{\Phi'}(\ell) \geq 0 \text{ for all } \ell \in \mathfrak{C}^\infty$$

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Fourier transformation leads to

$$\Phi'(-i\omega, i\omega) = M(-i\omega)^\top \Phi(-i\omega, i\omega) M(i\omega) \geq 0$$

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for all $\omega \in \mathbb{R}$

¡A frequency-domain inequality!

When is a system dissipative?

We just proved:

$$\mathrm{im}\ M(\tfrac{d}{dt}) \text{ is } \Phi\text{-dissipative}$$

if and only if

$$M(-i\omega)^\top \Phi(-i\omega, i\omega) M(i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}$$

Characterizations of dissipativity

Theorem: **The following conditions are equivalent:**

- $\int_{-\infty}^{+\infty} Q_\Phi(\ell)dt \geq 0$ for all $\mathfrak{C}^\infty$ compact-support ℓ;
- Q_Φ admits a storage function;
- Q_Φ admits a dissipation rate

Given Q_Φ, storage and dissipation are one-one:

$$\frac{d}{dt}Q_\Psi = Q_\Phi - Q_\Delta$$

$$(\zeta + \eta)\Psi(\zeta, \eta) = \Phi(\zeta, \eta) - \Delta(\zeta, \eta)$$

Example: mechanical systems

$$M\frac{d^2}{dt^2}q + D\frac{d}{dt}q + Kq = F \qquad \begin{bmatrix} F \\ q \end{bmatrix} = \begin{bmatrix} M\frac{d^2}{dt^2} + D\frac{d}{dt} + K \\ I_3 \end{bmatrix} \ell$$

Example: mechanical systems

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Supply rate: power

$$F^\top \left(\frac{d}{dt}q\right) = \left(M\frac{d^2}{dt^2}\ell + D\frac{d}{dt}\ell + K\ell\right)^\top \left(\frac{d}{dt}\ell\right)$$

corresponding to

$$\Phi(\zeta,\eta) = \frac{1}{2}(M\zeta^2 + D\zeta + K)^\top \eta + \frac{1}{2}\zeta(M\eta^2 + D\eta + K)$$

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$$\Phi(\zeta,\eta) = \tfrac{1}{2}(M\zeta^2 + D\zeta + K)^\top \eta + \tfrac{1}{2}\zeta(M\eta^2 + D\eta + K)$$

If dissipation inequality

$$\Phi(\zeta,\eta) = (\zeta+\eta)\Psi(\zeta,\eta) + \Delta(\zeta,\eta)$$

holds, then

$$\Phi(-\xi,\xi) = -\frac{1}{2}\xi^2(D^\top + D) = \Delta(-\xi,\xi)$$

$$\Longrightarrow \Delta(\zeta,\eta) = \frac{1}{2}(D^\top + D)\zeta\eta$$

Spectral factorization of $\Phi(-\xi,\xi)$ is key

Example: mechanical systems

$$M\frac{d^2}{dt^2}q + D\frac{d}{dt}q + Kq = F \qquad \begin{bmatrix} F \\ q \end{bmatrix} = \begin{bmatrix} M\frac{d^2}{dt^2} + D\frac{d}{dt} + K \\ I_3 \end{bmatrix} \ell$$

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$$\Delta(\zeta,\eta) = \tfrac{1}{2}(D^\top + D)\zeta\eta$$

Storage function

$$\Psi(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - \Delta(\zeta,\eta)}{\zeta+\eta} = \frac{1}{2}M\zeta\eta + \frac{1}{2}K$$

Total energy

Example: mechanical systems

$$M\frac{d^2}{dt^2}q + D\frac{d}{dt}q + Kq = F \qquad \begin{bmatrix} F \\ q \end{bmatrix} = \begin{bmatrix} M\frac{d^2}{dt^2} + D\frac{d}{dt} + K \\ I_3 \end{bmatrix} \ell$$

