$\mathscr{H}_\infty$-Control

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Outline

- The $\mathscr{H}_\infty$-Norm
- The $\mathscr{H}_\infty$-Control Problem
- $\mathscr{H}_\infty$-Analysis and the Bounded Real Lemma
- $\mathscr{H}_\infty$-Synthesis with LMIs
- $\mathscr{H}_\infty$-Synthesis with Riccati Equations
- $\mathscr{H}_2$-Analysis and Synthesis
- Mixed Controller Synthesis

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The $\mathscr{H}_\infty$-Norm

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$$\|M\| = \sup_{d \in \mathscr{L}_2(\mathbb{R}, \mathbb{R}^{n_d}),\, d \neq 0} \frac{\|Md\|_{\mathscr{L}_2(\mathbb{R}, \mathbb{R}^{n_e})}}{\|d\|_{\mathscr{L}_2(\mathbb{R}, \mathbb{R}^{n_d})}}.$$

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If $T(s)$ is the system's transfer matrix $C(sI - A)^{-1}B + D$ then

$$\|M\| = \|T\|_{\mathscr{H}_\infty} := \sup_{\omega \in \mathbb{R}} \sigma_{\max}(T(i\omega)).$$

The $\mathscr{H}_\infty$-Norm: Another Interpretation

Given a sinusoidal input $d(t) = d_0 e^{i\omega t}$, the response of the system up to transients is given by

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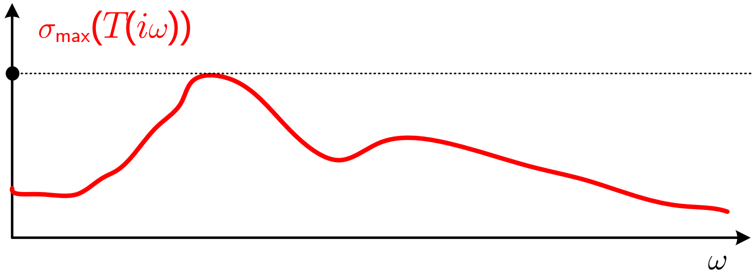
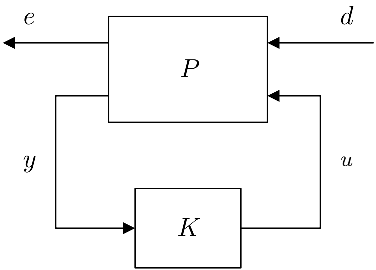
Analyze frequency-by-frequency attenuation with Bode-plot of T:



The $\mathscr{H}_\infty$-Control Problem

Given P determine a **stabilizing** controller K which minimizes the $\mathscr{H}_\infty$-norm of the closed-loop transfer matrix $\mathcal{T}(K)$:

minimize	$\|\mathcal{T}(K)\|_{\mathscr{H}_\infty}$
subject to	K stabilizes P



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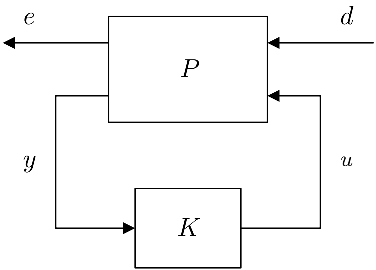

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- Due to energy-gain interpretation of the $\mathscr{H}_\infty$-norm: Optimal attenuation of disturbances $d \in \mathscr{L}_2(\mathbb{R}_+, \mathbb{R}^{n_d})$ at $e \in \mathscr{L}_2(\mathbb{R}_+, \mathbb{R}^{n_e})$.

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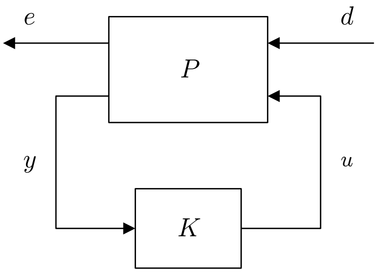

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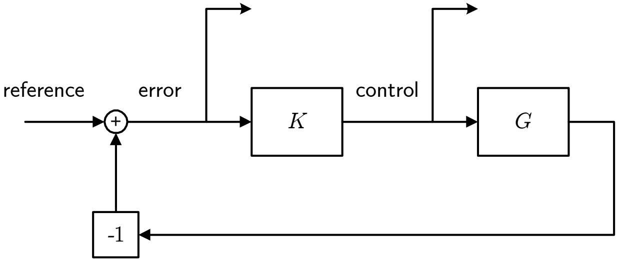
- This problem formulation is muuuuch more flexible, as we will only touch upon on the next couple of slides.

Example: Mixed Sensitivity Design

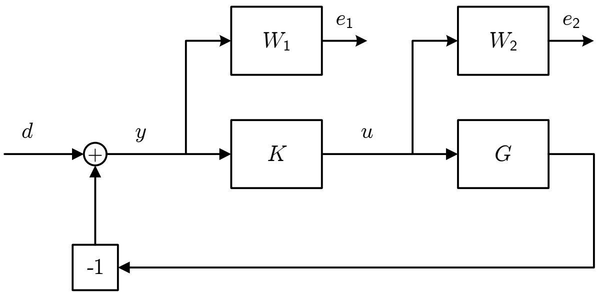
In a tracking problem, a major emphasis is laid on shaping the sensitivity (reference to tracking error), under the constraint that the control effort (reference to control) does not peak too much and rolls off at high frequencies. In view of this rough specs, consider



which indicates the relevant performance signals.

Example: Mixed Sensitivity Design

Choose a low-pass scalar weighting function w_1 and a constant or high-pass weight w_2. Define $W_1 = w_1 I$ and $W_2 = w_2 I$ and consider the following interconnection with weighted performance channels:



Then design a controller K which stabilizes this interconnection and minimizes the $\mathscr{H}_\infty$-norm of $d \to e = \text{col}(e_1, e_2)$.

Example: Mixed Sensitivity Design

Suppose the closed-loop $\mathscr{H}_\infty$-norm can be suppressed below γ. With

$$S = (I + GK)^{-1} \text{ and } R = KS = K(I + GK)^{-1}$$

this means that

$$\sigma_{\text{max}} \begin{pmatrix} w_1(i\omega)S(i\omega) \\ w_2(i\omega)R(i\omega) \end{pmatrix} \leq \gamma \text{ for all } \omega \in [0, \infty]$$

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$$\sigma_{\max}(S(i\omega)) \leq \frac{\gamma}{|w_1(i\omega)|}, \quad \sigma_{\max}(R(i\omega)) \leq \frac{\gamma}{|w_2(i\omega)|} \text{ for all } \omega \in [0, \infty].$$

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- If $\gamma \approx 1$, we achieve a high-pass characteristics for $\sigma_{\max}(S)$ and a low-pass characteristics for $\sigma_{\max}(R)$ as desired at the outset.

