Summer Course

Linear System Theory

Control

&

Matrix Computations

Monopoli, Italy **September 8-12, 2008**

Lecture 1

Models and Behaviors

Lecturer: Jan C. Willems

Outline

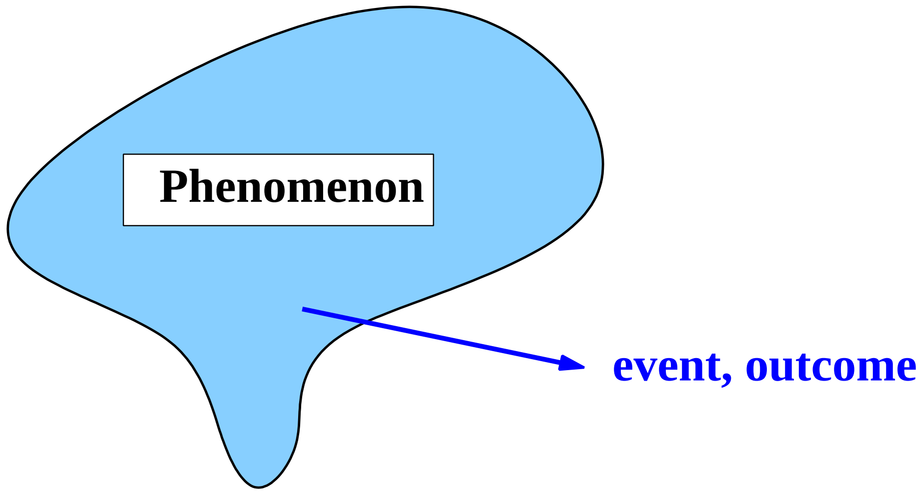
- Mathematical models
- The behavior
- Dynamical systems
- Linear time-invariant systems
- Kernel representations
- Latent variables
- The elimination theorem

Mathematical models

A bit of mathematics & philosophy

Mathematical models

Assume that we have a ‘real’ phenomenon that produces *‘events’*, *‘outcomes’*.



Mathematical models

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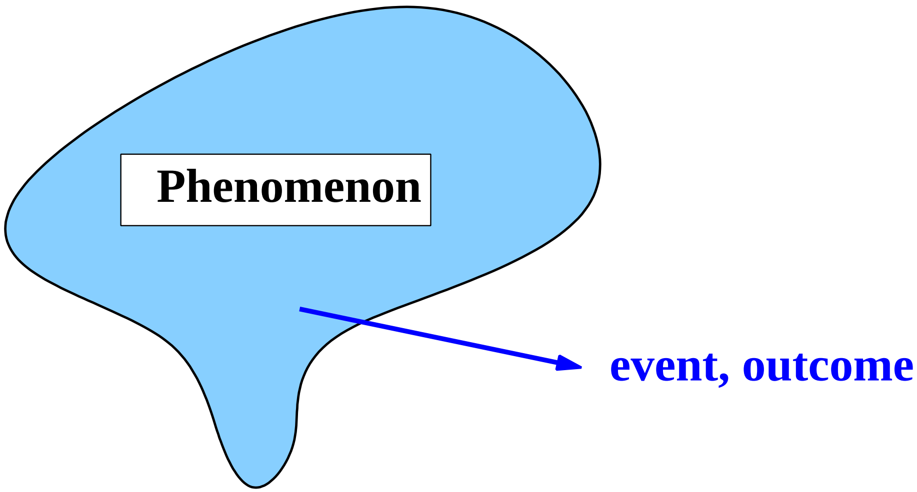

We view a deterministic mathematical model for a phenomenon as a prescription of which events can occur, and which events cannot occur.

Aim of this lecture

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- **In the second part, we apply this formalism to dynamical systems, especially to linear time-invariant differential systems.**

The universsum

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- **Do the events belong to a discrete set?**
  $\rightsquigarrow$ ***discrete event phenomena.***
- **Are the events real numbers, or vectors of real numbers?**
  $\rightsquigarrow$ ***continuous phenomena.***
- **Are the events functions of time?**
  $\rightsquigarrow$ ***dynamical phenomena.***
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The set where the events belong to is called the universum, denoted by $\mathscr{U}$.

Discrete event phenomena

Examples:

- **Words in a natural language**
  $\mathscr{U} = \{a, b, c, \ldots, x, y, z\}^{\mathtt{n}}$
  with $\mathtt{n} =$ **the number of letters in the longest word**