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Physically correct: $\frac{d}{dt}Q_\Psi + Q_\Delta$ **equals**

$$\underbrace{\left(\frac{d^2}{dt^2}q\right)^\top M\frac{d}{dt}q + q^\top K\frac{d}{dt}q}_{\frac{d}{dt}\left[\frac{1}{2}\left(\frac{d}{dt}q\right)^\top M\frac{d}{dt}q + \frac{1}{2}q^\top Kq\right]} + \frac{1}{2}\left(\frac{d}{dt}q\right)^\top (D + D^\top)\frac{d}{dt}q$$

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Supply rate equals

$$F^\top \frac{d}{dt}q = \left[\left(\frac{d^2}{dt^2}q\right)^\top M + \left(\frac{d}{dt}q\right)^\top D^\top + q^\top K\right]\frac{d}{dt}q$$

Outline

Characterizations of dissipativity

Dissipation and storage in an algebraic setting

Dissipation functions and spectral factorization

$$(\zeta+\eta)\boldsymbol{\Psi}(\zeta,\eta)+\boldsymbol{\Delta}(\zeta,\eta)=\boldsymbol{\Phi}(\zeta,\eta)$$

¿How to compute Δ and Ψ?

Dissipation functions and spectral factorization

$$(\zeta + \eta)\boldsymbol{\Psi}(\zeta, \eta) + \boldsymbol{\Delta}(\zeta, \eta) = \boldsymbol{\Phi}(\zeta, \eta)$$

¿How to compute $\boldsymbol{\Delta}$ and $\boldsymbol{\Psi}$?

Let $\zeta = -\xi$, $\eta = \xi$; then $\boldsymbol{\Delta}(-\xi, \xi) = \boldsymbol{\Phi}(-\xi, \xi)$

Dissipation functions and spectral factorization

$$(\zeta + \eta)\Psi(\zeta, \eta) + \Delta(\zeta, \eta) = \Phi(\zeta, \eta)$$

¿How to compute Δ and Ψ?

Let $\zeta = -\xi$, $\eta = \xi$; then $\Delta(-\xi, \xi) = \Phi(-\xi, \xi)$

Also, $Q_\Delta(\ell) \geq 0$ for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \Longrightarrow$ there exists square $D \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ such that

$$\Delta(\zeta, \eta) = D(\zeta)^\top D(\eta)$$

Dissipation functions and spectral factorization

$$(\zeta+\eta)\Psi(\zeta,\eta)+\Delta(\zeta,\eta)=\Phi(\zeta,\eta)$$

¿How to compute Δ and Ψ?

Let $\zeta=-\xi$, $\eta=\xi$; then $\Delta(-\xi,\xi)=\Phi(-\xi,\xi)$

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$$\Delta(\zeta,\eta)=D(\zeta)^\top D(\eta)$$

Spectral factorization: given $\Phi(-\xi,\xi)$, find square matrix D s.t.

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Let $\zeta=-\xi$, $\eta=\xi$; then $\Delta(-\xi,\xi)=\Phi(-\xi,\xi)$

Spectral factorization: given $\Phi(-\xi,\xi)$, find square matrix D s.t.

$$\Phi(-\xi,\xi)=D(-\xi)^\top D(\xi)$$

Solvable if and only if $\Phi(-i\omega,i\omega)\geq 0$ for all $\omega\in\mathbb{R}$.

¡Frequency domain condition for dissipativity!

Dissipation functions and spectral factorization

$$(\zeta+\eta)\Psi(\zeta,\eta)+\Delta(\zeta,\eta)=\Phi(\zeta,\eta)$$

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Spectral factorize $\Phi(-\xi,\xi)=D(-\xi)^\top D(\xi)$, define

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$$\Delta(\zeta,\eta):=D(\zeta)^\top D(\eta)$$

$\Phi(-\xi,\xi)=\Delta(-\xi,\xi)\Longrightarrow$ there exists $\Psi(\zeta,\eta)$ s.t.

$$\Phi(\zeta,\eta)-\Delta(\zeta,\eta)=(\zeta+\eta)\Psi(\zeta,\eta)$$

Then storage function is

$$\Psi(\zeta,\eta)=\frac{\Phi(\zeta,\eta)-\Delta(\zeta,\eta)}{\zeta+\eta}$$

Remarks

- **Many ways of spectral factorizing the same matrix**
  $\rightsquigarrow$ **many dissipation functions**
  $\rightsquigarrow$ **many storage functions.**