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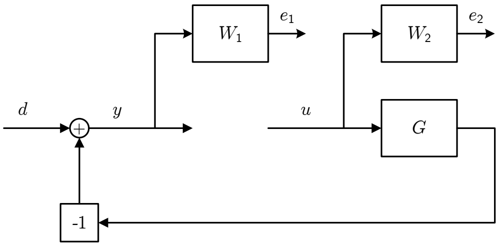
- If the optimal achievable norm is large, the imposed specifications are too tight and not achievable. The plots of

$$\sigma_{\max}\begin{pmatrix} w_1(i\omega)S(i\omega) \\ w_2(i\omega)R(i\omega) \end{pmatrix}, \ \sigma_{\max}(S(i\omega)), \ \frac{1}{|w_1(i\omega)|}, \ \sigma_{\max}(R(i\omega)), \ \frac{1}{|w_2(i\omega)|}$$

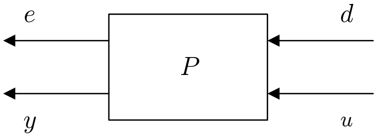
over frequency provide an indication about the frequency range on which the specifications were too tight. This information allows to adapt the weights for a next (better) design.

Example: Mixed Sensitivity Design

In summary: With a stabilizing controller we try to achieve a small minimal $\mathscr{H}_\infty$-norm for the open-loop interconnection



which is written in the so-called **generalized plant** format:



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System Descriptions and Problem Formulation

Open-loop interconnection and controller are described as

$$\begin{pmatrix} \dot{x} \\ \hline e \\ y \end{pmatrix} = \left(\begin{array}{c|cc} A & B_1 & B \\ \hline C_1 & D_1 & E \\ C & F & 0 \end{array}\right) \begin{pmatrix} x \\ \hline d \\ u \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \dot{x}_c \\ u \end{pmatrix} = \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix} \begin{pmatrix} x_c \\ y \end{pmatrix}.$$

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Controlled closed-loop system described with calligraphic matrices:

$$\begin{pmatrix} \dot{\xi} \\ e \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} \xi \\ d \end{pmatrix} \quad \text{and} \quad \mathcal{T}(s) = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}.$$

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Analysis: LMI-test for checking wether a controller achieves the specs.

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The celebrated **bounded real lemma** turns this “difficult-to-verify” **frequency domain inequality** into a genuine LMI.

$\mathcal{A}$ is stable and $\|\mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}\|_{\mathscr{H}_\infty} < \gamma$ holds iff there exist some $\mathcal{X} \succ 0$ such that

$$\begin{pmatrix} \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \mathcal{B}^T\mathcal{X} & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ \mathcal{C} & \mathcal{D} \end{pmatrix}^T P_\gamma \begin{pmatrix} 0 & I \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \prec 0. \qquad \text{(LMI)}$$

Proof of "if"

- The left-upper block of the LMI is $\mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} + \frac{1}{\gamma}\mathcal{C}^T\mathcal{C} \prec 0$. Hence $\mathcal{X} \succ 0$ implies that $\mathcal{A}$ is Hurwitz.

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- Finally, one easily checks for finite $\omega \in \mathbb{R}$ that

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Hence (LMI) implies for $\omega\in\mathbb{R}$ that

$$0\succ 0\begin{pmatrix}(i\omega I-\mathcal{A})^{-1}\mathcal{B}\\ I\end{pmatrix}^*\text{lhs of (LMI)}\begin{pmatrix}(i\omega I-\mathcal{A})^{-1}\mathcal{B}\\ I\end{pmatrix}=$$

$$\begin{pmatrix}I\\ \mathcal{T}(i\omega)\end{pmatrix}^*P_\gamma\begin{pmatrix}I\\ \mathcal{T}(i\omega)\end{pmatrix}.$$

Comments

- To show "only if" is harder, and very many different techniques of proof have been suggested in the literature.

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- Let $\mathcal{A}$ have no eigenvalues on the imaginary axis. Then the frequency-domain inequality holds iff (LMI) has a symmetric solution $\mathcal{X}$.

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- The result is best understood if embedded into dissipation theory for linear dynamical systems.

Rewriting (LMI)

Observe that

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if and only if (Schur)

$$\left(\begin{array}{cc|c} \mathcal{A}^T\mathcal{X}+\mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} & \mathcal{C}^T \\ \mathcal{B}^T\mathcal{X} & -\gamma I & \mathcal{D}^T \\ \hline \mathcal{C} & \mathcal{D} & -\gamma I \end{array}\right) \prec 0.$$

The latter inequality is just more convenient to work with!

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- $\mathscr{H}_\infty$-Analysis and the Bounded Real Lemma
- $\mathscr{H}_\infty$-Synthesis with LMIs
- $\mathscr{H}_\infty$-Synthesis with Riccati Equations
- $\mathscr{H}_2$-Analysis and Synthesis
- Mixed Controller Synthesis

State-Feedback Synthesis

Open-loop interconnection and controller are described as

$$\begin{aligned} \dot{x} &= Ax + B_1 d + Bu \\ e &= C_1 x + D_1 d + Eu \end{aligned} \quad \text{and} \quad u = D_K x.$$

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Stability of $\mathcal{A}$ and $\|\mathcal{T}\|_{\mathscr{H}_\infty} < \gamma$ iff exists $\mathcal{X}$ with

$$\mathcal{X} \succ 0, \begin{pmatrix} (A+BD_K)^T\mathcal{X}+\mathcal{X}(A+BD_K) & \mathcal{X}B_1 & (C_1+ED_K)^T \\ B_1^T\mathcal{X} & -\gamma I & D_1^T \\ C_1+ED_K & D_1 & -\gamma I \end{pmatrix} \prec 0.$$

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This is obviously **not an LMI** in the green variables. Remedy?

Static State-Feedback Synthesis

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- Perform the change of variables $Y = \mathcal{X}^{-1}$ and $M := D_K\mathcal{X}^{-1}$.
  This clearly results in an LMI in Y and M as shown on the next slide.

Static State-Feedback Synthesis

Synthesis inequalities for static state-feedback design:

$$Y \succ 0, \quad \begin{pmatrix} (AY+BM)^T+(AY+BM) & B_1 & (C_1Y+EM)^T \\ B_1^T & -\gamma I & D_1^T \\ (C_1Y+EM) & D_1 & -\gamma I \end{pmatrix} \prec 0.$$

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- Check whether the synthesis inequalities have solution Y, M.

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- If **yes** then $D_K = MY^{-1}$ is a stabilizing state-feedback gain which achieves $\|\mathcal{T}\|_{\mathscr{H}_\infty} < \gamma$.

Note that γ enters affinely. We can hence directly compute the optimal achievable $\mathscr{H}_\infty$-level by minimizing γ over the synthesis LMIs.

Output-Feedback Synthesis

Open-loop interconnection and controller are described as

$$\begin{pmatrix} \dot{x} \\ \hline e \\ y \end{pmatrix} = \left(\begin{array}{c|cc} A & B_1 & B \\ \hline C_1 & D_1 & E \\ C & F & 0 \end{array}\right) \begin{pmatrix} x \\ \hline d \\ u \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \dot{x}_c \\ u \end{pmatrix} = \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix} \begin{pmatrix} x_c \\ y \end{pmatrix}.$$

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This leads to controlled closed-loop system described with

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \left(\begin{array}{cc|c} A + BD_KC & BC_K & B_1 + BD_KF \\ B_KC & A_K & B_KF \\ \hline C_1 + ED_KC & EC_K & D_1 + ED_KF \end{array}\right).$$

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Need to solve the **nonlinear** matrix inequalities

$$\mathcal{X} \succ 0, \quad \begin{pmatrix} \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} & \mathcal{C}^T \\ \mathcal{B}^T\mathcal{X} & -\gamma I & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & -\gamma I \end{pmatrix} \prec 0.$$

Output-Feedback: Controller Parameter Change

According the partition of $\mathcal{A}$ introduce the following notations for the sub-blocks of $\mathcal{X}$ and its inverse:

$$\mathcal{X} = \begin{pmatrix} X & U \\ U^T & * \end{pmatrix}, \quad \mathcal{X}^{-1} = \begin{pmatrix} Y & V \\ V^T & * \end{pmatrix}.$$

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Recall for later that $YX + VU^T = I$.

