Summer Course

Linear System Theory
Control
&
Matrix Computations

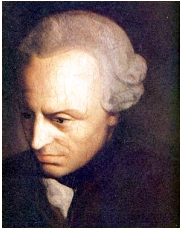
Monopoli September 8–12, 2008

Lecture 12: State and state construction

Lecturer: Paolo Rapisarda

University of Southampton, United Kingdom

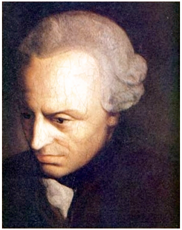
Some German



Zu einer neuen Wissenschaft, die gänzlich isoliert und die einzige ihrer Art ist, mit dem Vorurteil gehen, als könne man sie vermittelst seiner schon sonsts erworbenen vermeinten Kentnisse beuerteilen, obgleich die es eben sind, an deren Realität zuvor gänzlich gezweifelt werden muß, bringt nichts anderes zuwege, als daßman allenthalben das zu sehen glaubt, was einem schon sonst bekannt war, weil etwa die Ausdrücke jenem ähnlich lauten; nur daßeinem alles äußerst verunstaltet, widersinning und kauderwelsch vorkommen muß, weil man nicht die Gedanken des Verfassers, sondern immer nur seine eigene, durch lange Gewonhnheit zur Natur gewordene Denkungsart dabei zum Grunde legt.

(I. Kant, *Prolegomena zu einer jeden künftigen Metaphysik*, 1783)

Some German



To approach a new science - one that is entirely isolated and is the only one of its kind - with the prejudice that it can be judged by means of one's putative cognitions already otherwise obtained, even though it is precisely the reality of those that must first be completely called into question, results only in believing that one sees everywhere something that was already otherwise known, because the expressions perhaps sound similar; except that everything must seem to be extremely deformed, contradictory, and nonsensical, because one does not thereby make the author's thoughts fundamental, but always simply one's own, made natural through long habit.

(I. Kant, *Prolegomena to Any Future Metaphysics*, 1783)

Outline

Questions

- **Are state representations**

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

always a "natural" starting point for modeling?

Questions

- **Are state representations**

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

  always a "natural" starting point for modeling?

- **Mechanics $\rightsquigarrow$**
  2nd order differential equations;

  SYSID, transfer functions $\rightsquigarrow$
  high-order differential equations;

- **First principles and "tearing and zooming" modelling $\rightsquigarrow$**
  high-order differential equations, with auxiliary variables

- **Algebraic constraints among variables**

Questions

- **Are state representations**

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

always “natural"?

NO! NO! NO!

Questions

- **Are state representations**

$$\frac{d}{dt}x = Ax + Bu$$
$$y = Cx + Du$$

always “natural"?

NO! NO! NO!

- **Are state representations important?**

Questions

- **Are state representations**

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

always “natural"?

NO! NO! NO!

- **Are state representations important?**
- **First-order, input/output partition $\rightsquigarrow$ choice of initial condition, choice of input $\rightsquigarrow$ ease of simulation;**
- **First-order $\rightsquigarrow$ algorithms based on linear algebra**
- **Sometimes, “state” *is* natural: think electric circuits, position and momentum variables in mechanics, etc.**

Questions

- **Are state representations**

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

always “natural"?

NO! NO! NO!

- **Are state representations important?**

YES! YES! YES!

Issues

- **What is a first principles definition of “state"?**

Issues

- **What is a first principles definition of “state"?**

- **What does that imply for the equations?**

Issues

- **What is a first principles definition of “state"?**

- **What does that imply for the equations?**

- **How to construct a state from the equations?**

Issues

- **What is a first principles definition of “state"?**
- **What does that imply for the equations?**
- **How to construct a state from the equations?**
- **How to construct a state representation from the equations?**

Issues

- **What is a first principles definition of “state"?**
- **What does that imply for the equations?**
- **How to construct a state from the equations?**
- **How to construct a state representation from the equations?**

Higher-order differential equations

$\leadsto$

state representation

The basic idea

It’s the quarter final of the World Cup. You’re late...

The current score is what matters...

The basic idea

- **The state contains all the relevant information to determine the future behavior of the system**

The basic idea

- **The state contains all the relevant information to determine the future behavior of the system**

- **The state is the memory of the system**

The basic idea

- **The state contains all the relevant information to determine the future behavior of the system**
- **The state is the memory of the system**
- **Independence of past and future given the state: Markovianity**

The state property

$\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathcal{B}_{\text{full}})$ is a ***state system*** if

$$(w_1, x_1), (w_2, x_2) \in \mathcal{B}_{\text{full}} \text{ and } x_1(T) = x_2(T)$$
$$\Downarrow$$
$$(w_1, x_1) \underset{T}{\wedge} (w_2, x_2) \in \mathcal{B}_{\text{full}}$$

$\underset{T}{\wedge}$ is *concatenation at* T:

$$(f_1 \underset{T}{\wedge} f_2)(t) := \begin{cases} f_1(t) \text{ for } t < T \\ f_2(t) \text{ for } t \geq T \end{cases}$$

Graphically...

$$(w_1, x_1), (w_2, x_2) \in \mathcal{B}_{\text{full}} \text{ and } x_1(T) = x_2(T)$$

$$\Downarrow$$

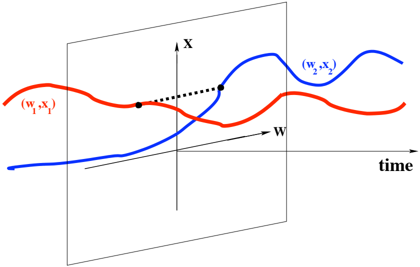
$$(w_1, x_1) \underset{T}{\wedge} (w_2, x_2) \in \mathcal{B}_{\text{full}}$$

Graphically...

$$(w_1, x_1), (w_2, x_2) \in \mathcal{B}_{\text{full}} \text{ and } x_1(T) = x_2(T)$$

$$\Downarrow$$

$$(w_1, x_1) \underset{T}{\wedge} (w_2, x_2) \in \mathcal{B}_{\text{full}}$$

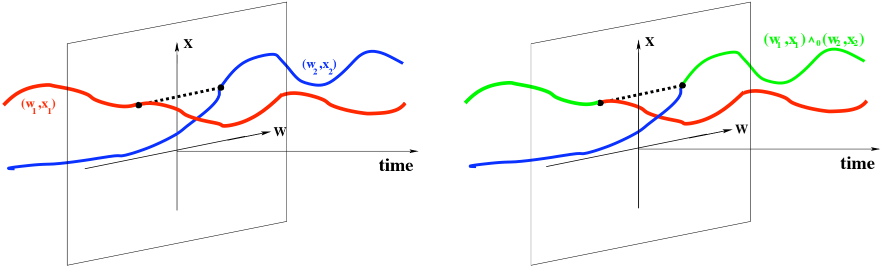


Graphically...

$$(w_1, x_1), (w_2, x_2) \in \mathcal{B}_{\text{full}} \textbf{ and } x_1(T) = x_2(T)$$

$$\Downarrow$$

$$(w_1, x_1) \underset{T}{\wedge} (w_2, x_2) \in \mathcal{B}_{\text{full}}$$



Example 1: discrete-time system

$\boldsymbol{\Sigma} = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{l}}, \mathcal{B}_{\text{full}})$, **with**

$$\mathcal{B}_{\text{full}} := \{(w, \ell) \mid F(\ell(t+1), \ell(t), w(t)) = 0 \text{ for all } t\}$$

where

$$\sigma : (\mathbb{R}^{\mathtt{l}})^{\mathbb{Z}} \to (\mathbb{R}^{\mathtt{l}})^{\mathbb{Z}}$$
$$(\sigma(\ell))(t) := \ell(t+1)$$

Example 1: discrete-time system

$\boldsymbol{\Sigma} = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{l}}, \mathcal{B}_{\text{full}})$, **with**

$$\mathcal{B}_{\text{full}} := \{(w, \ell) \mid F(\ell(t+1), \ell(t), w(t)) = 0 \text{ for all } t\}$$

Special case: input-state-output equations

$$\begin{aligned} \sigma x &= f(x, u) \\ y &= h(x, u) \\ w &= (u, y) \end{aligned}$$

Example 2: continuous-time system

$\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{l}}, \mathcal{B}_{\text{full}})$, **with**

$$\mathcal{B}_{\text{full}} := \{(w, \ell) \mid F \circ (\tfrac{d}{dt}\ell, \ell, w) = 0\}$$

Special case: input-state-output equations

$$\begin{aligned} \frac{d}{dt}x &= f(x, u) \\ y &= h(x, u) \\ w &= (u, y) \end{aligned}$$

Recapitulation

- **State equations are not always the most natural way of modeling systems...**

Recapitulation

- **State equations are not always the most natural way of modeling systems...**
- **...but they are important!**

Recapitulation

- **State equations are not always the most natural way of modeling systems...**
- **...but they are important!**
- **A first-principles definition of “state"**

Recapitulation

- **State equations are not always the most natural way of modeling systems...**
- **...but they are important!**
- **A first-principles definition of “state"**
- **A research program: from higher-order to state space equations**