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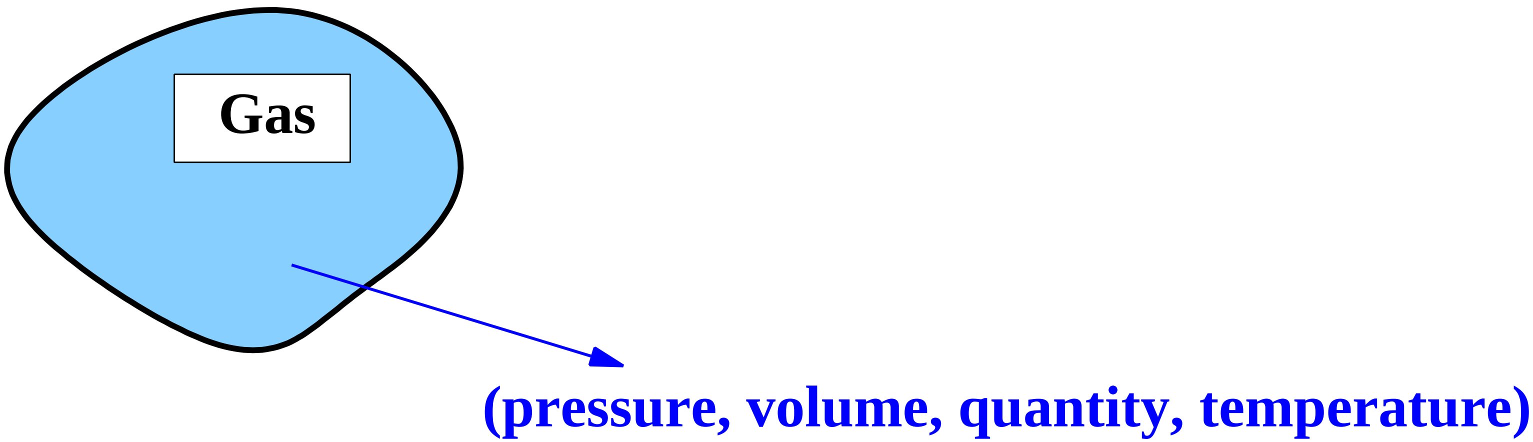
  Sentences in a natural language
- **DNA sequences**
- **Fortran code**
  LaTeX code
- **Error detecting and correcting codes**
  ISBN numbers

Continuous phenomena

Examples:

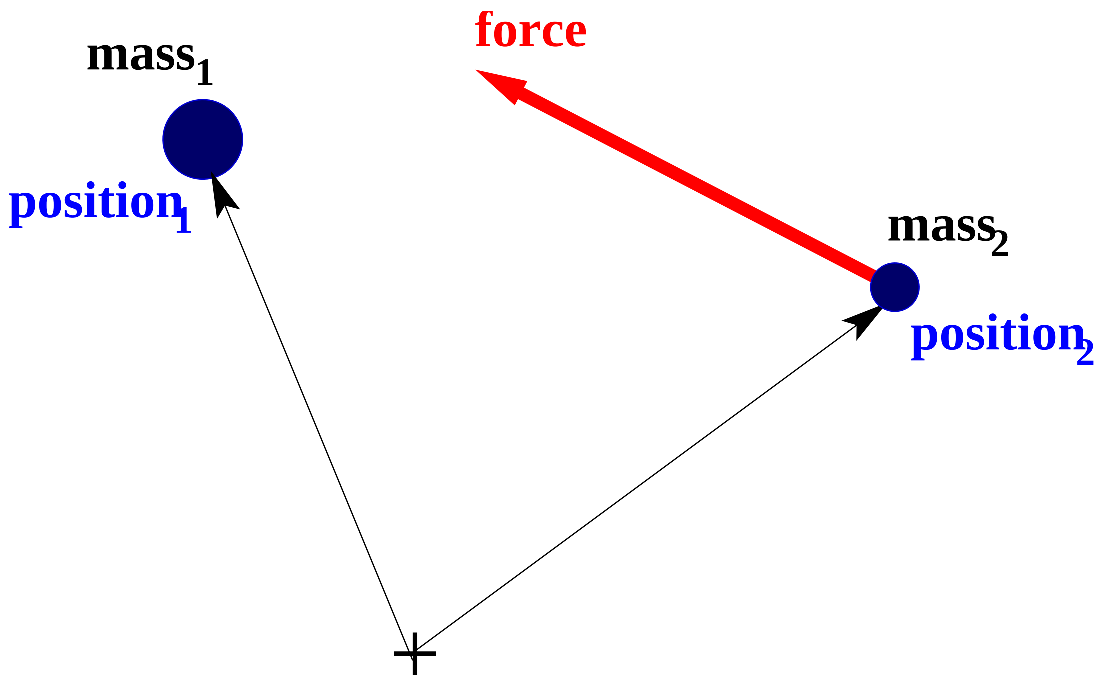
- **The pressure, volume, quantity, and temperature of a gas in a vessel**



$$\rightsquigarrow \quad \mathscr{U} = (0,\infty) \times (0,\infty) \times (0,\infty) \times (0,\infty)$$

Continuous phenomena

- **The gravitational attraction of two bodies**

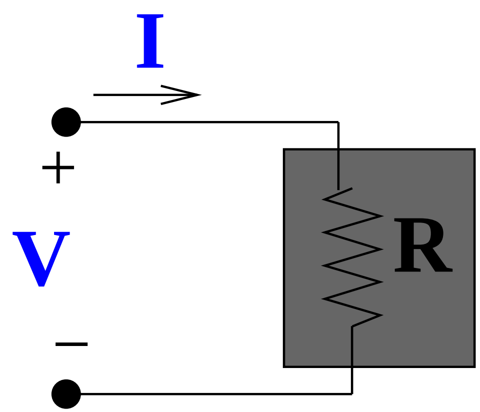


Event = mass$_1$, mass$_2$, position$_1$, position$_2$, force

$$\rightsquigarrow \quad \mathscr{U} = (0,\infty) \times (0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$$

Continuous phenomena

- **The voltage across and the current through a resistor**



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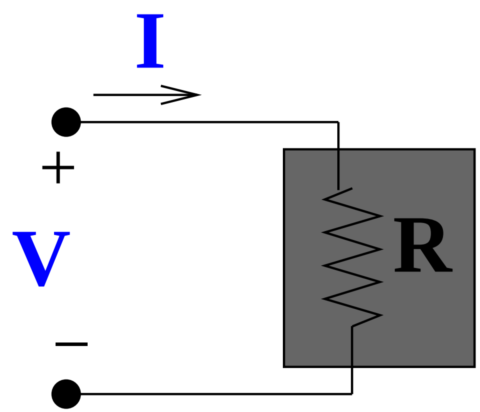