Let us define

$$\begin{pmatrix} K & L \\ M & N \end{pmatrix} = \begin{pmatrix} XAY & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} U & XB \\ 0 & I \end{pmatrix} \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix} \begin{pmatrix} V^T & 0 \\ CY & I \end{pmatrix}.$$

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This obviously transforms the variables $\mathcal{X}$, A_K, B_K, C_K, D_K into the **new variables** X, Y and K, L, M, N.

This transformation is motivated by its linearizing effect as shown next.

Output-Feedback: Block Transformation

With $\mathcal{Y} = \begin{pmatrix} Y & I \\ V^T & 0 \end{pmatrix}$ a short computation reveals

$$\mathcal{Y}^T \mathcal{X} \mathcal{Y} = \begin{pmatrix} Y & I \\ I & X \end{pmatrix},$$

$$\left(\begin{array}{c|c} \mathcal{Y}^T(\mathcal{X}\mathcal{A})\mathcal{Y} & \mathcal{Y}^T(\mathcal{X}\mathcal{B}) \\ \hline \mathcal{C}\mathcal{Y} & \mathcal{D} \end{array} \right) =$$

$$= \left(\begin{array}{cc|c} AY + BM & A + BNC & B_1 + BNF \\ K & XA + LC & XB_1 + LF \\ \hline C_1Y + EM & C_1 + ENC & D_1 + ENF \end{array} \right).$$

Observe the **affine** dependence on X, Y and K, L, M, N!

Output-Feedback: Congruence Transformation

For necessity: Can assume w.l.o.g. that $\mathcal{Y}$ has full column rank.

For sufficiency: Make sure that $\mathcal{Y}$ is square and non-singular.

Transform

$$\mathcal{X} \succ 0, \quad \begin{pmatrix} \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} & \mathcal{C}^T \\ \mathcal{B}^T\mathcal{X} & -\gamma I & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & -\gamma I \end{pmatrix} \prec 0$$

by congruence with matrices $\mathcal{Y}$ and $\mathrm{diag}(\mathcal{Y}, I, I)$ into

$$\mathcal{Y}^T\mathcal{X}\mathcal{Y} \succ 0, \quad \begin{pmatrix} \mathcal{Y}^T(\mathcal{A}^T\mathcal{X})\mathcal{Y} + \mathcal{Y}^T(\mathcal{X}\mathcal{A})\mathcal{Y} & \mathcal{Y}^T(\mathcal{X}\mathcal{B}) & \mathcal{Y}^T\mathcal{C}^T \\ (\mathcal{B}^T\mathcal{X})\mathcal{Y} & -\gamma I & \mathcal{D}^T \\ \mathcal{C}\mathcal{Y} & \mathcal{D} & -\gamma I \end{pmatrix} \prec 0.$$

Substitute formulas on previous slide to obtain **synthesis inequalities**.

Output-Feedback: Synthesis Inequalities

There exists a controller that renders $\mathcal{A}$ Hurwitz and which achieves $\|\mathcal{T}\|_{\mathscr{H}_\infty} < \gamma$ iff there exist X, Y and K, L, M, N such that

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0,$$

$$\left(\begin{array}{cc|cc} \text{sym}(AY+BM) & (A+BNC)+K^T & (B_1+BNF) & (C_1Y+EM)^T \\ (A+BNC)+K & \text{sym}(XA+LC) & (XB_1+LF) & (C_1+ENC)^T \\ \hline (B_1+BNF)^T & (XB_1+LF)^T & -\gamma I & (D_1+ENF)^T \\ (C_1Y+EM) & (C_1+ENC) & (D_1+ENF) & -\gamma I \end{array}\right) \prec 0$$

where we use the abbreviation $\text{sym}(A) = A^T + A$.

These are LMIs in X, Y and K, L, M, N! Also γ enters affinely!

Output-Feedback: Controller Construction

- Solve synthesis inequalities to determine X, Y and K, L, M, N.

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- A_K has the same size as A (full order controller).
- Freedom in choosing U, V only affects the controller realization (and not its transfer matrix). Example choice: $U = X$, $V = X^{-1} - Y$.

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The proposed linearizing change of variables can be applied to many other performance specifications which can be expressed by LMIs.

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- Obtain synthesis inequalities by substitution

$$\mathcal{Y}^T\mathcal{X}\mathcal{Y} \to \boldsymbol{X}(v), \quad \begin{pmatrix} \mathcal{Y}^T[\mathcal{X}\mathcal{A}]\mathcal{Y} & \mathcal{Y}^T[\mathcal{X}\mathcal{B}] \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \to \begin{pmatrix} \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix}$$

with affine $\boldsymbol{A}(v)$, $\boldsymbol{B}(v)$, $\boldsymbol{C}(v)$, $\boldsymbol{D}(v)$ in new variables v on next slide.

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- Works **both** in continuous-time and discrete-time in identical fashion.

Variables and Blocks

State-Feedback: $v = \left(\ Y,\ \ M\ \right)$ and

$$\boldsymbol{X}(v) = Y, \quad \begin{pmatrix} \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix} = \begin{pmatrix} AY + BM & B_1 \\ C_1 Y + EM & D_1 \end{pmatrix}$$

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Output-Feedback: $v = \left(X, \quad Y, \quad K, \quad L, \quad M, \quad N \right)$ and

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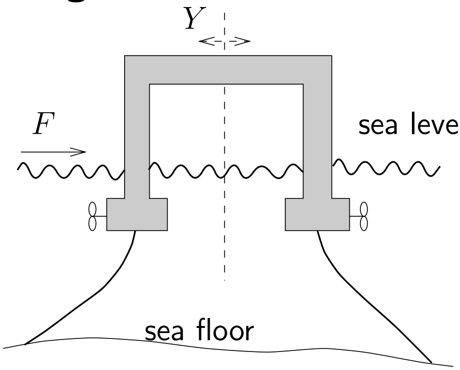
$$\begin{pmatrix} \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix} = \left(\begin{array}{cc|c} AY + BM & A + BNC & B_1 + BNF \\ K & XA + LC & XB_1 + LF \\ \hline C_1Y + EM & C_1 + ENC & D_1 + ENF \end{array}\right)$$

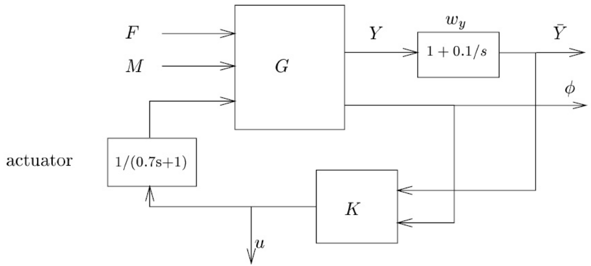
Example: Floating Platform



- Minimize drift Y resulting from lateral force F. Controller should act on low-frequency component of force only.
- Suppress resonance of $M \rightarrow \phi$ (moment to vertical angle).
- Keep thruster actuation bounded.