Outline

First-order discrete-time representations

Theorem: **A ‘complete’ latent variable system**

$$\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{x}}, \mathcal{B}_{\text{full}})$$

is a state system iff $\mathcal{B}_{\text{full}}$ **can be described by**

$$F \circ (\sigma x, x, w) = 0$$

First-order discrete-time representations

Theorem: **A 'complete' latent variable system**

$$\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{x}}, \mathcal{B}_{\text{full}})$$

is a state system iff $\mathcal{B}_{\text{full}}$ can be described by

$$F \circ (\sigma x, x, w) = 0$$

0-th order in w, 1st order in x

First-order discrete-time representations

Theorem: **A ‘complete’ latent variable system**

$$\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{x}}, \mathcal{B}_{\mathrm{full}})$$

is a state system iff $\mathcal{B}_{\mathrm{full}}$ **can be described by**

$$F \circ (\sigma x, x, w) = 0$$

Linear case:

$$E\sigma x + Fx + Gw = 0$$

First-order discrete-time representations

Theorem: **A 'complete' latent variable system**

$$\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{x}}, \mathcal{B}_{\text{full}})$$

is a state system iff $\mathcal{B}_{\text{full}}$ **can be described by**

$$F \circ (\sigma x, x, w) = 0$$

Linear case:

$$E\sigma x + Fx + Gw = 0$$

1st order in *x* is equivalent to state property!

State construction: basic idea

Problem: **Given kernel or image or hybrid description, find a state representation**

$$E\sigma x + Fx + Gw = 0$$

State construction: basic idea

Problem: **Given kernel or image or hybrid description, find a state representation**

$$E\sigma x + Fx + Gw = 0$$

First **compute polynomial operator in the shift acting on system variables, inducing a state variable:**

$$X(\sigma)w = x \qquad X(\sigma)\ell = x \qquad X(\sigma)\begin{bmatrix} w \\ \ell \end{bmatrix} = x$$

State construction: basic idea

Problem: **Given kernel or image or hybrid description, find a state representation**

$$E\sigma x + Fx + Gw = 0$$

First **compute polynomial operator in the shift acting on system variables, inducing a state variable:**

$$X(\sigma)w = x \qquad X(\sigma)\ell = x \qquad X(\sigma)\begin{bmatrix} w \\ \ell \end{bmatrix} = x$$

Then **use original eqs. and** X **to obtain 1st order representation.**

State construction: basic idea

Problem: **Given kernel or image or hybrid description, find a state representation**

$$E\sigma x + Fx + Gw = 0$$

First **compute polynomial operator in the shift acting on system variables, inducing a state variable:**

$$X(\sigma)w = x \qquad X(\sigma)\ell = x \qquad X(\sigma)\begin{bmatrix} w \\ \ell \end{bmatrix} = x$$

Then **use original eqs. and X to obtain 1st order representation.**

$X(\sigma)$ is called a state map

State maps for kernel representations

$X \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ **induces a** ***state map*** $X(\sigma)$ **for** $\mathtt{ker}(R(\sigma))$ **if the behavior** $\mathcal{B}_{\text{full}}$ **with latent variable** x**, consisting of all** (w, x) **such that**

$$\begin{aligned} R(\sigma)w &= 0 \\ X(\sigma)w &= x \end{aligned}$$

satisfies the state property.

State maps for kernel representations

$X \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ induces a **state map $X(\sigma)$** for $\mathtt{ker}(R(\sigma))$ if the behavior $\mathcal{B}_{\text{full}}$ with latent variable x, consisting of all (w, x) such that

$$\begin{aligned} R(\sigma)w &= 0 \\ X(\sigma)w &= x \end{aligned}$$

satisfies the state property.

- State maps exist

State maps for kernel representations

$X \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ induces a *state map $X(\sigma)$* for $\mathtt{ker}(R(\sigma))$ if the behavior $\mathcal{B}_{\text{full}}$ with latent variable x, consisting of all (w, x) such that

$$\begin{aligned} R(\sigma)w &= 0 \\ X(\sigma)w &= x \end{aligned}$$

satisfies the state property.

- **State maps exist**
- **Minimality**

State maps for kernel representations

$X \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ **induces a** ***state map $X(\sigma)$*** **for** $\mathtt{ker}(R(\sigma))$ **if the behavior** $\mathcal{B}_{\text{full}}$ **with latent variable** x**, consisting of all** (w, x) **such that**

$$\begin{aligned} R(\sigma)w &= 0 \\ X(\sigma)w &= x \end{aligned}$$

satisfies the state property.

- **State maps exist**
- **Minimality**
- **State map** $\rightsquigarrow$ **state representation**

Minimal state maps

State system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathcal{B}_f)$ is (state)-minimal if every other state system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}'}, \mathcal{B}_f')$ with same external behavior is such that $\mathtt{n}' \geq \mathtt{n}$

Minimal state dimension: $\mathtt{n}(\mathcal{B})$, McMillan degree of $\mathcal{B}$

Minimal state maps

State system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathcal{B}_f)$ is (state)-minimal if every other state system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}'}, \mathcal{B}_f')$ with same external behavior is such that $\mathtt{n}' \geq \mathtt{n}$

Minimal state dimension: $\mathtt{n}(\mathcal{B})$, McMillan degree of $\mathcal{B}$

Minimal state maps

State system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathcal{B}_f)$ is (state)-minimal if every other state system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}'}, \mathcal{B}'_f)$ with same external behavior is such that $\mathtt{n}' \geq \mathtt{n}$

Minimal state dimension: $\mathtt{n}(\mathcal{B})$, McMillan degree of $\mathcal{B}$

Minimal state map $\leadsto$ minimal state variable

Minimal state maps

State system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathcal{B}_f)$ is (state)-minimal if every other state system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}'}, \mathcal{B}_f')$ with same external behavior is such that $\mathtt{n}' \geq \mathtt{n}$

Minimal state dimension: $\mathtt{n}(\mathcal{B})$, McMillan degree of $\mathcal{B}$

Minimal state map $\rightsquigarrow$ minimal state variable

Minimality of $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathcal{B}_f)$ equivalent with:

- **Trimness: for every $x_0 \in \mathbb{R}^{\mathtt{n}}$ exists $(w, x) \in \mathcal{B}_f$ such that $x(0) = x_0$;**

Minimal state maps

State system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathcal{B}_f)$ is (state)-minimal if every other state system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}'}, \mathcal{B}'_f)$ with same external behavior is such that $\mathtt{n}' \geq \mathtt{n}$

Minimal state dimension: $\mathtt{n}(\mathcal{B})$, McMillan degree of $\mathcal{B}$

Minimal state map $\rightsquigarrow$ minimal state variable

Minimality of $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathcal{B}_f)$ equivalent with:

- **Trimness: for every $x_0 \in \mathbb{R}^{\mathtt{n}}$ exists $(w, x) \in \mathcal{B}_f$ such that $x(0) = x_0$;**
- **Observability of x from $w \iff$ exists $X \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ s.t. $x = X\left(\frac{d}{dt}\right) w$.**

Minimal state maps

State system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathcal{B}_f)$ is (state)-minimal if every other state system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}'}, \mathcal{B}'_f)$ with same external behavior is such that $\mathtt{n}' \geq \mathtt{n}$

Minimal state dimension: $\mathtt{n}(\mathcal{B})$, McMillan degree of $\mathcal{B}$

Minimal state map $\rightsquigarrow$ minimal state variable

Minimality of $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathcal{B}_f)$ equivalent with:

- **Trimness: for every $x_0 \in \mathbb{R}^{\mathtt{n}}$ exists $(w, x) \in \mathcal{B}_f$ such that $x(0) = x_0$;**
- **Observability of x from $w \iff$ exists $X \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ s.t. $x = X\left(\frac{d}{dt}\right) w$.**

¡Minimal state variable induced by state map!

Example

$$\mathcal{B} = \{ w \mid r(\sigma) w = 0 \}$$

where $r \in \mathbb{R}[\xi]$, $\deg(r) = n$.