Event = (voltage, current) $\rightsquigarrow$ $\mathscr{U} = \mathbb{R}^2$

- **Price/demand** $\rightsquigarrow$ $\mathscr{U} = [0, \infty) \times [0, \infty)$
  Price/supply $\rightsquigarrow$ $\mathscr{U} = [0, \infty) \times [0, \infty)$
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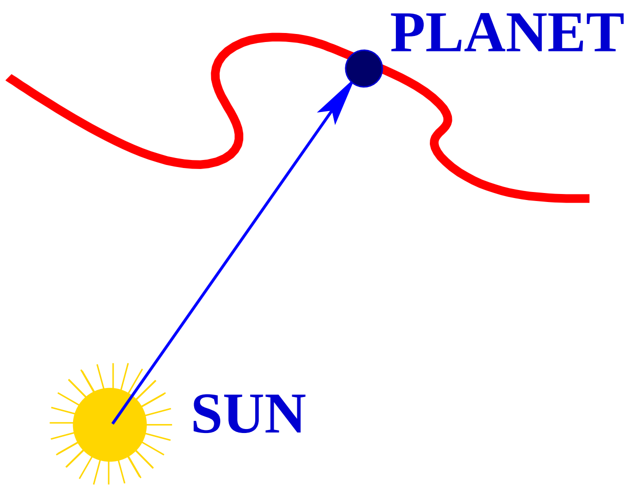
Dynamical phenomena

Dynamical phenomena $\rightsquigarrow$ this course.

Examples:

- **Planetary motion**



The events are maps from $\mathbb{R}$ to $\mathbb{R}^3$

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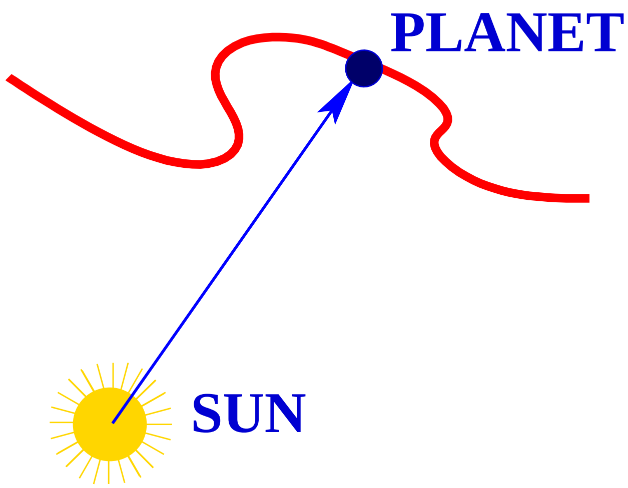

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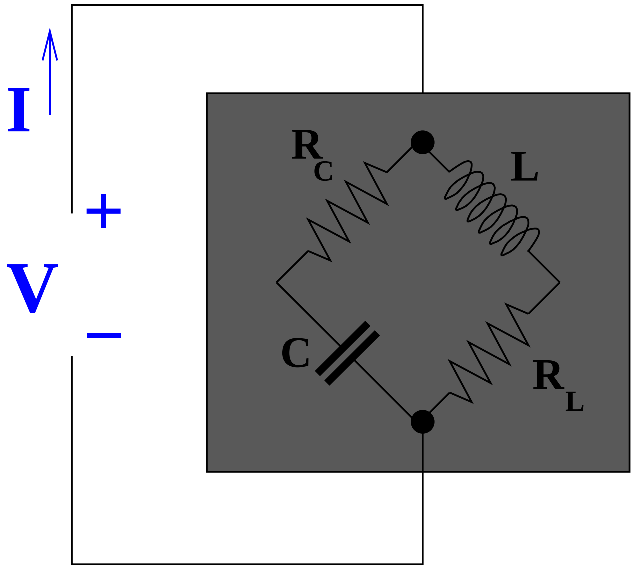
$$\rightsquigarrow \qquad \mathscr{U} = \{w : \mathbb{R} \to \mathbb{R}^3\} \quad = \left(\mathbb{R}^3\right)^{\mathbb{R}}$$

$A^B :=$ the set of maps from A to $B = \{f : A \to B\}$

Dynamical phenomena

- The voltage across and the current into an electrical port with ‘dynamics’