Example: Floating Platform

With actuator dynamics we use the following interconnection structure:

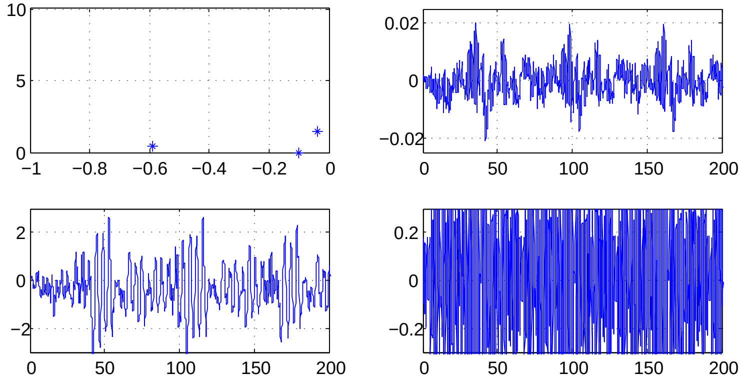


- Keep $|Y(t)|$ below $2.5cm$ and $|\phi(t)|$ below 3^o.
- Thruster actuation $|u(t)|$ should stay below 0.3.
- Push resonance peak of $M \to \phi$ down below 1.5.

Example: Output-Feedback Design

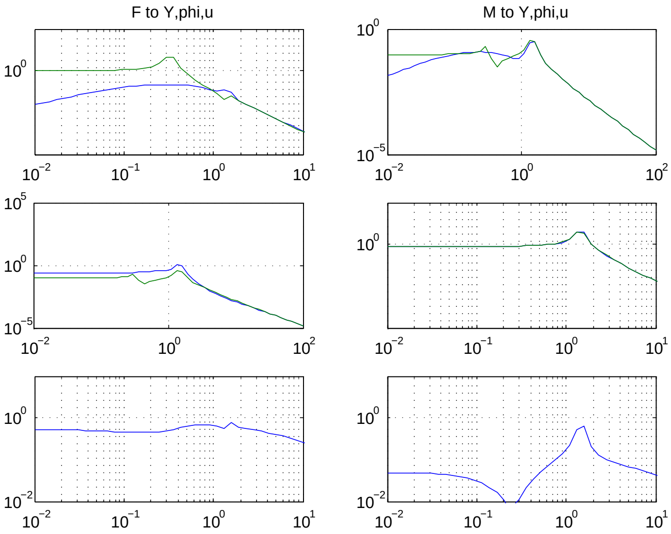
$\mathscr{H}_\infty$ design with LMI's for $\begin{pmatrix} F \\ M \end{pmatrix} \to \begin{pmatrix} \bar{Y} \\ 0.1\,\phi \\ 0.5\,u \end{pmatrix}$.

Closed-loop poles and time-domain specifications:



Example: Output-Feedback Design

Frequency domain-domain characteristics:



Outline

- The $\mathscr{H}_\infty$-Norm
- The $\mathscr{H}_\infty$-Control Problem
- $\mathscr{H}_\infty$-Analysis and the Bounded Real Lemma
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- $\mathscr{H}_\infty$-Synthesis with Riccati Equations
- $\mathscr{H}_2$-Analysis and Synthesis
- Mixed Controller Synthesis

From LMIs to Riccati Equations

Consider the specific open-loop system

$$\begin{pmatrix} \dot{x} \\ \hline e \\ y \end{pmatrix} = \left(\begin{array}{c|cc} A & B_1 & B \\ \hline C_1 & 0 & E \\ C & F & 0 \end{array}\right) \begin{pmatrix} x \\ \hline d \\ u \end{pmatrix} \quad \text{with} \quad \begin{cases} E^T C_1 = 0, \quad E^T E = I \\ B_1 F^T = 0, \quad F F^T = I \end{cases}$$

Carsten Scherer

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which has the following additional properties:

- (A, B_2) is stabilizable and (A, C_2) is detectable.
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Also consider $\gamma = 1$ and design a strictly proper controller ($D_K = 0$).

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Much stronger than required but convenient for simple derivation.

Simplfied Synthesis Inequalities

Since $D_K = 0$ iff $N = 0$, the synthesis LMIs from slide 22 simplify to

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0,$$

$$\left(\begin{array}{cc|cc} \text{sym}(AY+BM) & (A+BNC)+K^T & B_1 & (C_1Y+EM)^T \\ (A+BNC)+K & \text{sym}(XA+LC) & (XB_1+LF) & C_1^T \\ \hline (B_1+BNF)^T & (XB_1+LF)^T & -I & 0 \\ (C_1Y+EM) & C_1 & 0 & -I \end{array}\right) \prec 0.$$

Due to the particular way in which K enters, it can be easily eliminated.

Intermezzo: Elimination

There exists some K with

$$\begin{pmatrix} Q & S + K^T \\ S^T + K & R \end{pmatrix} \prec 0$$

if and only if

$$Q \prec 0 \;\; \text{and} \;\; R \prec 0.$$

Proof. "Only if" is obvious. Choose $K = -S^T$ to prove "If".

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There exists some L with

$$Q + (L + S)(L + S)^T \prec 0$$

if and only if

$$Q \prec 0.$$

Proof. "Only if" is obvious. Choose $L = -S$ to prove "If".

Simplfied Synthesis Inequalities

The resulting synthesis inequalities are

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0,$$

$$\left(\begin{array}{c|cc} (XA+LC)^T + (XA+LC) & (XB_1+LF) & C_1^T \\ \hline (XB_1+LF)^T & -I & 0 \\ C_1 & 0 & -I \end{array}\right) \prec 0,$$

$$\left(\begin{array}{c|cc} (AY+BM)^T + (AY+BM) & B_1 & (C_1Y+EM)^T \\ \hline B_1^T & -I & 0 \\ (C_1Y+EM) & 0 & -I \end{array}\right) \prec 0.$$

Towards Riccati Inequalities

Taking the Schur complements leads to

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0,$$

$$(XA+LC)^T+(XA+LC)+(XB_1+LF)(XB_1+LF)^T+C_1^TC_1 \prec 0,$$

$$(AY+BM)^T+(AY+BM)+B_1B_1^T+(C_1Y+EM)^T(C_1Y+EM) \prec 0.$$

Towards Riccati Inequalities

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$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0,$$

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$$(AY+BM)^T+(AY+BM)+B_1B_1^T+(C_1Y+EM)^T(C_1Y+EM) \prec 0.$$

The latter inequalities can be rearranged to

$$A^TX + XA + XB_1B_1^TX + C_1^TC_1 - C^TC + (L + C^T)(L + C^T)^T \prec 0,$$

$$AY + YA^T + YC_1^TC_1Y + B_1B_1^T - BB^T + (M + B^T)^T(M + B^T) \prec 0.$$

Now we can also eliminate L and M.

Solution in Terms of Riccati Inequalities

There exists a controller that renders $\mathcal{A}$ Hurwitz and which achieves $\|\mathcal{T}\|_{\mathscr{H}_\infty} < 1$ iff there exist X, Y such that

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0,$$

$$A^T X + XA + XB_1B_1^T X + C_1^T C_1 - C^T C \prec 0,$$

$$AY + YA^T + YC_1^T C_1 Y + B_1 B_1^T - BB^T \prec 0.$$

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- The Riccati inequalities can be turned into LMIs by Schur. These synthesis inequalities only involve X, Y. (Reduced complexity!)