(Minimal) state map induced by

$$\begin{bmatrix} 1 \\ \xi \\ \vdots \\ \xi^{n-1} \end{bmatrix} \rightsquigarrow \begin{bmatrix} w \\ \sigma w \\ \vdots \\ \sigma^{n-1} w \end{bmatrix}$$

The state property revisited

A *linear* system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathcal{B}_{\text{full}})$ with latent variable x is a state system if

$$(w, x) \in \mathcal{B}_{\text{full}} \text{ and } x(T) = 0$$
$$\Downarrow$$
$$(0, 0) \underset{T}{\wedge} (w, x) \in \mathcal{B}_{\text{full}}$$

The state property revisited

A *linear* system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathcal{B}_{\text{full}})$ with latent variable x is a state system if

$$(w, x) \in \mathcal{B}_{\text{full}} \text{ and } x(T) = 0$$
$$\Downarrow$$
$$(0, 0) \underset{T}{\wedge} (w, x) \in \mathcal{B}_{\text{full}}$$

- **Time-invariance $\Longrightarrow$ can choose $T = 0$;**

The state property revisited

A *linear* system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathcal{B}_{\text{full}})$ with latent variable *x* is a state system if

$$(w, x) \in \mathcal{B}_{\text{full}} \text{ and } x(T) = 0$$
$$\Downarrow$$
$$(0, 0) \underset{T}{\wedge} (w, x) \in \mathcal{B}_{\text{full}}$$

- **Time-invariance $\Longrightarrow$ can choose $T = 0$;**
- **Concatenability with zero trajectory is key;**

The state property revisited

A *linear* system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathcal{B}_{\text{full}})$ with latent variable x is a state system if

$$(w, x) \in \mathcal{B}_{\text{full}} \text{ and } x(T) = 0$$
$$\Downarrow$$
$$(0, 0) \underset{T}{\wedge} (w, x) \in \mathcal{B}_{\text{full}}$$

- **Time-invariance $\Longrightarrow$ can choose $T = 0$;**
- **Concatenability with zero trajectory is key;**
- **$x = X(\sigma)w$ and $w \in \mathcal{B} \Longrightarrow$ concatenability of w with zero is key.**

When is $w \in \mathcal{B}$ concatenable with zero?

$$R_0 w + R_1 \sigma w + \ldots + R_L \sigma^L w = 0$$

...	0	0	$w(0)$	$w(1)$	$w(2)$	$w(3)$	...
...	$k = -2$	$k = -1$	$k = 0$	$k = 1$	$k = 2$	$k = 3$	...

When is $\boldsymbol{w} \in \mathcal{B}$ concatenable with zero?

$$R_0 w + R_1 \sigma w + \ldots + R_L \sigma^L w = 0$$

$\ldots$	0	0	R_0	R_1	R_2	R_3	$\ldots$
$\ldots$	0	0	$w(0)$	$w(1)$	$w(2)$	$w(3)$	$\ldots$
$\ldots$	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$\ldots$

$$R_0 w(0) + R_1 w(1) + \ldots + R_L w(L) = 0$$

When is $w \in \mathcal{B}$ concatenable with zero?

$$R_0 w + R_1 \sigma w + \ldots + R_L \sigma^L w = 0$$

$\ldots$	0	R_0	R_1	R_2	R_3	R_4	$\ldots$
$\ldots$	0	0	$w(0)$	$w(1)$	$w(2)$	$w(3)$	$\ldots$
$\ldots$	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$\ldots$

$$R_0 w(0) + R_1 w(1) + \ldots + R_L w(L) = 0$$

$$R_1 w(0) + R_2 w(1) + \ldots + R_L w(L-1) = 0$$

When is $w \in \mathcal{B}$ concatenable with zero?

$$R_0 w + R_1 \sigma w + \ldots + R_L \sigma^L w = 0$$

$\ldots$	R_0	R_1	R_2	R_3	R_4	R_5	$\ldots$
$\ldots$	0	0	$w(0)$	$w(1)$	$w(2)$	$w(3)$	$\ldots$
$\ldots$	$k = -2$	$k = -1$	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$\ldots$

$$R_0 w(0) + R_1 w(1) + \ldots + R_L w(L) = 0$$

$$R_1 w(0) + R_2 w(1) + \ldots + R_L w(L-1) = 0$$

$$R_2 w(0) + R_3 w(1) + \ldots + R_L w(L-2) = 0$$

When is $\boldsymbol{w} \in \mathcal{B}$ concatenable with zero?

$$R_0 w + R_1 \sigma w + \ldots + R_L \sigma^L w = 0$$

$\ldots$	0	0	$w(0)$	$w(1)$	$w(2)$	$w(3)$	$\ldots$
$\ldots$	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$\ldots$

$$R_0 w(0) + R_1 w(1) + \ldots + R_L w(L) = 0$$

$$R_1 w(0) + R_2 w(1) + \ldots + R_L w(L-1) = 0$$

$$R_2 w(0) + R_3 w(1) + \ldots + R_L w(L-2) = 0$$

$$\vdots \qquad = \vdots$$

When is $w \in \mathcal{B}$ concatenable with zero?

$$R_0 w + R_1 \sigma w + \ldots + R_L \sigma^L w = 0$$

...	R_{L-3}	R_{L-2}	R_{L-1}	R_L	0	0	...
...	0	0	$w(0)$	$w(1)$	$w(2)$	$w(3)$	...
...	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	...

$$R_0 w(0) + R_1 w(1) + \ldots + R_L w(L) = 0$$

$$R_1 w(0) + R_2 w(1) + \ldots + R_L w(L-1) = 0$$

$$R_2 w(0) + R_3 w(1) + \ldots + R_L w(L-2) = 0$$

$$\vdots \qquad = \vdots$$

$$R_{L-1} w(0) + R_L w(1) = 0$$

When is $w \in \mathcal{B}$ concatenable with zero?

$$R_0 w + R_1 \sigma w + \ldots + R_L \sigma^L w = 0$$

$\ldots$	R_{L-2}	R_{L-1}	R_L	0	0	0	$\ldots$
$\ldots$	0	0	$w(0)$	$w(1)$	$w(2)$	$w(3)$	$\ldots$
$\ldots$	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$\ldots$

$$R_0 w(0) + R_1 w(1) + \ldots + R_L w(L) = 0$$
$$R_1 w(0) + R_2 w(1) + \ldots + R_L w(L-1) = 0$$
$$R_2 w(0) + R_3 w(1) + \ldots + R_L w(L-2) = 0$$
$$\vdots \qquad = \vdots$$
$$R_{L-1} w(0) + R_L w(1) = 0$$
$$R_L w(0) = 0$$

The shift-and-cut map

$$\sigma_+ : \mathbb{R}[\xi] \longrightarrow \mathbb{R}[\xi]$$

$$\sigma_+(\sum_{i=0}^{n} p_i \xi^i) := \sum_{i=0}^{n-1} p_{i+1} \xi^i$$

"Divide by ξ and take polynomial part"

Extended componentwise to vectors and matrices

Example

$$\boldsymbol{R}(\xi) = \boldsymbol{R}_0 + \boldsymbol{R}_1\xi + \ldots + \boldsymbol{R}_{L-1}\xi^{L-1} + \boldsymbol{R}_L\xi^L$$

Example

$$R(\xi) = R_0 + R_1\xi + \ldots + R_{L-1}\xi^{L-1} + R_L\xi^L$$

$$\sigma_+(R(\xi)) = R_1 + \ldots + R_{L-1}\xi^{L-2} + R_L\xi^{L-1}$$

Example

$$R(\xi) = R_0 + R_1\xi + \ldots + R_{L-1}\xi^{L-1} + R_L\xi^L$$

$$\sigma_+(R(\xi)) = R_1 + \ldots + R_{L-1}\xi^{L-2} + R_L\xi^{L-1}$$

$$\sigma_+^2(R(\xi)) = R_2 + \ldots + R_{L-1}\xi^{L-3} + R_L\xi^{L-2}$$

Example

$$R(\xi) = R_0 + R_1\xi + \ldots + R_{L-1}\xi^{L-1} + R_L\xi^L$$

$$\sigma_+(R(\xi)) = R_1 + \ldots + R_{L-1}\xi^{L-2} + R_L\xi^{L-1}$$

$$\sigma_+^2(R(\xi)) = R_2 + \ldots + R_{L-1}\xi^{L-3} + R_L\xi^{L-2}$$

$$\vdots = \vdots$$

Example

$$R(\xi) = R_0 + R_1\xi + \ldots + R_{L-1}\xi^{L-1} + R_L\xi^L$$

$$\sigma_+(R(\xi)) = R_1 + \ldots + R_{L-1}\xi^{L-2} + R_L\xi^{L-1}$$

$$\sigma_+^2(R(\xi)) = R_2 + \ldots + R_{L-1}\xi^{L-3} + R_L\xi^{L-2}$$

$$\vdots = \vdots$$

$$\sigma_+^L(R(\xi)) = R_L$$

Shift-and-cut and concatenability with zero

w is concatenable with zero $\Leftrightarrow$

$$
\begin{aligned}
(\sigma_+(R)(\sigma)w)(0) &= 0 \\
(\sigma_+^2(R)(\sigma)w)(0) &= 0 \\
\vdots &= \vdots \\
(\sigma_+^L(R)(\sigma)w)(0) &= 0
\end{aligned}
$$

$\mathrm{col}((\sigma_+^i(R))_{i=1,\ldots,L}(\sigma)$ is a state map!

Shift-and-cut and concatenability with zero

w is concatenable with zero $\Leftrightarrow$

$$\begin{aligned}
(\sigma_+(R)(\sigma)w)(0) &= 0\\
(\sigma_+^2(R)(\sigma)w)(0) &= 0\\
\vdots &= \vdots\\
(\sigma_+^L(R)(\sigma)w)(0) &= 0
\end{aligned}$$

$\text{col}((\sigma_+^i(R))_{i=1,\ldots,L}(\sigma)$ is a state map!