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  $\rightsquigarrow$ **many dissipation functions**
  $\rightsquigarrow$ **many storage functions.**

- **Set of storage functions is convex:**

$$Q_{\Psi_1}, Q_{\Psi_2} \text{ storage functions and } \alpha \in [0,1]$$
$$\Longrightarrow \alpha Q_{\Psi_1} + (1-\alpha) Q_{\Psi_2} \text{ is storage function}$$

Maximal and minimal storage functions

Let $\mathcal{B} \in \mathfrak{L}^{\mathtt{w}}$ be controllable and Φ-dissipative. There exist storage functions Q_{Ψ_-} and Q_{Ψ_+} such that for any storage function Q_Ψ it holds

$$Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}$$

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Q_{Ψ_-}s is minimal-, Q_{Ψ_+} is maximal storage function

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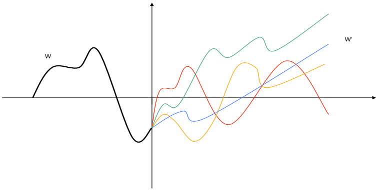
$$Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}$$

Q_{Ψ_-} is available storage:

$$Q_{\Psi_-}(w)(0) = \sup_{\substack{w' \text{ s.t.} \\ w \wedge w' \in \mathcal{B}}} \left(-\int_0^\infty Q_\Phi(w')dt \right)$$



Maximum amount of energy extractable from system.

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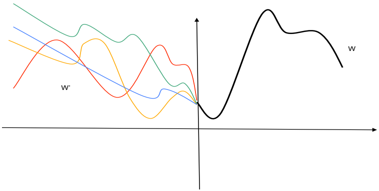
Q_{Ψ_+} is required supply:

$$Q_{\Psi_+}(w)(0) = \inf_{\substack{w' \text{ s.t.} \\ w' \wedge w \in \mathcal{B}}} \left(\int_{-\infty}^{0} Q_\Phi(w')\,dt \right)$$



Minimum energy needed to produce w from $t = 0$

Spectral factorization and extremal storage functions

If $\det \Phi(-\xi,\xi) \neq 0$ and $\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$, there exist H, A s.t.

$$\Phi(-\xi,\xi) = H(-\xi)^\top H(\xi) = A(-\xi)^\top A(\xi)$$

where

$$\det(H(\lambda)) = 0 \Longrightarrow \lambda \in \mathbb{C}_-^0 \quad \text{("semi-Hurwitz polynomial")}$$

$$\det(A(\lambda)) = 0 \Longrightarrow \lambda \in \mathbb{C}_+^0 \quad \text{("semi-anti-Hurwitz polynomial")}$$

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In this case,

$$\Psi_-(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - H(\zeta)^\top H(\eta)}{\zeta+\eta}$$

$$\Psi_+(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - A(\zeta)^\top A(\eta)}{\zeta+\eta}$$

Storage functions and the state

Circuit theory folklore: state variables are associated with energy storing elements (capacitors, inductors)

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Physics: potential energy in a field dependent on position (and velocity/acceleration)

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Circuit theory folklore: state variables are associated with energy storing elements (capacitors, inductors)

Physics: potential energy in a field dependent on position (and velocity/acceleration)

¿Can we give rational foundation to the intuition that “storage" is related with “memory"?

Storage functions and the state

Theorem: Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. Assume that $\mathcal{B} = \mathrm{im}\,(M(\frac{d}{dt}))$ is Σ-dissipative.

Let $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be a storage function, and let $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ be a state map for $\mathcal{B}$.

Then $\exists\ K = K^\top \in \mathbb{R}^{\bullet\times\bullet}$, $E = E^\top \in \mathbb{R}^{\bullet\times\bullet}$ such that

$$\Psi(\zeta,\eta) = X(\zeta)^\top K X(\eta)$$

$$\Delta(\zeta,\eta) = \begin{bmatrix} M(\zeta) \\ X(\zeta) \end{bmatrix}^\top E \begin{bmatrix} M(\eta) \\ X(\eta) \end{bmatrix}$$

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¡The storage function is a quadratic function of the state!