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- The Riccati inequalities can be turned into LMIs by Schur. These synthesis inequalities only involve X, Y. (Reduced complexity!)
- Since (A, C_1) is observable and (A, B_1) is controllable, we can replace the Riccati inequalities by the corresponding equations.

Intermezzo: Hamiltonians and Riccati

Set $R = B_1 B_1^T$ and $Q = C_1^T C_1 - C^T C$. Equivalent are:

- The Hamiltonian $\begin{pmatrix} A & R \\ -Q & -A^T \end{pmatrix}$ has no eigenvalue in $\mathbb{C}^0$.

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X_+ is related to the solution set of the ARI as follows:

- **Largest:** Any solution X of the ARI satisfies $X \prec X_+$.

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- The algebraic Riccati inequality $A^T X + XA + XRX + Q \prec 0$ (ARI) has a symmetric solution X.

X_+ is related to the solution set of the ARI as follows:

- **Largest:** Any solution X of the ARI satisfies $X \prec X_+$.
- **Can come arbitrarily close:** For all $\epsilon > 0$ there exists a solution X of the ARI which satisfies $X_+ - \epsilon I \prec X$.

Solution in Terms of Riccati Equations

There exists a controller that renders $\mathcal{A}$ Hurwitz and which achieves $\|\mathcal{T}\|_{\mathscr{H}_\infty} < 1$ iff the Riccati equations

$$A^T X + XA + XB_1B_1^T X + C_1^T C_1 - C^T C = 0,$$

$$AY + YA^T + YC_1^T C_1 Y + B_1 B_1^T - BB^T = 0$$

have anti-stabilizing solutions X_+ and Y_+ which satisfy

$$\begin{pmatrix} X_+ & I \\ I & Y_+ \end{pmatrix} \succ 0.$$

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- These conditions can be verified with standard Riccati solvers.

Solution in Terms of Riccati Equations

There exists a controller that renders $\mathcal{A}$ Hurwitz and which achieves $\|\mathcal{T}\|_{\mathscr{H}_\infty} < 1$ iff the Riccati equations

$$A^T X + XA + XB_1B_1^TX + C_1^TC_1 - C^TC = 0,$$

$$AY + YA^T + YC_1^TC_1Y + B_1B_1^T - BB^T = 0$$

have anti-stabilizing solutions X_+ and Y_+ which satisfy

$$\begin{pmatrix} X_+ & I \\ I & Y_+ \end{pmatrix} \succ 0.$$

- These conditions can be verified with standard Riccati solvers.
- Result is usually formulated for the inverses $P = X_+{}^{-1}$ and $Q = Y_+{}^{-1}$ as shown next.

Solution in Terms of Indefinite Riccati Equations

There exists a controller that renders $\mathcal{A}$ Hurwitz and which achieves $\|\mathcal{T}\|_{\mathscr{H}_\infty} < 1$ iff the **indefinite** Riccati equations

$$AP + PA^T + B_1B_1^T + P(C_1^TC_1 - C^TC)P = 0,$$

$$A^TQ + QA + C_1^TC_1 + Q(B_1B_1^T - BB^T)Q = 0$$

have **stabilizing solutions** P and Q which satisfy

$$P \succ 0, \quad Q \succ 0, \quad \max|\lambda(PQ)| < 1.$$

If all conditions are satisfied, a suitable controller is given by

$$\left[\begin{array}{c|c} A + (B_1B_1^T - B_2B_2^T)Q + (Q - P^{-1})^{-1}C_2^TC_2 & -(Q - P^{-1})^{-1}C_2^T \\ \hline -B_2^TQ & 0 \end{array}\right].$$

Doyle, Glover, Khargonekar, Francis (1989)

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$\mathscr{H}_2$-Performance

Suppose $\mathcal{A}$ is Hurwitz and $\mathcal{D} = 0$. The $\mathscr{H}_2$-norm of $\mathcal{T}$ is defined as

$$\|\mathcal{T}\|_{\mathscr{H}_2} := \sqrt{\frac{1}{2\pi}\,\text{trace}\int_{-\infty}^{\infty} \mathcal{T}(i\omega)^*\mathcal{T}(i\omega)\,d\omega}.$$

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Easily computed by either one of following formulas:

$$\|\mathcal{T}\|_2^2 = \mathrm{trace}\left(\mathcal{C}P_0\mathcal{C}^T\right) \quad \text{where} \quad \mathcal{A}P_0 + P_0\mathcal{A}^T + \mathcal{B}\mathcal{B}^T = 0.$$

$$\|\mathcal{T}\|_2^2 = \mathrm{trace}\left(\mathcal{B}^T Q_0\mathcal{B}\right) \quad \text{where} \quad \mathcal{A}^T Q_0 + Q_0\mathcal{A} + \mathcal{C}^T\mathcal{C} = 0.$$

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$$\|\mathcal{T}\|_2^2 = \mathrm{trace}\left(\mathcal{B}^TQ_0\mathcal{B}\right) \quad \text{where} \quad \mathcal{A}^TQ_0 + Q_0\mathcal{A} + \mathcal{C}^T\mathcal{C} = 0.$$

Why? Recall that $Q_0 = \int_0^\infty e^{\mathcal{A}^Tt}\mathcal{C}^T\mathcal{C}e^{\mathcal{A}t}\,dt$ and apply Parseval:

$$\|\mathcal{T}\|_2^2 = \mathrm{trace}\int_0^\infty [\mathcal{C}e^{\mathcal{A}t}\mathcal{B}]^T[\mathcal{C}e^{\mathcal{A}t}\mathcal{B}]\,dt =$$

$$= \mathrm{trace}\,\mathcal{B}^T\left[\int_0^\infty e^{\mathcal{A}^Tt}\mathcal{C}^T\mathcal{C}e^{\mathcal{A}t}\,dt\right]\mathcal{B} = \mathrm{trace}\left(\mathcal{B}^TQ_0\mathcal{B}\right).$$

Carsten Scherer

Stochastic Interpretation of $\mathscr{H}_2$-Norm

Let d be white noise in $\dot{\xi} = \mathcal{A}\xi + \mathcal{B}d$.

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If d is white noise and $\dot{\xi} = \mathcal{A}\xi + \mathcal{B}d$, $e = \mathcal{C}\xi$ then

$$\lim_{t\to\infty} E[e(t)^T e(t)] = \|\mathcal{T}\|_{\mathscr{H}_2}^2.$$

The squared $\mathscr{H}_2$-norm equals the asymptotic variance of output.

Deterministic Interpretation of $\mathscr{H}_2$-Norm

Let d_k be a standard unit vector and denote the output of

$$\dot{\xi}(t) = \mathcal{A}\xi(t), \quad e = \mathcal{C}\xi, \quad \xi(0) = \mathcal{B}d_k$$

by $e_k(.)$. This is just the response to an impulse in channel k.