From kernel representation to state map

Denote $\mathrm{col}((\sigma_+^i(R))_{i=1,\ldots,L} =: \Sigma_R$.

Theorem: Let $\mathcal{B} = \mathtt{ker}(R(\sigma))$. Then

$$\begin{aligned} R(\sigma)w &= 0 \\ \Sigma_R(\sigma)w &= x \end{aligned}$$

is a **state representation** of $\mathcal{B}$ with **state variable** x.

Recapitulation

- **State** $\rightsquigarrow$ **first order equations in *x*, zeroth in *w***

Recapitulation

- **State $\rightsquigarrow$ first order equations in x, zeroth in w**
- **State map: acts on variables, yields state**

Recapitulation

- **State $\rightsquigarrow$ first order equations in *x*, zeroth in *w***
- **State map: acts on variables, yields state**
- **Concatenability is key**

Recapitulation

- **State $\rightsquigarrow$ first order equations in x, zeroth in w**
- **State map: acts on variables, yields state**
- **Concatenability is key**
- **Shift-and-cut operation**

Recapitulation

- **State $\rightsquigarrow$ first order equations in *x*, zeroth in *w***
- **State map: acts on variables, yields state**
- **Concatenability is key**
- **Shift-and-cut operation**
- **Shift-and-cut state map**

There's more than the shift-and-cut state map

Many systems of equations are equivalent to shift-and-cut ones:

$$\begin{aligned}
(\sigma_+(R)(\sigma)w)(0) &= 0\\
(\sigma_+^2(R)(\sigma)w)(0) &= 0\\
\vdots \qquad &= \vdots\\
(\sigma_+^L(R)(\sigma)w)(0) &= 0
\end{aligned}$$

There's more than the shift-and-cut state map

Many systems of equations are equivalent to shift-and-cut ones:

$$\begin{aligned}
(\sigma_+(R)(\sigma)w)(0) &= 0\\
(\sigma_+^2(R)(\sigma)w)(0) &= 0\\
\vdots \quad &= \vdots\\
(\sigma_+^L(R)(\sigma)w)(0) &= 0
\end{aligned}$$

$\Longrightarrow$ **different state maps are possible!**

There's more than the shift-and-cut state map

Many systems of equations are equivalent to shift-and-cut ones:

$$\begin{aligned}
(\sigma_+(R)(\sigma)w)(0) &= 0\\
(\sigma_+^2(R)(\sigma)w)(0) &= 0\\
\vdots \quad &= \vdots\\
(\sigma_+^L(R)(\sigma)w)(0) &= 0
\end{aligned}$$

$\Longrightarrow$ **different state maps are possible!**

¿How to characterize this nonuniqueness?

Example: scalar systems

$$r_0 w + r_1 \sigma w + \ldots + \sigma^n w = 0$$

Example: scalar systems

$$r_0 w + r_1 \sigma w + \ldots + \sigma^n w = 0$$

Observe w concatenable with zero iff $w = 0$. Indeed,

$$\begin{aligned}
\sigma_+^n(r)(\sigma)w &= w \\
\sigma_+^{n-1}(r)(\sigma)w &= r_{n-1}w + \sigma w \\
\vdots &= \vdots \\
\sigma_+(r)(\sigma)w &= r_1 w + \ldots + \sigma^{n-1} w
\end{aligned}$$

Example: scalar systems

$$r_0 w + r_1 \sigma w + \ldots + \sigma^n w = 0$$

Observe w concatenable with zero iff $w = 0$. Indeed,

$$\begin{aligned}
\sigma_+^n(r)(\sigma)w &= w \\
\sigma_+^{n-1}(r)(\sigma)w &= r_{n-1}w + \sigma w \\
\vdots &= \vdots \\
\sigma_+(r)(\sigma)w &= r_1 w + \ldots + \sigma^{n-1} w
\end{aligned}$$

Zero at $t = 0$ iff $(\sigma^k w)(0) = 0$ for $k = 0, \ldots, n-1$.

Example: scalar systems

$$r_0 w + r_1 \sigma w + \ldots + \sigma^n w = 0$$

Note also “shift-and-cut state map" different from “standard state map"

$$\begin{array}{ll} w & w \\ r_{n-1} w + \sigma w & \sigma w \\ \vdots & \vdots \\ r_1 w + \ldots + \sigma^{n-1} w & \sigma^{n-1} w \end{array}$$

Example: scalar systems

$$r_0 w + r_1 \sigma w + \ldots + \sigma^n w = 0$$

Note also "shift-and-cut state map" different from "standard state map"

$$\begin{matrix} w & & w \\ r_{n-1} w + \sigma w & & \sigma w \\ \vdots & & \vdots \\ r_1 w + \ldots + \sigma^{n-1} w & & \sigma^{n-1} w \end{matrix}$$

...although each is "equivalent" to the other...

Algebraic characterization

Theorem: Let $\mathcal{B} = \mathtt{ker}(R(\sigma))$, and define Σ_R as above. Define

$$\begin{aligned} \Xi_R &:= \{ f \in \mathbb{R}^{1\times \mathtt{w}}[\xi] \mid \exists\, \alpha \in \mathbb{R}^{1\times \bullet} \text{ s.t. } f = \alpha \Sigma_R \} \\ \langle R \rangle &:= \{ f \in \mathbb{R}^{1\times \mathtt{w}}[\xi] \mid \exists\, g \in \mathbb{R}^{1\times \bullet}[\xi] \text{ s.t. } f = gR \} \end{aligned}$$

Algebraic characterization

Theorem: Let $\mathcal{B} = \mathtt{ker}(R(\sigma))$, and define Σ_R as above. Define

$$\begin{aligned} \Xi_R &:= \{f \in \mathbb{R}^{1\times \mathtt{w}}[\xi] \mid \exists\, \alpha \in \mathbb{R}^{1\times \bullet} \text{ s.t. } f = \alpha \Sigma_R\} \\ \langle R \rangle &:= \{f \in \mathbb{R}^{1\times \mathtt{w}}[\xi] \mid \exists\, g \in \mathbb{R}^{1\times \bullet}[\xi] \text{ s.t. } f = gR\} \end{aligned}$$

$X \in \mathbb{R}^{\bullet\times \mathtt{w}}[\xi]$ is state map for $\mathcal{B}$

if and only if

$$\mathrm{rowspan}_{\mathbb{R}}(X) \oplus \langle R \rangle = \Xi_R + \langle R \rangle$$

Algebraic characterization

Theorem: Let $\mathcal{B} = \mathtt{ker}(R(\sigma))$, and define Σ_R as above. Define

$$\Xi_R := \{f \in \mathbb{R}^{1\times \mathtt{w}}[\xi] \mid \exists\, \alpha \in \mathbb{R}^{1\times \bullet} \text{ s.t. } f = \alpha\Sigma_R\}$$
$$\langle R\rangle := \{f \in \mathbb{R}^{1\times \mathtt{w}}[\xi] \mid \exists\, g \in \mathbb{R}^{1\times \bullet}[\xi] \text{ s.t. } f = gR\}$$

$X \in \mathbb{R}^{\bullet\times \mathtt{w}}[\xi]$ is state map for $\mathcal{B}$

if and only if

$$\mathrm{rowspan}_{\mathbb{R}}(X) \oplus \langle R\rangle = \Xi_R + \langle R\rangle$$

X is minimal state map if and only if its rows are basis for complementary subspace of $\langle R\rangle$ in $\Xi_R + \langle R\rangle$.

Example

$$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u$$

$$\left[\xi^2 + 2\xi + 3 \quad | \quad -\xi - 3\right]$$

Example

$$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u \qquad \left[\xi^2 + 2\xi + 3 \quad | \quad -\xi - 3\right]$$

$$\sigma_+ \rightsquigarrow \left[\xi + 2 \quad | \quad -1\right] \rightsquigarrow \left[\sigma + 2 \quad | \quad -1\right]$$

Example

$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u$ $\quad$ $\begin{bmatrix} \xi^2 + 2\xi + 3 & | & -\xi - 3 \end{bmatrix}$

$$\sigma_+ \rightsquigarrow \begin{bmatrix} \xi + 2 & | & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \sigma + 2 & | & -1 \end{bmatrix}$$

If $(y, u) \in \mathcal{B}$, then for all $g \in \mathbb{R}[\xi]$

$$\begin{aligned} \begin{bmatrix} \sigma + 2 & | & -1 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} &= \begin{bmatrix} \sigma + 2 & | & -1 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \\ &+ \underbrace{g(\sigma) \begin{bmatrix} \sigma^2 + 2\sigma + 3 & | & -\sigma - 3 \end{bmatrix}}_{=0 \text{ on } \mathcal{B}} \begin{bmatrix} y \\ u \end{bmatrix} \end{aligned}$$

Example

$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u$ $\quad$ $\left[\xi^2 + 2\xi + 3 \;\middle|\; -\xi - 3\right]$

$$\sigma_+ \rightsquigarrow \left[\xi + 2 \;\middle|\; -1\right] \rightsquigarrow \left[\sigma + 2 \;\middle|\; -1\right]$$