The events are maps from $\mathbb{R}$ to $\mathbb{R}^2$

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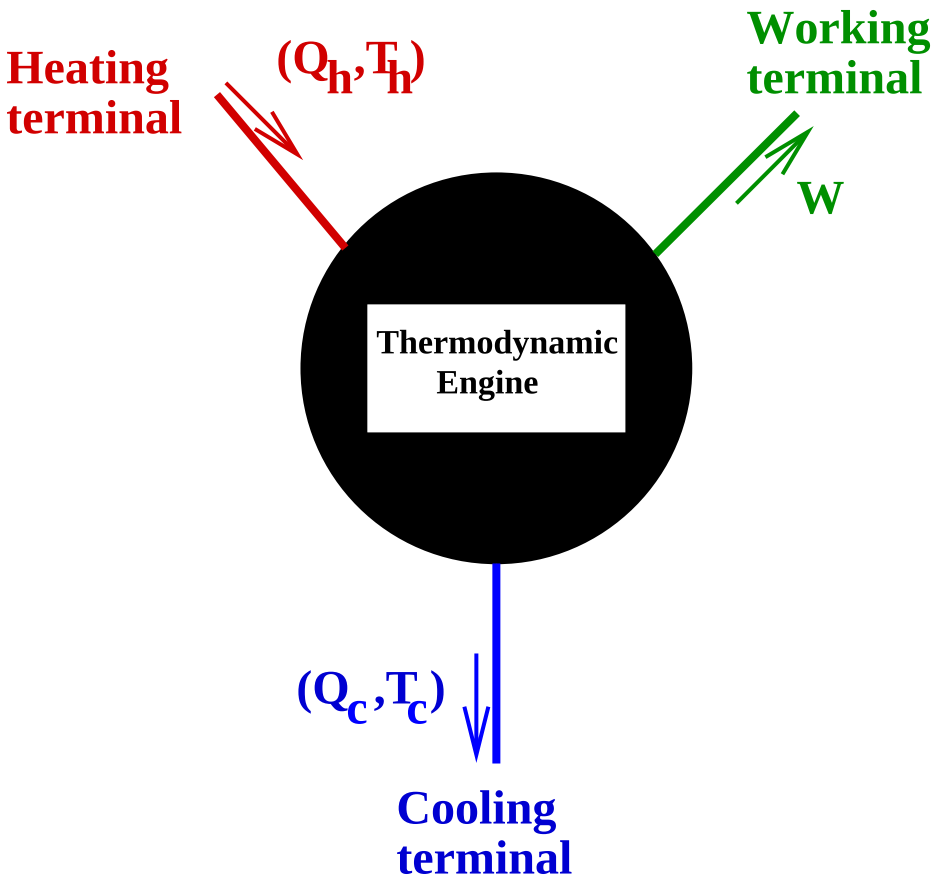
Dynamical phenomena

- **Heat flows, temperatures, and work in a thermodynamic system**

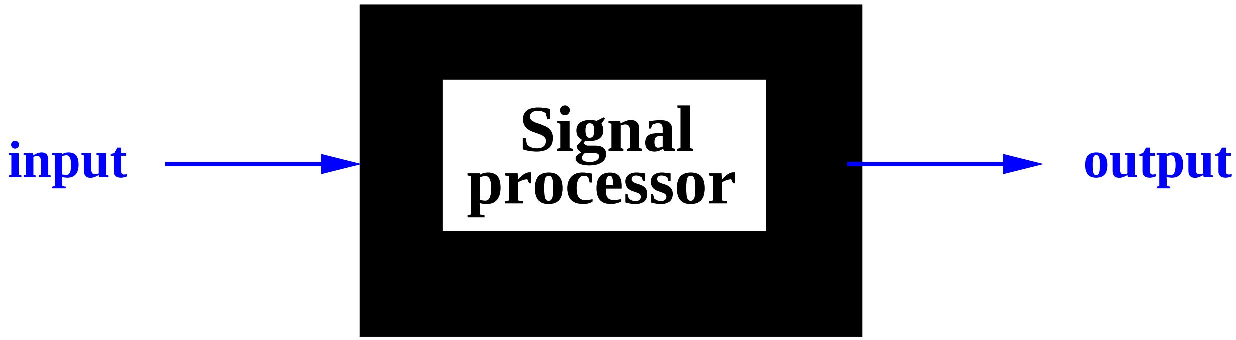


Events: maps from $\mathbb{R}$ **to** $[0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty) \times \mathbb{R}$

$$\rightsquigarrow \qquad \mathscr{U} = \{(Q_h, T_h, Q_c, T_c, W) : \mathbb{R} \to \cdots\}$$

Dynamical phenomena

- **The input and the output of a signal processor**



Events: maps from $\mathbb{Z}$ **to** $\mathbb{R} \times \mathbb{R}$

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Dynamical phenomena

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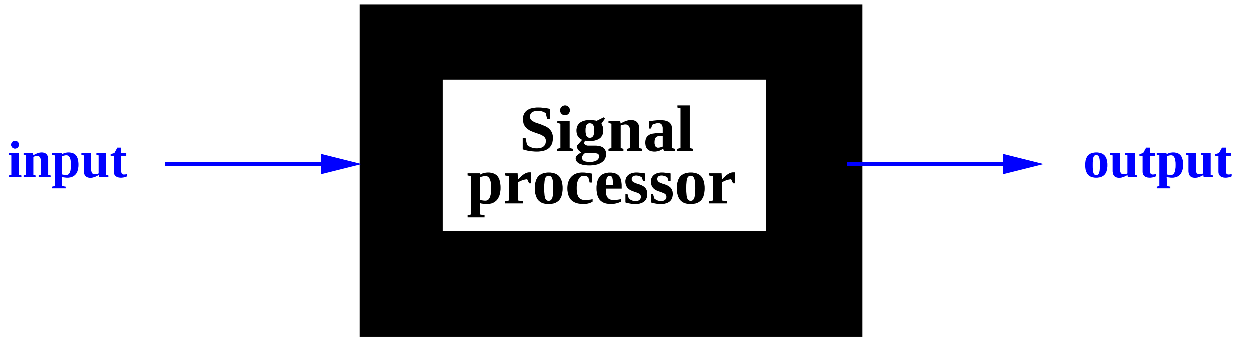