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Since $e_k(t) = \mathcal{C}e^{\mathcal{A}t}\mathcal{B}d_k$ we infer

$$\int_0^\infty e_k(t)^T e_k(t)\, dt = d_k^T \mathcal{B}^T \left[\int_0^\infty e^{\mathcal{A}^T t}\mathcal{C}^T\mathcal{C}e^{\mathcal{A}t}\, dt\right]\mathcal{B}d_k = d_k^T(\mathcal{B}^T Q_0 \mathcal{B})d_k.$$

Deterministic Interpretation of $\mathscr{H}_2$-Norm

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Since $e_k(t) = \mathcal{C}e^{\mathcal{A}t}\mathcal{B}d_k$ we infer

$$\int_0^\infty e_k(t)^T e_k(t)\,dt = d_k^T \mathcal{B}^T \left[\int_0^\infty e^{\mathcal{A}^T t}\mathcal{C}^T\mathcal{C}e^{\mathcal{A}t}\,dt\right]\mathcal{B}d_k = d_k^T(\mathcal{B}^T Q_0 \mathcal{B})d_k.$$

After summing over k the right-hand side is $\operatorname{trace}(\mathcal{B}^T Q_0 \mathcal{B}) = \|\mathcal{T}\|_{\mathscr{H}_2}^2$.

Squared $\mathscr{H}_2$-norm is energy sum of transients of output responses:

$$\sum_k \int_0^\infty \|e_k(t)\|^2\,dt = \|\mathcal{T}\|_{\mathscr{H}_2}^2.$$

$\mathscr{H}_2$-Analysis by LMIs

If $\mathcal{A}$ is Hurwitz it is easy to see that

$$\mathcal{A}P_0 + P_0\mathcal{A}^T + \mathcal{B}\mathcal{B}^T = 0 \ \text{ implies } \ \operatorname{trace}(\mathcal{C}P_0\mathcal{C}^T) < \gamma^2$$

iff there exists some $\mathcal{P}$ with

$$\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^T + \mathcal{B}\mathcal{B}^T \prec 0 \ \text{ and } \ \operatorname{trace}(\mathcal{C}\mathcal{P}\mathcal{C}^T) < \gamma^2.$$

$\mathscr{H}_2$-Analysis by LMIs

If $\mathcal{A}$ is Hurwitz it is easy to see that

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$\mathcal{A}$ is Hurwitz and $\|\mathcal{T}\|_{\mathscr{H}_2} < \gamma$ iff $\mathcal{D} = 0$ and there exists a $\mathcal{P}$ with

$$\mathcal{P} \succ 0,\; \mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^T + \mathcal{B}\mathcal{B}^T \prec 0,\; \sum_{k=1}^{n_e} \mathcal{C}_k\mathcal{P}\mathcal{C}_k^T < \gamma^2.$$

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For a fixed controller these are LMIs in $\mathcal{P}$. However they are not in the correct format in order to apply our general procedure.

Schur and congruence allow to rearrange these for $\mathcal{X} = \gamma\mathcal{P}^{-1}$ into ...

General Procedure: Illustration for $\mathscr{H}_2$-Synthesis

... these equivalent versions:

$$\begin{pmatrix} \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \mathcal{B}^T\mathcal{X} & -\gamma I \end{pmatrix} \prec 0, \quad \begin{pmatrix} \gamma & \mathcal{C}_1 & \cdots & \mathcal{C}_{n_e} \\ \mathcal{C}_1^T & \mathcal{X} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_{n_e}^T & 0 & \cdots & \mathcal{X} \end{pmatrix} \succ 0.$$

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Formal congruence trafo with $\mathrm{diag}(\mathcal{Y}, I)$ and $\mathrm{diag}(1, \mathcal{Y}, \ldots, \mathcal{Y})$:

There exists a controller which renders $\mathcal{A}$ Hurwitz and closed-loop $\mathscr{H}_2$-norm smaller than γ iff exist v such that $\boldsymbol{D}(v) = 0$ and

$$\begin{pmatrix} \boldsymbol{A}(v)^T + \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{B}(v)^T & -\gamma I \end{pmatrix} \prec 0, \quad \begin{pmatrix} \gamma & \boldsymbol{C}_1(v) & \cdots & \boldsymbol{C}_{n_e}(v) \\ \boldsymbol{C}_1(v)^T & \boldsymbol{X}(v) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{C}_{n_e}(v)^T & 0 & \cdots & \boldsymbol{X}(v) \end{pmatrix} \succ 0.$$

Solution in Terms of Riccati Equations

Under the assumptions on slide 30 one can derive, in a similar fashion as we sketched for $\mathscr{H}_\infty$, the following classical $\mathscr{H}_2$-synthesis result.

Determine the (existing) stabilizing solutions of the Riccati equations

$$AP + PA^T + B_1B_1^T + PC^TCP = 0,$$

$$A^TQ + QA + C_1^TC_1 + QBB^TQ = 0.$$

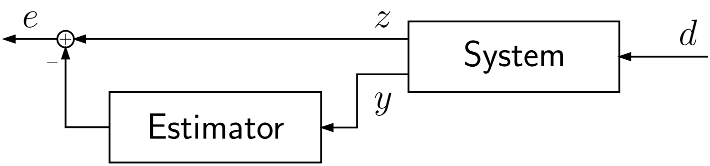
Then the unique optimal $\mathscr{H}_2$-controller is given by

$$\left[\begin{array}{c|c} A - B_2B_2^TQ - PC_2^TC_2 & PC_2^T \\ \hline -B_2^TQ & 0 \end{array}\right].$$

Doyle, Glover, Khargonekar, Francis (1989)

Estimation Problems

Interconnection for estimation:



All LMI results apply to **estimator** synthesis! It's just a special case!

- Find estimator which minimizes $\mathscr{H}_\infty$-norm of $w \to e$ ...

... render energy gain as small as possible.

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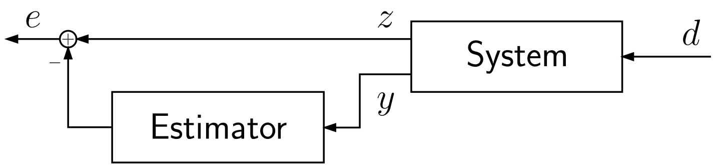


All LMI results apply to **estimator** synthesis! It's just a special case!

- Find estimator which minimizes $\mathscr{H}_\infty$-norm of $w \to e$ ...

  ... render energy gain as small as possible.

- Find estimator which minimizes $\mathscr{H}_2$-norm of $w \to e$ ...

  ... optimally reduce asymptotic variance against white noise.

Outline

- The $\mathscr{H}_\infty$-Norm
- The $\mathscr{H}_\infty$-Control Problem
- $\mathscr{H}_\infty$-Analysis and the Bounded Real Lemma
- $\mathscr{H}_\infty$-Synthesis with LMIs
- $\mathscr{H}_\infty$-Synthesis with Riccati Equations
- $\mathscr{H}_2$-Analysis and Synthesis
- Mixed Controller Synthesis

Multi-Objective Control

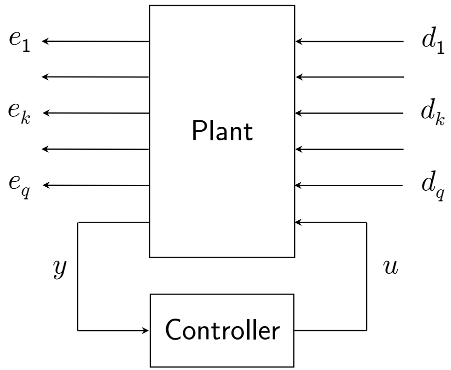
Design controller which achieves multiple objectives on different channels of closed-loop system:

- Loop-shaping:

$$\|\mathcal{T}_{d_1\to e_1}\|_{\mathscr{H}_\infty} < \gamma_1$$

- Disturbance attenuation:

$$\|\mathcal{T}_{d_2\to e_2}\|_{\mathscr{H}_2} < \gamma_2$$



No loss of generality: Relevant channels are $d_k \to e_k$, $k = 1, \ldots, q$.