If $(y, u) \in \mathcal{B}$, then for all $g \in \mathbb{R}[\xi]$

$$\begin{aligned}\left[\sigma + 2 \;\middle|\; -1\right]\begin{bmatrix} y \\ u \end{bmatrix} &= \left[\sigma + 2 \;\middle|\; -1\right]\begin{bmatrix} y \\ u \end{bmatrix} \\ &+ \underbrace{g(\sigma)\left[\sigma^2 + 2\sigma + 3 \;\middle|\; -\sigma - 3\right]}_{=0 \text{ on } \mathcal{B}}\begin{bmatrix} y \\ u \end{bmatrix}\end{aligned}$$

‘equivalence modulo R’

Example

$$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u \qquad \begin{bmatrix} \xi^2 + 2\xi + 3 & | & -\xi - 3 \end{bmatrix}$$

$$\begin{aligned} \sigma_+ &\rightsquigarrow \begin{bmatrix} \xi + 2 & | & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \sigma + 2 & | & -1 \end{bmatrix} \\ \sigma_+^2 &\rightsquigarrow \begin{bmatrix} 1 & | & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & | & 0 \end{bmatrix} \end{aligned}$$

Example

$$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u \qquad \begin{bmatrix}\xi^2 + 2\xi + 3 & | & -\xi - 3\end{bmatrix}$$

$$\begin{aligned}\sigma_+ &\rightsquigarrow \begin{bmatrix}\xi + 2 & | & -1\end{bmatrix} \rightsquigarrow \begin{bmatrix}\sigma + 2 & | & -1\end{bmatrix}\\ \sigma_+^2 &\rightsquigarrow \begin{bmatrix}1 & | & 0\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & | & 0\end{bmatrix}\end{aligned}$$

If $(y, u) \in \mathcal{B}$, then for all $g \in \mathbb{R}[\xi]$

$$\begin{aligned}\begin{bmatrix}1 & | & 0\end{bmatrix}\begin{bmatrix}y\\u\end{bmatrix} &= \begin{bmatrix}1 & | & 0\end{bmatrix}\begin{bmatrix}y\\u\end{bmatrix}\\ &+ \underbrace{g(\sigma)\begin{bmatrix}\sigma^2 + 2\sigma + 3 & | & -\sigma - 3\end{bmatrix}}_{=0 \text{ on } \mathcal{B}}\begin{bmatrix}y\\u\end{bmatrix}\end{aligned}$$

Example

$$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u \qquad \begin{bmatrix} \xi^2 + 2\xi + 3 & | & -\xi - 3 \end{bmatrix}$$

$$\sigma_+ \rightsquigarrow \begin{bmatrix} \xi + 2 & | & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \sigma + 2 & | & -1 \end{bmatrix}$$

$$\sigma_+^2 \rightsquigarrow \begin{bmatrix} 1 & | & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & | & 0 \end{bmatrix}$$

$$\Xi_R = \{\alpha \begin{bmatrix} \xi + 2 & | & -1 \end{bmatrix} + \beta \begin{bmatrix} 1 & | & 0 \end{bmatrix} \mid \alpha, \beta \in \mathbb{R}\}$$

Example

$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u$ $\quad$ $\begin{bmatrix} \xi^2 + 2\xi + 3 & | & -\xi - 3 \end{bmatrix}$

$$\sigma_+ \rightsquigarrow \begin{bmatrix} \xi + 2 & | & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \sigma + 2 & | & -1 \end{bmatrix}$$

$$\sigma_+^2 \rightsquigarrow \begin{bmatrix} 1 & | & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & | & 0 \end{bmatrix}$$

$$\Xi_R = \{\alpha \begin{bmatrix} \xi + 2 & | & -1 \end{bmatrix} + \beta \begin{bmatrix} 1 & | & 0 \end{bmatrix} \mid \alpha, \beta \in \mathbb{R}\}$$

Any set of generators of $\Xi_R \rightsquigarrow$ a state map

Example

$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u$ $\quad$ $\left[\xi^2 + 2\xi + 3 \quad | \quad -\xi - 3\right]$

$$\sigma_+ \rightsquigarrow \left[\xi + 2 \quad | \quad -1\right] \rightsquigarrow \left[\sigma + 2 \quad | \quad -1\right]$$

$$\sigma_+^2 \rightsquigarrow \left[1 \quad | \quad 0\right] \rightsquigarrow \left[1 \quad | \quad 0\right]$$

$$\Xi_R = \{\alpha \left[\xi + 2 \quad | \quad -1\right] + \beta \left[1 \quad | \quad 0\right] \mid \alpha, \beta \in \mathbb{R}\}$$

A basis of $\Xi_R \rightsquigarrow$ a minimal state map

Example

$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u$ $\left[\xi^2 + 2\xi + 3 \;\middle|\; -\xi - 3\right]$

$$\Xi_R = \{\alpha \left[\xi + 2 \;\middle|\; -1\right] + \beta \left[1 \;\middle|\; 0\right] \mid \alpha, \beta \in \mathbb{R}\}$$

Observe that

$$X(\xi) = \begin{bmatrix} \xi + 2 & | & -1 \\ 1 & | & 0 \end{bmatrix} + f(\xi)R(\xi), \qquad f \in \mathbb{R}^{2\times 1}[\xi]$$

takes same values of

$$\begin{bmatrix} \xi + 2 & | & -1 \\ 1 & | & 0 \end{bmatrix}$$

on $\mathtt{ker}\ R(\sigma)$.

State maps for hybrid representations

$$R(\sigma)w = M(\sigma)\ell$$

$$(0,0) \underset{0}{\wedge} (w,\ell) \in \mathcal{B}_{\text{full}} \Longrightarrow 0 \underset{0}{\wedge} w \in \mathcal{B}$$

$$\Downarrow$$

State map for $\mathcal{B}_{\text{full}}$ is also state map for $\mathcal{B}$

$$\Downarrow$$

Use shift-and-cut on $\begin{bmatrix} R & -M \end{bmatrix}$

State maps for hybrid representations

$$R(\sigma)w = M(\sigma)\ell$$

$$(0,0) \underset{0}{\wedge} (w,\ell) \in \mathcal{B}_{\text{full}} \Longrightarrow 0 \underset{0}{\wedge} w \in \mathcal{B}$$

$$\Downarrow$$

State map for $\mathcal{B}_{\text{full}}$ is also state map for $\mathcal{B}$

$$\Downarrow$$

Use shift-and-cut on $\begin{bmatrix} R & -M \end{bmatrix}$

$\Longrightarrow$ **only sufficient: state for $\mathcal{B}_{\text{full}}$ vs. state for $\mathcal{B}$.**

“Reduction" needed for characterization.

Special case: image representations

$$\boldsymbol{w} = \boldsymbol{M}(\sigma)\ell$$

Special case: image representations

$$w = M(\sigma)\ell$$

- $\mathcal{B}$ **has observable image representation iff** **controllable**

Special case: image representations

$$w = M(\sigma)\ell$$

- $\mathcal{B}$ has observable image representation iff **controllable**
- W.l.o.g. $M = \begin{bmatrix} D \\ N \end{bmatrix}$ with ND^{-1} proper

Special case: image representations

$$w = M(\sigma)\ell$$

- $\mathcal{B}$ **has observable image representation iff controllable**

- **W.l.o.g.** $M = \begin{bmatrix} D \\ N \end{bmatrix}$ **with** ND^{-1} **proper**

- **input** $\longrightarrow$ / **output** $\longrightarrow$ $\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} D(\sigma) \\ N(\sigma) \end{bmatrix} \ell$

State maps for image representations

Theorem: Let $\mathcal{B} = \mathrm{im}(M(\sigma))$, with $M = \begin{bmatrix} D \\ N \end{bmatrix}$, ND^{-1} proper. Then

$$\begin{aligned} w &= M(\sigma)\ell \\ x &= X(\sigma)\ell \end{aligned}$$

is **state representation** of $\mathcal{B}$ with **state variable** x **IFF**

$$\text{row span } (X) = \text{row span } (\Sigma_D)$$

State maps for image representations

Theorem: Let $\mathcal{B} = \mathrm{im}(M(\sigma))$, with $M = \begin{bmatrix} D \\ N \end{bmatrix}$, ND^{-1} proper. Then

$$\begin{aligned} w &= M(\sigma)\ell \\ x &= X(\sigma)\ell \end{aligned}$$

is **state representation** of $\mathcal{B}$ with **state variable** x **IFF**

$$\begin{aligned} \text{row span } (X) &= \text{row span } (\Sigma_D) \\ &= \{r \in \mathbb{R}^{1\times 1}[\xi] \mid rD^{-1} \text{ strictly proper}\} \end{aligned}$$

Recapitulation

- **State maps are not unique**

Recapitulation

- **State maps are not unique**
- **Algebraic characterization of state maps**

Recapitulation

- **State maps are not unique**
- **Algebraic characterization of state maps**
- **State maps for hybrid representations**

Outline

On the space of solutions

$\mathfrak{C}^\infty$-solutions too restrictive: no step, no ramp, etc.

$$\mathcal{L}_1^{\text{loc}} := \{f : \mathbb{R} \to \mathbb{R}^{\mathtt{w}} \mid \int_K |f| dx \text{ finite } \forall \text{ compact } K \subset \mathbb{R}\}$$

On the space of solutions

$\mathfrak{C}^\infty$**-solutions too restrictive: no step, no ramp, etc.**

$$\mathcal{L}_1^{\text{loc}} := \{f : \mathbb{R} \to \mathbb{R}^{\mathtt{w}} \mid \int_K |f| dx \text{ finite } \forall \text{ compact } K \subset \mathbb{R}\}$$

Equality in the sense of distributions:

$$R\left(\tfrac{d}{dt}\right) w = 0 \quad \Leftrightarrow \quad \int_{-\infty}^{+\infty} w(t)^\top (R(-\tfrac{d}{dt})^\top f)(t) dt = 0$$

for all testing functions f.