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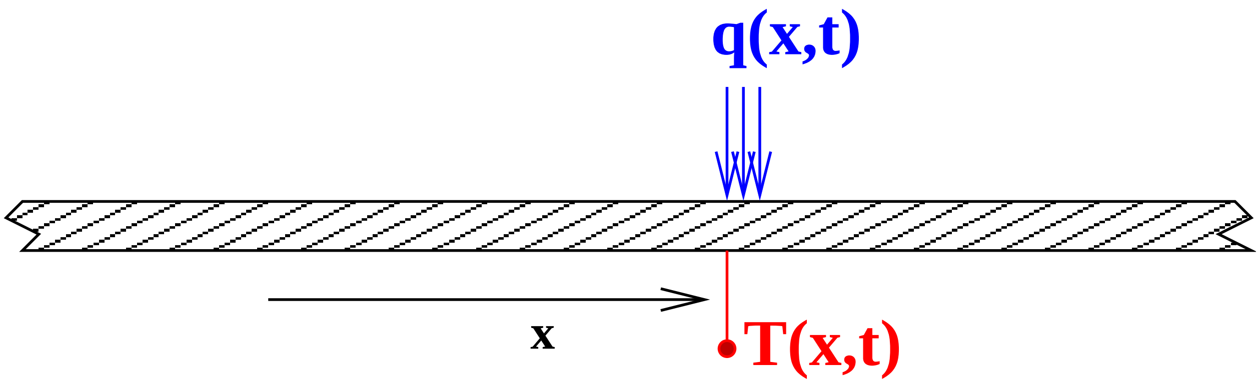
- **Variables associated with mechanical devices, electrical instruments, chemical systems, multi-domain constructs, economic processes, ... Phenomena with ‘memory’.**

Distributed phenomena

- Temperature profile of, and heat absorbed by, a rod



Events: maps from $\mathbb{R} \times \mathbb{R}$ **to** $[0,\infty) \times \mathbb{R}$

$$\rightsquigarrow \qquad \mathscr{U} = \{(T,q) : \mathbb{R}^2 \to [0,\infty) \times \mathbb{R}\}$$

Distributed phenomena

- **EM fields.**

  In each point of space & at each time, there is an

electric field	$\vec{E}(t,x,y,z)$
magnetic field	$\vec{B}(t,x,y,z)$
current density	$\vec{j}(t,x,y,z)$
charge density	$\rho(t,x,y,z)$

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$$\rightsquigarrow \qquad \mathscr{U} = \{(\vec{E},\vec{B},\vec{j},\rho) : \mathbb{R}^4 \to \mathbb{R}^{10}\}$$

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- **Images**
- **Phenomena in which things happen simultaneously at different points in space**

A model is a subset: the ‘behavior’

The behavior

Given is a phenomenon with universum $\mathscr{U}$.
Without further scrutiny, any event in $\mathscr{U}$ is possible.

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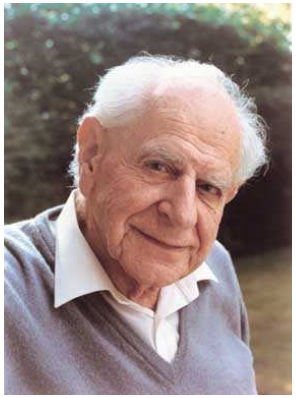
A model is a subset $\mathscr{B}$ of $\mathscr{U}$

$\mathscr{B}$ is called *the behavior* of the model

The behavior

Every "good" scientific theory is prohibition:
it forbids certain things to happen...
The more a theory forbids, the better it is.

Karl Popper
Conjectures and Refutations:
The Growth of Scientific Knowledge
Routhledge, 1963



Karl Popper
(1902-1994)

Examples

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  For example, SPQR $\notin \mathscr{B}$**.**

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  Sentences in a natural language.
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- **DNA sequences.** $\mathscr{B} =$???
- **LaTeX code.** $\mathscr{B} =$ **all LaTeX files that 'run'.**

Discrete event phenomena

- **32-bit binary strings with a parity check.**

$$\mathscr{U} = \{0,1\}^{32}$$

$$\mathscr{B} = \left\{ a_1 a_2 \cdots a_{31} a_{32} \mid a_k \in \{0,1\} \textbf{ and } a_{32} \overset{(\mathrm{mod}\ 2)}{=} \sum_{k=1}^{31} a_k \right\}$$

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$\mathscr{B}$ **can be expressed in other ways. For example,**

$$\mathscr{B} = \{ a_1 a_2 \cdots a_{31} a_{32} \mid a_k \in \{0,1\} \text{ and } \sum_{k=1}^{32} a_k \overset{(\text{mod } 2)}{=} 0 \}$$

$$\mathscr{B} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{31} \\ a_{32} \end{bmatrix} \mid \exists \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{30} \\ b_{31} \end{bmatrix} \text{ s.t. } \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{31} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & \cdots & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{30} \\ b_{31} \end{bmatrix} \right\}$$

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input/output representation

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kernel representation

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image representation

Examples:

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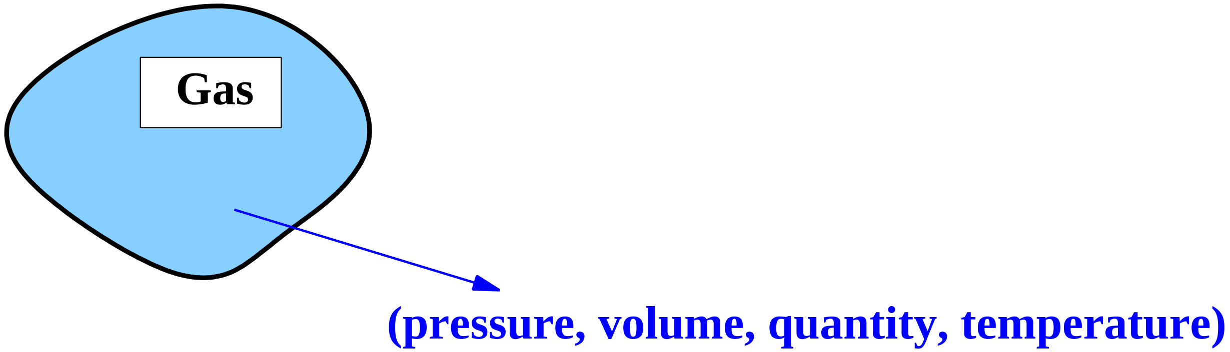
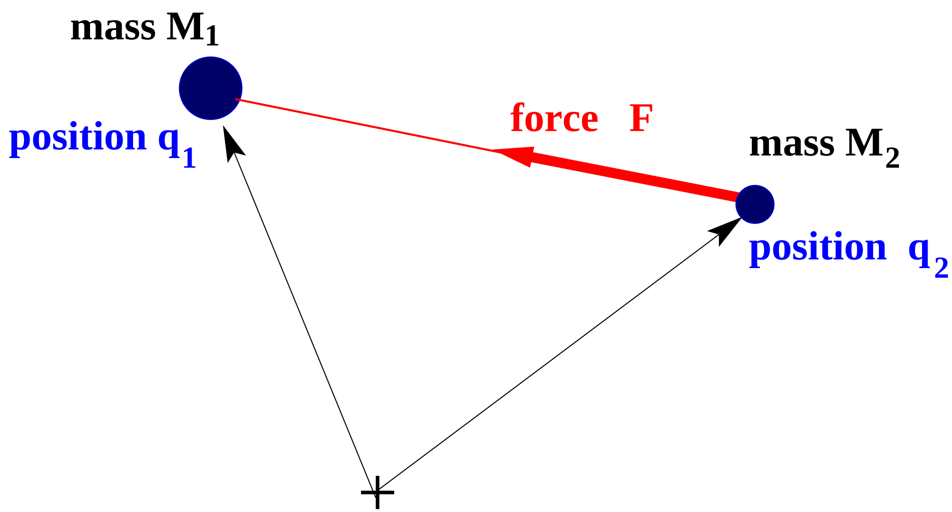


$$\mathscr{U} = (0,\infty) \times (0,\infty) \times (0,\infty) \times (0,\infty)$$

Gas law: $\mathscr{B} = \{(P,V,N,T) \in \mathscr{U} \mid PV = NT\}$

Continuous phenomena

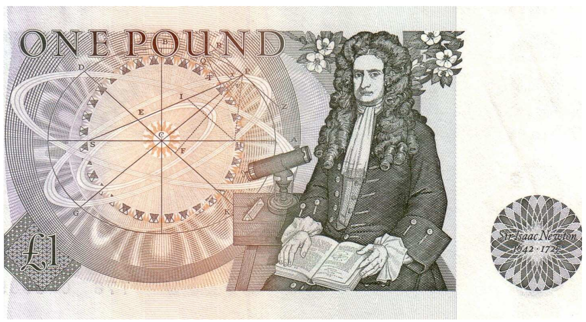
- **The gravitational attraction of two bodies**



$$\rightsquigarrow \quad \mathscr{U} = (0,\infty) \times (0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$$

$$\mathscr{B} = \left\{ \vec{F} = \frac{M_1 M_2 \vec{1}_{M_2 \to M_1}}{\|\vec{q}_1 - \vec{q}_2\|^2} \right\}$$

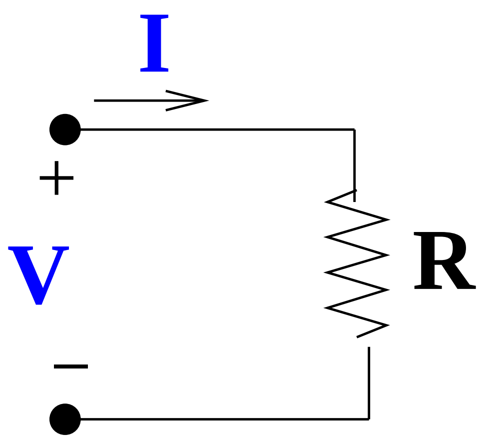
'inverse square law'



Isaac Newton, 1642-1727

Continuous phenomena

- **The voltage across and the current through a resistor**



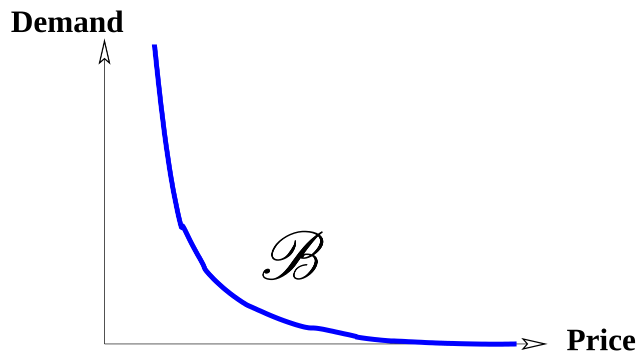
Event = (voltage, current) $\rightsquigarrow \quad \mathscr{U} = \mathbb{R}^2$