Multi-Channel System Description

Open-loop system and controller:

$$\begin{pmatrix} \dot{x} \\ \hline e_1 \\ \vdots \\ e_q \\ \hline y \end{pmatrix} \left(\begin{array}{c|ccc|c} A & B_1 & \cdots & B_q & B \\ \hline C_1 & D_1 & \cdots & D_{1q} & E_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_q & D_{q1} & \cdots & D_q & E_q \\ \hline C & F_1 & \cdots & F_q & 0 \end{array}\right) \begin{pmatrix} x \\ \hline d_1 \\ \vdots \\ d_q \\ \hline u \end{pmatrix}, \quad \begin{pmatrix} \dot{x}_c \\ u \end{pmatrix} = \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix} \begin{pmatrix} x_c \\ y \end{pmatrix}.$$

Controlled closed-loop system:

$$\begin{pmatrix} \dot{\xi} \\ \hline e_1 \\ \vdots \\ e_q \end{pmatrix} = \left(\begin{array}{c|ccc} \mathcal{A} & \mathcal{B}_1 & \cdots & \mathcal{B}_q \\ \hline \mathcal{C}_1 & \mathcal{D}_1 & \cdots & \mathcal{D}_{1q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_q & \mathcal{D}_{q1} & \cdots & \mathcal{D}_q \end{array}\right) \begin{pmatrix} \xi \\ \hline d_1 \\ \vdots \\ d_q \end{pmatrix}.$$

Multi-Objective $\mathscr{H}_2/\mathscr{H}_\infty$-Control

Find controller such that $\mathcal{A}$ is Hurwitz and

$$\|\mathcal{C}_1(sI-\mathcal{A})^{-1}\mathcal{B}_1+\mathcal{D}_1\|_{\mathscr{H}_\infty}<\gamma_1,\quad \|\mathcal{C}_2(sI-\mathcal{A})^{-1}\mathcal{B}_2+\mathcal{D}_2\|_{\mathscr{H}_2}<\gamma_2.$$

Related analysis conditions: $\mathcal{D}_2=0$ and

$$\begin{pmatrix} \mathcal{A}^T\mathcal{X}_1+\mathcal{X}_1\mathcal{A} & \mathcal{X}_1\mathcal{B}_1 & \mathcal{C}_1^T \\ \mathcal{B}_1^T\mathcal{X}_1 & -\gamma_1 I & \mathcal{D}_1^T \\ \mathcal{C}_1 & \mathcal{D}_1 & -\gamma_1 I \end{pmatrix} \prec 0$$

$$\begin{pmatrix} \mathcal{A}^T\mathcal{X}_2+\mathcal{X}_2\mathcal{A} & \mathcal{X}_2\mathcal{B}_2 \\ \mathcal{B}_2^T\mathcal{X}_2 & -\gamma_2 I \end{pmatrix} \prec 0,\quad \begin{pmatrix} \gamma & (\mathcal{C}_2)_1 & \cdots \\ (\mathcal{C}_2)_1^T & \mathcal{X}_2 & \cdots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix} \succ 0.$$

In general need $\mathcal{X}_1\neq\mathcal{X}_2$. Untractable in state-space.

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In general need $\mathcal{X}_1\neq\mathcal{X}_2$. Untractable in state-space.

Relaxation: Introduce **extra constraint** $\mathcal{X}_1=\mathcal{X}_2$.

Mixed-Objective $\mathscr{H}_2/\mathscr{H}_\infty$-Control

Find controller such that $\mathcal{D}_2 = 0$ and that there exists $\mathcal{X}$ with

$$\begin{pmatrix} \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_1 & \mathcal{C}_1^T \\ \mathcal{B}_1^T\mathcal{X} & -\gamma_1 I & \mathcal{D}_1^T \\ \mathcal{C}_1 & \mathcal{D}_1 & -\gamma_1 I \end{pmatrix} \prec 0$$

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Solvability of mixed problem **implies** stability of $\mathcal{A}$ and the desired norm inequalities. Can hence conclude in general that

$$\text{Minimal mixed } \gamma_2 \quad \geq \quad \text{Minimal multi-objective } \gamma_2.$$

$\mathcal{X}_1 = \mathcal{X}_2$ often implies that there is a gap and the inequality is strict.

Solution of Mixed $\mathscr{H}_2/\mathscr{H}_\infty$-Control

But $\mathcal{X}_1 = \mathcal{X}_2$ implies tractability: Can apply general procedure!

Mixed synthesis conditions: $\boldsymbol{D}_2(v) = 0$ and

$$\begin{pmatrix} \boldsymbol{A}(v)^T + \boldsymbol{A}(v) & \boldsymbol{B}_1(v) & \boldsymbol{C}_1(v)^T \\ \boldsymbol{B}_1(v)^T & -\gamma_1 I & \boldsymbol{D}_1(v)^T \\ \boldsymbol{C}_1(v) & \boldsymbol{D}_1(v) & -\gamma_1 I \end{pmatrix} \prec 0$$

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Can be solved by standard algorithms ...

... controller construction as usual ...

... controller order **identical** to order of system!

Carsten Scherer

Extensions

- For fixed α_1, α_2 and variable γ_1, γ_2, optimize $\alpha_1\gamma_1 + \alpha_2\gamma_2$.

  Analyze trade-off between specifications by playing with α_1, α_2.

- **Improve relaxation** with tuning parameter $\alpha > 0$: $\mathcal{X}_1 = \alpha\mathcal{X}_2$.

  Line-search over α. Might reduce conservatism.

- Can include more than two LMI performance on different channels.

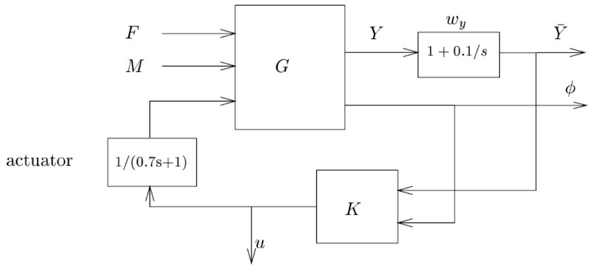
  Never forget conservatism.

- Possible to include other type of constraints.

  Important example: Closed-loop poles in **convex** LMI region.

Example: Floating Platform

With actuator dynamics we use the following interconnection structure:

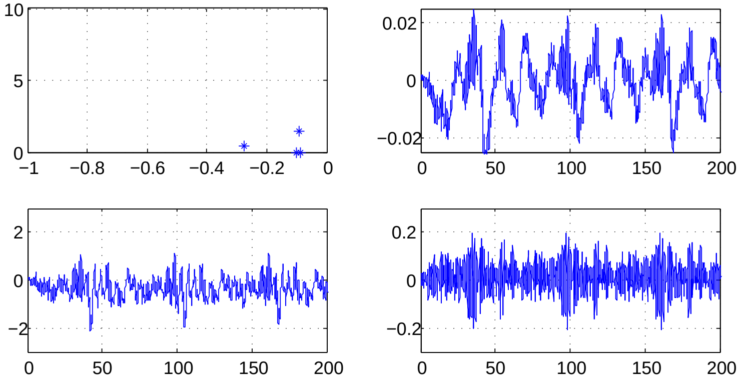


- Keep $|Y(t)|$ below $2.5cm$ and $|\phi(t)|$ below 3^o.
- Thruster actuation $|u(t)|$ should stay below 0.3.
- Push resonance peak of $M \to \phi$ down below 1.5.