On the space of solutions

$\mathfrak{C}^\infty$**-solutions too restrictive: no step, no ramp, etc.**

$$\mathcal{L}_1^{\text{loc}} := \{f : \mathbb{R} \to \mathbb{R}^{\mathtt{w}} \mid \int_K |f| dx \text{ finite } \forall \text{ compact } K \subset \mathbb{R}\}$$

Equality in the sense of distributions:

$$R\left(\frac{d}{dt}\right) w = 0 \qquad \Leftrightarrow \qquad \int_{-\infty}^{+\infty} w(t)^\top (R(-\tfrac{d}{dt})^\top f)(t) dt = 0$$

for all testing functions f.

Testing function: $\mathfrak{C}^\infty$ with compact support (a 'blip')

On the space of solutions

$\mathfrak{C}^\infty$**-solutions too restrictive: no step, no ramp, etc.**

$$\mathcal{L}_1^{\text{loc}} := \{ f : \mathbb{R} \to \mathbb{R}^{\mathtt{w}} \mid \int_K |f| dx \text{ finite } \forall \text{ compact } K \subset \mathbb{R} \}$$

Equality in the sense of distributions:

$$R\left(\frac{d}{dt}\right) w = 0 \quad \Leftrightarrow \quad \int_{-\infty}^{+\infty} w(t)^\top (R(-\tfrac{d}{dt})^\top f)(t) dt = 0$$

for all testing functions f.

From now,

$$\mathtt{ker}\, R\left(\frac{d}{dt}\right) := \left\{ w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{d}{dt}\right) w = 0 \text{ in the sense of distributions} \right\}$$

The state property revisited

$\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathcal{B}_{\text{full}})$ **is a state system if**

$$(w_1, x_1), (w_2, x_2) \in \mathcal{B}_{\text{full}} \text{ and } x_1(T) = x_2(T)$$

and x_1, x_2 continuous at T

$$\Downarrow$$

$$(w_1, x_1) \underset{T}{\wedge} (w_2, x_2) \in \mathcal{B}_{\text{full}}$$

'State map' $\rightsquigarrow X\left(\frac{d}{dt}\right)$

From kernel representation to state map

Denote $\mathrm{col}((\sigma_+^i(R)))_{i=1,\ldots,L} =: \Sigma_R$.

Theorem: **Let** $\mathcal{B} = \mathtt{ker}(R\left(\frac{d}{dt}\right))$. **Then**

$$\begin{aligned} R\left(\frac{d}{dt}\right) w &= 0 \\ \Sigma_R\left(\frac{d}{dt}\right) w &= x \end{aligned}$$

is a state representation of $\mathcal{B}$ **with state variable** x.

¿How to prove it?

When is $w \in \mathcal{B}$ concatenable with zero?

$$
\begin{aligned}
0 \underset{0}{\wedge} w \in \mathcal{B} \quad &\Longleftrightarrow \quad \int_{-\infty}^{+\infty} (0 \underset{0}{\wedge} w)(t)^\top (R(-\frac{d}{dt})^\top f)(t)dt = 0 \\
&\Longleftrightarrow \quad \int_{0}^{+\infty} w(t)^\top (R(-\frac{d}{dt})^\top f)(t)dt = 0
\end{aligned}
$$

for all testing functions f

When is $w \in \mathcal{B}$ concatenable with zero?

$$0 \underset{0}{\wedge} w \in \mathcal{B} \iff \int_{-\infty}^{+\infty} (0 \underset{0}{\wedge} w)(t)^\top (R(-\frac{d}{dt})^\top f)(t)dt = 0$$
$$\iff \int_{0}^{+\infty} w(t)^\top (R(-\frac{d}{dt})^\top f)(t)dt = 0$$

for all testing functions f

Integrating repeatedly by parts on f yields:

$$\sum_{k=1}^{\deg(R)} \sum_{j=k}^{\deg(R)} (-1)^{k-1} (\frac{d^{j-k}}{dt^{j-k}} w)(0)^\top R_j^\top (\frac{d^{k-1}}{dt^{k-1}} f)(0)$$
$$+ \int_0^{+\infty} (R\left(\frac{d}{dt}\right) w)(t)^\top f(t)dt = 0$$

When is $w \in \mathcal{B}$ concatenable with zero?

$$0 \underset{0}{\wedge} w \in \mathcal{B} \iff \int_{-\infty}^{+\infty} (0 \underset{0}{\wedge} w)(t)^\top (R(-\frac{d}{dt})^\top f)(t)dt = 0$$

$$\iff \int_{0}^{+\infty} w(t)^\top (R(-\frac{d}{dt})^\top f)(t)dt = 0$$

for all testing functions f

Integrating repeatedly by parts on f yields:

$$\sum_{k=1}^{\deg(R)} \sum_{j=k}^{\deg(R)} (-1)^{k-1} (\tfrac{d^{j-k}}{dt^{j-k}} w)(0)^\top R_j^\top (\tfrac{d^{k-1}}{dt^{k-1}} f)(0)$$

$$+ \int_0^{+\infty} (R\left(\tfrac{d}{dt}\right) w)(t)^\top f(t)dt = 0$$

$w \in \mathcal{B}$ concatenable with zero if and only if...

$$\sum_{k=1}^{\deg(R)} \sum_{j=k}^{\deg(R)} (-1)^{k-1} \left(\frac{d^{j-k}}{dt^{j-k}} w\right)(0)^\top R_j^\top \left(\frac{d^{k-1}}{dt^{k-1}} f\right)(0) = 0$$

$$\Updownarrow$$

$$\begin{bmatrix} f(0) \\ \left(\frac{d}{dt} f\right)(0) \\ \vdots \\ (-1)^{\deg(R)-1}\left(\frac{d^{\deg(R)-1}}{dt^{\deg(R)-1}} f\right)(0) \end{bmatrix}^\top \left(\Sigma_R\left(\frac{d}{dt}\right) w\right)(0) = 0$$

$$\Updownarrow$$

$$\left(\Sigma_R\left(\frac{d}{dt}\right) w\right)(0) = 0$$

The shift-and-cut state map!

Furthermore...

- **Algebraic characterization**, **minimality**: as in discrete-time case

Furthermore...

- **Algebraic characterization, minimality: as in discrete-time case**

- **Systems with latent variables: shift-and-cut on full equations; “reduction"**

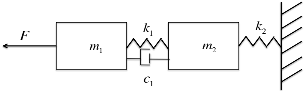
Furthermore...

- **Algebraic characterization**, **minimality**: as in discrete-time case

- **Systems with latent variables**: shift-and-cut on full equations; “reduction"

- **State equations**: also first order in state variable and zeroth in *w*, as in discrete-time case

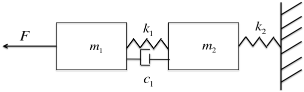
Example- 1



$$m_1 \frac{d^2 w_1}{dt^2} + c_1 \left(\frac{d}{dt} w_1 - \frac{d}{dt} w_2 \right) + k_1 (w_1 - w_2) - F = 0$$

$$-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + c_1 \left(\frac{d}{dt} w_2 - \frac{d}{dt} w_1 \right) + (k_1 + k_2) w_2 = 0$$

Example- 1

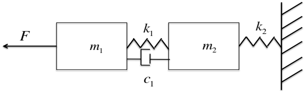


$$m_1 \frac{d^2 w_1}{dt^2} + c_1 \left(\frac{d}{dt} w_1 - \frac{d}{dt} w_2 \right) + k_1 (w_1 - w_2) - F = 0$$

$$-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + c_1 \left(\frac{d}{dt} w_2 - \frac{d}{dt} w_1 \right) + (k_1 + k_2) w_2 = 0$$

$$R(\xi) := \begin{bmatrix} m_1\xi^2 + c_1\xi + k_1 & -c_1\xi - k_1 & -1 \\ -c_1\xi - k_1 & m_2\xi^2 + c_1\xi + k_1 + k_2 & 0 \end{bmatrix}$$