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¡The dissipation function
is a quadratic function of the state and of the input!

Recapitulation

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- **Characterization of dissipativity, dissipation and storage functions;**

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Recapitulation

- **Characterization of dissipativity, dissipation and storage functions;**
- **Spectral factorization and storage functions;**
- **Extremal storage functions;**
- **Storage function is a function of the state.**

Part IV: Dissipativity and state representations

Outline

The linear matrix inequality

The algebraic Riccati equation

Setting the stage

$$\mathcal{B} = \left\{ (x, u) \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{n}+\mathtt{m}}) \mid \frac{d}{dt} x = Ax + Bu \right\}$$

$\mathcal{B}$ **controllable** $\iff$ (A, B) **controllable**

Setting the stage

$$\mathcal{B} = \left\{ (x, u) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}+\mathtt{m}}) \mid \frac{d}{dt}x = Ax + Bu \right\}$$

$\mathcal{B}$ **controllable** $\Longleftrightarrow$ (A, B) **controllable**

Observable image representation of $\mathcal{B}$**:**

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} X(\frac{d}{dt}) \\ U(\frac{d}{dt}) \end{bmatrix} \ell$$

Setting the stage

Observable image representation of $\mathcal{B}$:

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} X(\frac{d}{dt}) \\ U(\frac{d}{dt}) \end{bmatrix} \ell$$

$\mathcal{B}$ is dissipative with respect to

$$\Sigma := \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix} \rightsquigarrow x^\top Q x + 2x^\top S^\top u + u^\top R u$$

Setting the stage

Observable image representation of $\mathcal{B}$:

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$$\Sigma := \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix} \rightsquigarrow x^\top Q x + 2x^\top S^\top u + u^\top R u$$

Leads to

$$\Phi(\zeta, \eta) := \begin{bmatrix} X(\zeta)^\top & U(\zeta)^\top \end{bmatrix} \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix} \begin{bmatrix} X(\eta) \\ U(\eta) \end{bmatrix}$$

acting on $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$.

The linear matrix inequality

Theorem. **The following conditions are equivalent:**

1. $\mathcal{B}$ **is Σ-dissipative;**
2. $\int_{-\infty}^{+\infty} Q_\Phi \geq 0$;
3. $\exists\ K = K^\top \in \mathbb{R}^{\mathtt{n}}$ **s.t. linear matrix inequality (LMI)**

$$\begin{bmatrix} Q - A^\top K - KA & -KB + S^\top \\ -B^\top K + S & R \end{bmatrix} \geq 0$$

holds.

If any of the above conditions hold, then $x^\top K x$ is a storage function for $\mathcal{B}$.

Outline

The linear matrix inequality

The algebraic Riccati equation

The algebraic Riccati equation

Assume $\det \Phi(-\xi, \xi) \neq 0$. Then there exists F of full row rank $\mathtt{m}$ s.t.

$$\begin{bmatrix} Q - A^\top K - KA & -KB + S^\top \\ -B^\top K + S & R \end{bmatrix} = F^\top F$$

The algebraic Riccati equation

Assume det $\Phi(-\xi,\xi) \neq 0$. Then there exists F of full row rank $\mathtt{m}$ s.t.

$$\begin{bmatrix} Q - A^\top K - KA & -KB + S^\top \\ -B^\top K + S & R \end{bmatrix} = F^\top F$$

Assume $R > 0$, and write Schur complement of R:

$$Q - A^\top K - KA - (-KB + S^\top)R^{-1}(-BK + S) = 0$$

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Remarks

- **State-space case as special case;**
- **First-order aspect and other (historical, etc.) reasons $\leadsto$ efficient algorithms;**
- **Optimal control, filtering, etc. applications of ARE.**

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- **First principles approach to dissipation theory;**
- **Two-variable polynomial matrices and the calculus of bilinear- and differential forms;**
- **Answers (algorithmic!) to: “when is a system dissipative?", “how to compute a dissipation function?", etc.**
- **Algebraic Riccati equation, LMIs, etc. as *special case* of higher-order approach.**