'Ohm's law' $\quad \mathscr{B} = \{(V, I) \mid V = RI\}$

Georg Ohm, 1789 – 1854

Continuous phenomena

- **Price/demand:**
  $\mathscr{U} = [0,\infty) \times [0,\infty)$



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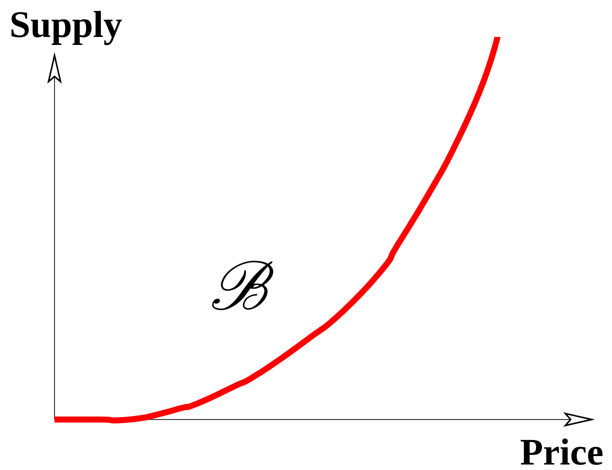

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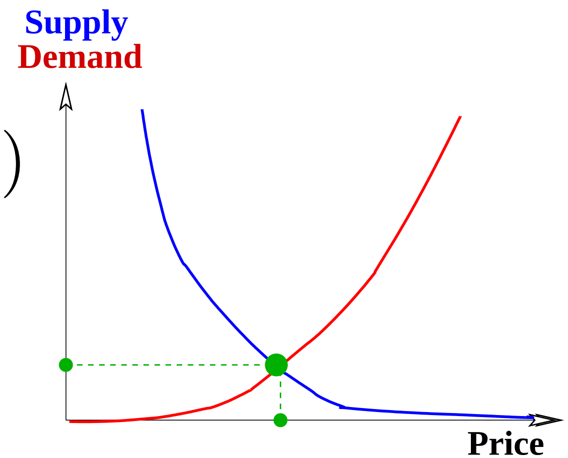

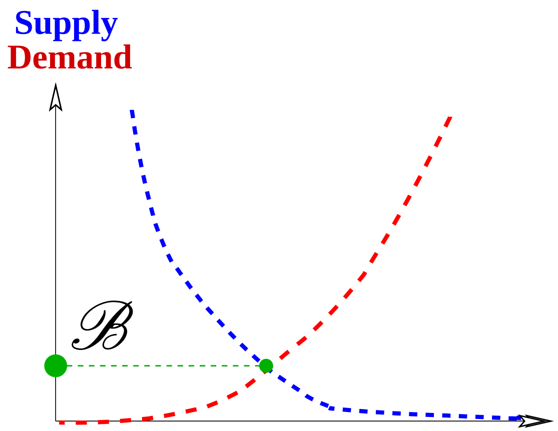

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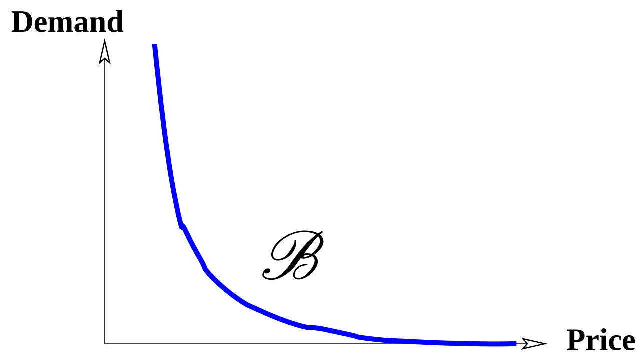

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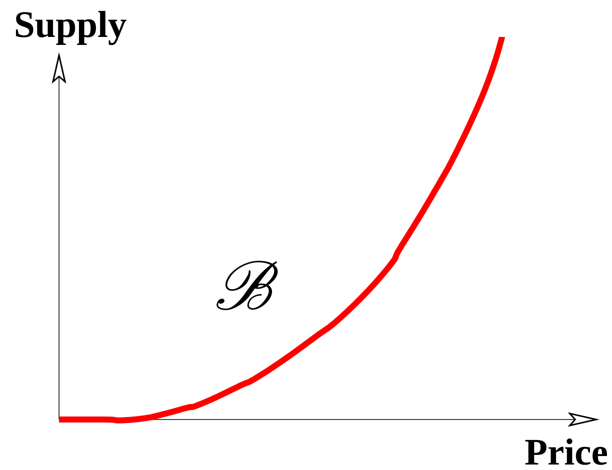