Example- 1



$$\begin{aligned} m_1 \frac{d^2 w_1}{dt^2} + c_1 \left(\frac{d}{dt} w_1 - \frac{d}{dt} w_2 \right) + k_1 (w_1 - w_2) - F &= 0 \\ -k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + c_1 \left(\frac{d}{dt} w_2 - \frac{d}{dt} w_1 \right) + (k_1 + k_2) w_2 &= 0 \end{aligned}$$

$$R(\xi) := \begin{bmatrix} m_1\xi^2 + c_1\xi + k_1 & -c_1\xi - k_1 & -1 \\ -c_1\xi - k_1 & m_2\xi^2 + c_1\xi + k_1 + k_2 & 0 \end{bmatrix}$$

Shift-and-cut:

$$\begin{aligned} \sigma_+(R(\xi)) &= \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \\ -c_1 & m_2\xi + c_1 & 0 \end{bmatrix} \\ \sigma_+^2(R(\xi)) &= \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{bmatrix} \end{aligned}$$

Example- 2

$$X(\xi) := \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \\ -c_1 & m_2\xi + c_1 & 0 \\ m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{bmatrix}$$

- **linearly independent rows over $\mathbb{R}$;**
- **spans** `rowspan`**(Ξ_R)**

Example- 2

$$X(\xi) := \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \\ -c_1 & m_2\xi + c_1 & 0 \\ m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{bmatrix}$$

- **linearly independent rows over $\mathbb{R}$;**
- **spans $\texttt{rowspan}(\Xi_R)$**

$X(\xi)$ is a state map

Example- 2

$$X(\xi) := \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \\ -c_1 & m_2\xi + c_1 & 0 \\ m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{bmatrix}$$

- **linearly independent rows over $\mathbb{R}$;**
- **spans $\texttt{rowspan}(\Xi_R)$**

$X(\xi)$ is a state map

$$\begin{aligned} X_1(\xi) + f(\xi)R(\xi) &= \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \end{bmatrix} \\ &+ f_1(\xi)\begin{bmatrix} m_1\xi^2 + c_1\xi + k_1 & -c_1\xi - k_1 & -1 \end{bmatrix} \\ &+ f_2(\xi)\begin{bmatrix} -c_1\xi - k_1 & m_2\xi^2 + c_1\xi + k_1 + k_2 & 0 \end{bmatrix} \end{aligned}$$

induces the same value as $X_1(\frac{d}{dt})$ on any trajectory of $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)$.

Example- 3

Classical mechanics 'state':

$$X_c(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ \xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \xi & 0 \end{bmatrix}$$

Shift-and-cut state map:

$$X(\xi) := \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \\ -c_1 & m_2\xi + c_1 & 0 \\ m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{bmatrix}$$

Example- 3

Classical mechanics 'state':

$$X_c(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ \xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \xi & 0 \end{bmatrix}$$

Shift-and-cut state map:

$$X(\xi) := \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \\ -c_1 & m_2\xi + c_1 & 0 \\ m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{bmatrix}$$

$$X(\xi) = \underbrace{\begin{bmatrix} c_1 & m_1 & -c_1 & 0 \\ -c_1 & 0 & c_1 & m_2 \\ m_1 & 0 & 0 & 0 \\ 0 & 0 & m_2 & 0 \end{bmatrix}}_{=:T} X_c(\xi)$$

T is nonsingular ('state isomorphism theorem').

Recapitulation

- **$\mathcal{L}_1^{\text{loc}}$-trajectories**

Recapitulation

- **$\mathcal{L}_1^{\text{loc}}$-trajectories**

- **Equality in the sense of distributions + concatenability with zero = state maps!**

Recapitulation

- **$\mathcal{L}_1^{\text{loc}}$-trajectories**
- **Equality in the sense of distributions + concatenability with zero = state maps!**
- **Algebra is analogous to discrete-time case.**

Outline

From kernel representation to state representation

$$\boldsymbol{R} \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi] \rightsquigarrow \textbf{state map } \boldsymbol{X} \in \mathbb{R}^{\mathtt{n}\times\mathtt{w}}[\xi]$$

From kernel representation to state representation

$$R \in \mathbb{R}^{g \times w}[\xi] \leadsto \text{state map } X \in \mathbb{R}^{n \times w}[\xi]$$

Find:

$E, F \in \mathbb{R}^{(n+g) \times n}$, $G \in \mathbb{R}^{(n+g) \times w}$

$T \in \mathbb{R}^{(n+g) \times g}[\xi]$ with $\mathrm{rank}(T(\lambda)) = g \ \forall \lambda \in \mathbb{C}$

satisfying

$$E\xi X(\xi) + FX(\xi) + G = T(\xi)R(\xi)$$

From kernel representation to state representation

$$R \in \mathbb{R}^{g \times w}[\xi] \rightsquigarrow \text{state map } X \in \mathbb{R}^{n \times w}[\xi]$$

Find:

$E, F \in \mathbb{R}^{(n+g) \times n}$, $G \in \mathbb{R}^{(n+g) \times w}$

$T \in \mathbb{R}^{(n+g) \times g}[\xi]$ with $\operatorname{rank}(T(\lambda)) = g \ \forall \lambda \in \mathbb{C}$

satisfying

$$E\xi X(\xi) + FX(\xi) + G = T(\xi)R(\xi)$$

Linear equations, Gröbner bases computations!

From I/O representation to I/O/S representation

I/O representation $R = \begin{bmatrix} P & -Q \end{bmatrix}$ $\rightsquigarrow$ **state map** $\begin{bmatrix} X_y & X_u \end{bmatrix}$

Find:

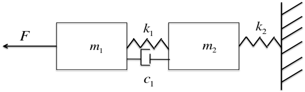
$A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times p}$, $D \in \mathbb{R}^{p \times m}$

$T \in \mathbb{R}^{(n+p) \times p}[\xi]$ with $\operatorname{rank}(T(\lambda)) = g \ \forall \lambda \in \mathbb{C}$

satisfying

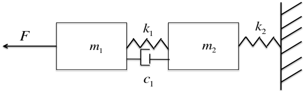
$$\begin{bmatrix} \xi X_y(\xi) & \xi X_u(\xi) \\ I_p & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_y(\xi) & X_u(\xi) \\ 0 & I_m \end{bmatrix} + T(\xi) R(\xi)$$

Example



$$
\begin{aligned}
m_1 \frac{d^2 w_1}{dt^2} + c_1 \left(\frac{d}{dt} w_1 - \frac{d}{dt} w_2 \right) + k_1 (w_1 - w_2) - F &= 0 \\
-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + c_1 \left(\frac{d}{dt} w_2 - \frac{d}{dt} w_1 \right) + (k_1 + k_2) w_2 &= 0
\end{aligned}
$$

Example



$$m_1 \frac{d^2 w_1}{dt^2} + c_1 \left(\frac{d}{dt} w_1 - \frac{d}{dt} w_2 \right) + k_1 (w_1 - w_2) - F = 0$$

$$-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + c_1 \left(\frac{d}{dt} w_2 - \frac{d}{dt} w_1 \right) + (k_1 + k_2) w_2 = 0$$

$$X(\xi) := \begin{bmatrix} m_1 \xi + c_1 & -c_1 & 0 \\ -c_1 & m_2 \xi + c_1 & 0 \\ m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{bmatrix} \qquad R(\xi) = \begin{bmatrix} m_1 \xi^2 + c_1 \xi + k_1 & -c_1 \xi - k_1 & -1 \\ -c_1 \xi - k_1 & m_2 \xi^2 + c_1 \xi + k_1 + k_2 & 0 \end{bmatrix}$$

Example

$$X(\xi) := \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \\ -c_1 & m_2\xi + c_1 & 0 \\ m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{bmatrix} \qquad R(\xi) = \begin{bmatrix} m_1\xi^2 + c_1\xi + k_1 & -c_1\xi - k_1 & -1 \\ -c_1\xi - k_1 & m_2\xi^2 + c_1\xi + k_1 + k_2 & 0 \end{bmatrix}$$

Observe that

$$\xi X_1(\xi) \quad = \quad \xi \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \end{bmatrix} = R_1(\xi) + \begin{bmatrix} 0 & 0 & -\frac{k_1}{m_1} & \frac{k_1}{m_2} \end{bmatrix} X(\xi) + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Example

$$X(\xi) := \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \\ -c_1 & m_2\xi + c_1 & 0 \\ m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{bmatrix} \qquad R(\xi) = \begin{bmatrix} m_1\xi^2 + c_1\xi + k_1 & -c_1\xi - k_1 & -1 \\ -c_1\xi - k_1 & m_2\xi^2 + c_1\xi + k_1 + k_2 & 0 \end{bmatrix}$$

Observe that

$$\begin{aligned} \xi X_1(\xi) &= \xi \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \end{bmatrix} = R_1(\xi) + \begin{bmatrix} 0 & 0 & -\frac{k_1}{m_1} & \frac{k_1}{m_2} \end{bmatrix} X(\xi) + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\ \xi X_2(\xi) &= \xi \begin{bmatrix} -c_1 & m_2\xi + c_1 & 0 \end{bmatrix} = R_2(\xi) + \begin{bmatrix} 0 & 0 & \frac{k_1}{m_1} & -\frac{k_1+k_2}{m_2} \end{bmatrix} X(\xi) \end{aligned}$$