Example: Mixed Design

$\mathscr{H}_\infty$-bound 0.8: $\begin{pmatrix} F \\ M \end{pmatrix} \to \begin{pmatrix} \bar{Y} \\ 0.1\,\phi \end{pmatrix}$. $\mathscr{H}_2$-minimization: $\begin{pmatrix} F \\ M \end{pmatrix} \to u$.

Closed-loop poles and time-domain specifications:



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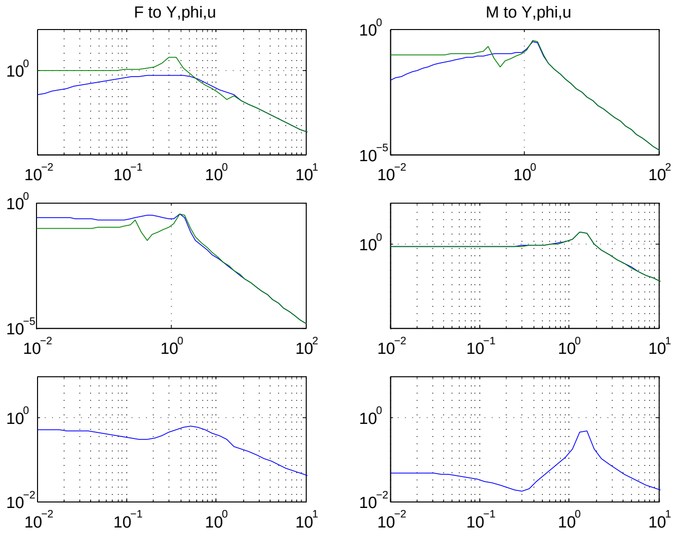
Frequency domain-domain characteristics:



TU Delft

Reminder: Eigenvalues in LMI Region

All eigenvalues of $A \in \mathbb{R}^{n\times n}$ are contained in the LMI-region

$$\left\{ z \in \mathbb{C} : \begin{pmatrix} I \\ zI \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} I \\ zI \end{pmatrix} \prec 0 \right\}$$

if and only if there exists a $K \succ 0$ such that

$$\begin{pmatrix} I \\ A \otimes I \end{pmatrix}^T \begin{pmatrix} K \otimes Q & K \otimes S \\ K \otimes S^T & K \otimes R \end{pmatrix} \begin{pmatrix} I \\ A \otimes I \end{pmatrix} \prec 0.$$

Beautiful generalization of standard stability test!

Gahinet, Chilali (1996)

Closed-loop Poles in Convex LMI-Region

Eigenvalues of $\mathcal{A}$ in LMI-region defined by Q, R, S iff exists $\mathcal{X}$ with

$$\mathcal{X} \succ 0, \quad \begin{pmatrix} I \\ \mathcal{A} \otimes I \end{pmatrix}^T \begin{pmatrix} \mathcal{X} \otimes Q & \mathcal{X} \otimes S \\ \mathcal{X} \otimes S^T & \mathcal{X} \otimes R \end{pmatrix} \begin{pmatrix} I \\ \mathcal{A} \otimes I \end{pmatrix} \prec 0$$

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or equivalently

$$\mathcal{X} \succ 0, \quad \mathcal{X} \otimes Q + (\mathcal{X}\mathcal{A}) \otimes S + (\mathcal{A}^T \mathcal{X}) \otimes S^T + (\mathcal{A}^T \mathcal{X} \mathcal{A}) \otimes R \prec 0.$$

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LMIs equivalent to (Schur and properties of Kronecker product):

$$\begin{pmatrix} \mathcal{X} \otimes Q + (\mathcal{X}\mathcal{A}) \otimes S + (\mathcal{A}^T \mathcal{X}) \otimes S^T & (\mathcal{A}^T \mathcal{X}) \otimes T \\ (\mathcal{X}\mathcal{A}) \otimes T^T & -\mathcal{X} \otimes I \end{pmatrix} \prec 0.$$

Formal congruence trafo with $\operatorname{diag}(\mathcal{Y} \otimes I, \mathcal{Y} \otimes I)$. Done!

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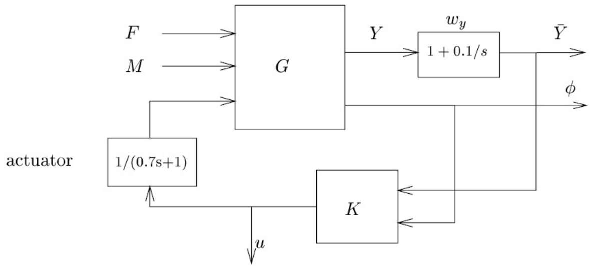

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Example: Mixed + Pole-Placement

Reduce resonance by pushing resonating pole away from axis.

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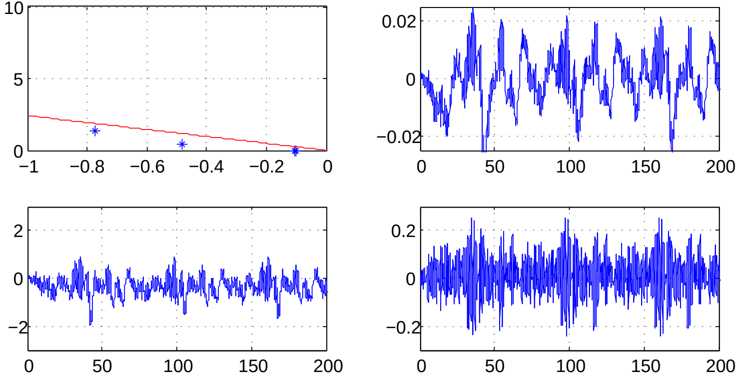

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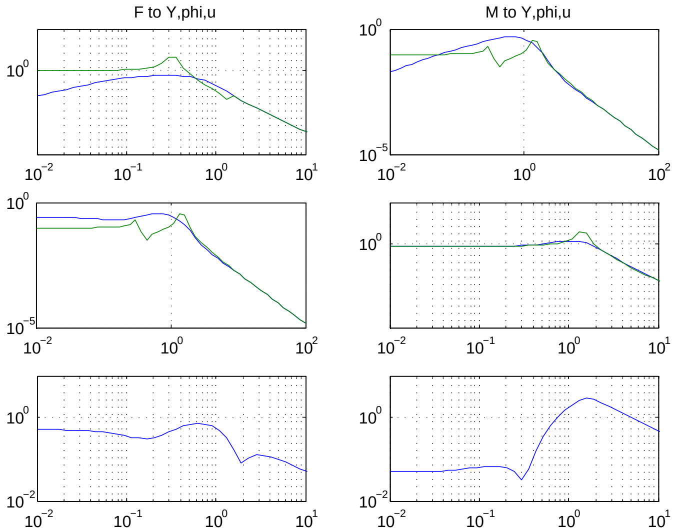

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