Example

$$X(\xi) := \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \\ -c_1 & m_2\xi + c_1 & 0 \\ m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{bmatrix} \qquad R(\xi) = \begin{bmatrix} m_1\xi^2 + c_1\xi + k_1 & -c_1\xi - k_1 & -1 \\ -c_1\xi - k_1 & m_2\xi^2 + c_1\xi + k_1 + k_2 & 0 \end{bmatrix}$$

Observe that

$$\begin{aligned}
\xi X_1(\xi) &= \xi \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \end{bmatrix} = R_1(\xi) + \begin{bmatrix} 0 & 0 & -\frac{k_1}{m_1} & \frac{k_1}{m_2} \end{bmatrix} X(\xi) + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\
\xi X_2(\xi) &= \xi \begin{bmatrix} -c_1 & m_2\xi + c_1 & 0 \end{bmatrix} = R_2(\xi) + \begin{bmatrix} 0 & 0 & \frac{k_1}{m_1} & -\frac{k_1+k_2}{m_2} \end{bmatrix} X(\xi) \\
\xi X_3(\xi) &= \xi \begin{bmatrix} m_1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{c_1}{m_1} & \frac{c_1}{m_2} \end{bmatrix} X(\xi)
\end{aligned}$$

Example

$$X(\xi) := \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \\ -c_1 & m_2\xi + c_1 & 0 \\ m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{bmatrix} \qquad R(\xi) = \begin{bmatrix} m_1\xi^2 + c_1\xi + k_1 & -c_1\xi - k_1 & -1 \\ -c_1\xi - k_1 & m_2\xi^2 + c_1\xi + k_1 + k_2 & 0 \end{bmatrix}$$

Observe that

$$\begin{aligned}
\xi X_1(\xi) &= \xi \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \end{bmatrix} = R_1(\xi) + \begin{bmatrix} 0 & 0 & -\frac{k_1}{m_1} & \frac{k_1}{m_2} \end{bmatrix} X(\xi) + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\
\xi X_2(\xi) &= \xi \begin{bmatrix} -c_1 & m_2\xi + c_1 & 0 \end{bmatrix} = R_2(\xi) + \begin{bmatrix} 0 & 0 & \frac{k_1}{m_1} & -\frac{k_1+k_2}{m_2} \end{bmatrix} X(\xi) \\
\xi X_3(\xi) &= \xi \begin{bmatrix} m_1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{c_1}{m_1} & \frac{c_1}{m_2} \end{bmatrix} X(\xi) \\
\xi X_4(\xi) &= \xi \begin{bmatrix} 0 & m_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \frac{c_1}{m_1} & -\frac{c_1}{m_2} \end{bmatrix} X(\xi)
\end{aligned}$$

Example

$$X(\xi) := \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \\ -c_1 & m_2\xi + c_1 & 0 \\ m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{bmatrix} \qquad R(\xi) = \begin{bmatrix} m_1\xi^2 + c_1\xi + k_1 & -c_1\xi - k_1 & -1 \\ -c_1\xi - k_1 & m_2\xi^2 + c_1\xi + k_1 + k_2 & 0 \end{bmatrix}$$

Observe that

$$\begin{aligned}
\xi X_1(\xi) &= \xi \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \end{bmatrix} = R_1(\xi) + \begin{bmatrix} 0 & 0 & -\frac{k_1}{m_1} & \frac{k_1}{m_2} \end{bmatrix} X(\xi) + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\
\xi X_2(\xi) &= \xi \begin{bmatrix} -c_1 & m_2\xi + c_1 & 0 \end{bmatrix} = R_2(\xi) + \begin{bmatrix} 0 & 0 & \frac{k_1}{m_1} & -\frac{k_1+k_2}{m_2} \end{bmatrix} X(\xi) \\
\xi X_3(\xi) &= \xi \begin{bmatrix} m_1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{c_1}{m_1} & \frac{c_1}{m_2} \end{bmatrix} X(\xi) \\
\xi X_4(\xi) &= \xi \begin{bmatrix} 0 & m_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \frac{c_1}{m_1} & -\frac{c_1}{m_2} \end{bmatrix} X(\xi)
\end{aligned}$$

State equations are

$$\frac{d}{dt} X\left(\frac{d}{dt}\right) w = \begin{bmatrix} 0 & 0 & -\frac{k_1}{m_1} & \frac{k_1}{m_2} \\ 0 & 0 & \frac{k_1}{m_1} & -\frac{k_1+k_2}{m_2} \\ 1 & 0 & -\frac{c_1}{m_1} & \frac{c_1}{m_2} \\ 0 & 1 & \frac{c_1}{m_1} & -\frac{c_1}{m_2} \end{bmatrix} X\left(\frac{d}{dt}\right) w + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} w$$

On the choice of state map

State map
+
system equations $\rightsquigarrow$ **state-space equations**

On the choice of state map

State map + **system equations** $\rightsquigarrow$ **state-space equations**

On the choice of state map

State map
+
system equations $\rightsquigarrow$ **state-space equations**

$$\left(\frac{d^2}{dt^2} + 2\frac{d}{dt} + 3\right)y = \left(\frac{d}{dt} + 3\right)u$$

$$\begin{bmatrix}\xi^2 + 2\xi + 3 & -\xi - 3\end{bmatrix}$$

On the choice of state map

State map
+
system equations $\rightsquigarrow$ **state-space equations**

$$\left(\frac{d^2}{dt^2} + 2\frac{d}{dt} + 3\right)y = \left(\frac{d}{dt} + 3\right)u$$

$$\begin{bmatrix}\xi^2 + 2\xi + 3 & -\xi - 3\end{bmatrix}$$

Take $X(\xi) = \begin{bmatrix} 1 & 0 \\ \xi + 2 & -1 \end{bmatrix}$ **('reverse shift-and-cut').**

Then

$$A = \begin{bmatrix} -2 & 1 \\ -3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

'observer canonical form'

On the choice of state map

State map
+
system equations $\rightsquigarrow$ **state-space equations**

$$\left(\frac{d^2}{dt^2} + 2\frac{d}{dt} + 3\right)y = \left(\frac{d}{dt} + 3\right)u$$

$$\begin{bmatrix}\xi^2 + 2\xi + 3 & -\xi - 3\end{bmatrix}$$

Take $X(\xi) = \begin{bmatrix}1 & 0 \\ \xi & -1\end{bmatrix}$**. Then**

$$A = \begin{bmatrix}0 & 1 \\ -3 & -2\end{bmatrix} \quad B = \begin{bmatrix}1 \\ 1\end{bmatrix}$$

$$C = \begin{bmatrix}1 & 0\end{bmatrix} \quad D = \begin{bmatrix}0\end{bmatrix}$$

‘observable canonical form’

Example

In mechanical example,

$$X(\xi) = \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \\ -c_1 & m_2\xi + c_1 & 0 \\ m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{bmatrix} \quad \rightsquigarrow \quad A = \begin{bmatrix} 0 & 0 & -\frac{k_1}{m_1} & \frac{k_1}{m_2} \\ 0 & 0 & \frac{k_1}{m_1} & -\frac{k_1+k_2}{m_2} \\ 1 & 0 & -\frac{c_1}{m_1} & \frac{c_1}{m_2} \\ 0 & 1 & \frac{c_1}{m_1} & -\frac{c_1}{m_2} \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Example

In mechanical example,

$$X(\xi) = \begin{bmatrix} m_1\xi + c_1 & -c_1 & 0 \\ -c_1 & m_2\xi + c_1 & 0 \\ m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{bmatrix} \quad \rightsquigarrow \quad A = \begin{bmatrix} 0 & 0 & -\frac{k_1}{m_1} & \frac{k_1}{m_2} \\ 0 & 0 & \frac{k_1}{m_1} & -\frac{k_1+k_2}{m_2} \\ 1 & 0 & -\frac{c_1}{m_1} & \frac{c_1}{m_2} \\ 0 & 1 & \frac{c_1}{m_1} & -\frac{c_1}{m_2} \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

With classical mechanics state map:

$$X_c(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ \xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \xi & 0 \end{bmatrix} \quad \rightsquigarrow \quad A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{c_1}{m_1} & \frac{k_1}{m_1} & \frac{c_1}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{c_1}{m_2} & -\frac{k_1+k_2}{m_2} & -\frac{c_1}{m_2} \end{bmatrix}, B = \begin{bmatrix} \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix}$$

Summary

Summary

- **The state is constructed!**

Summary

- **The state is constructed!**
- **Axiom of state**

Summary

- **The state is constructed!**
- **Axiom of state**
- **Concatenability with zero**

Summary

- **The state is constructed!**
- **Axiom of state**
- **Concatenability with zero**
- **State maps**

Summary

- **The state is constructed!**
- **Axiom of state**
- **Concatenability with zero**
- **State maps**
- **State maps $\rightsquigarrow$ state-space equations**

Summary

- **The state is constructed!**
- **Axiom of state**
- **Concatenability with zero**
- **State maps**
- **State maps $\rightsquigarrow$ state-space equations**
- **Algorithms!**