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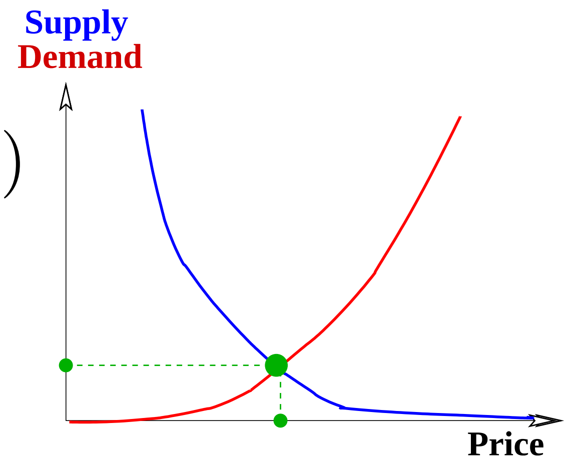
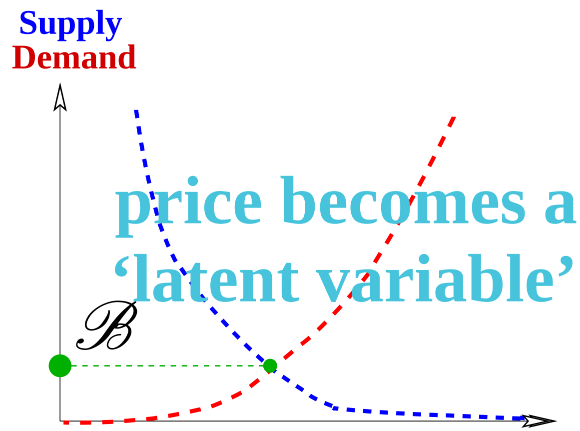
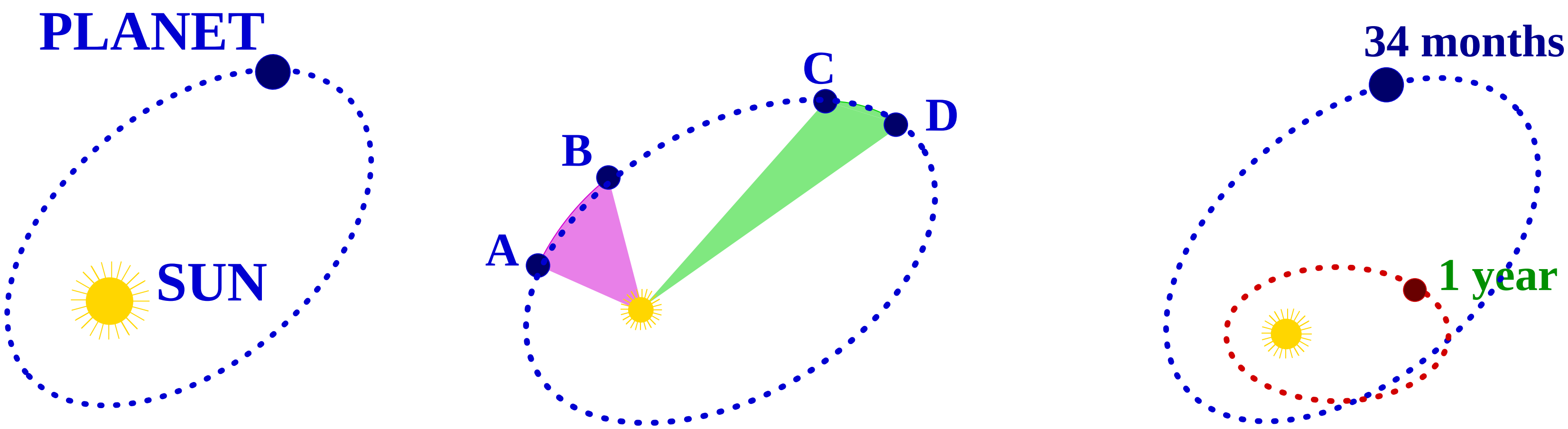




Dynamical phenomena

- **Planetary motion** $\mathscr{U} = \left(\mathbb{R}^3\right)^{\mathbb{R}}$



Kepler's laws $\rightsquigarrow \mathscr{B}$

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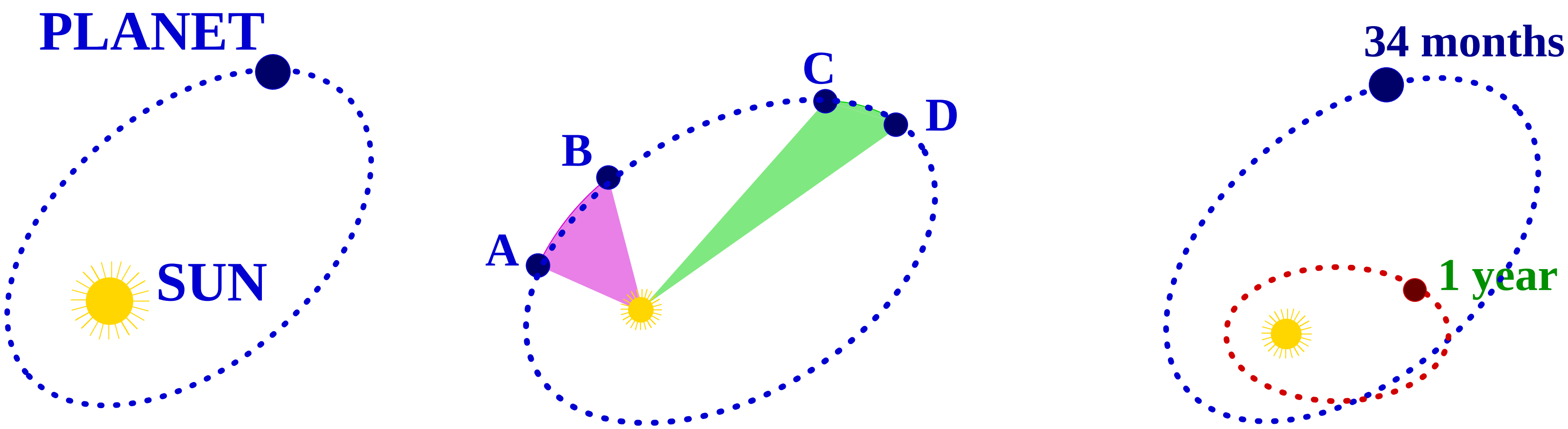


Kepler's laws $\rightsquigarrow$ $\mathscr{B}$ = the orbits $\mathbb{R} \to \mathbb{R}^3$ with:

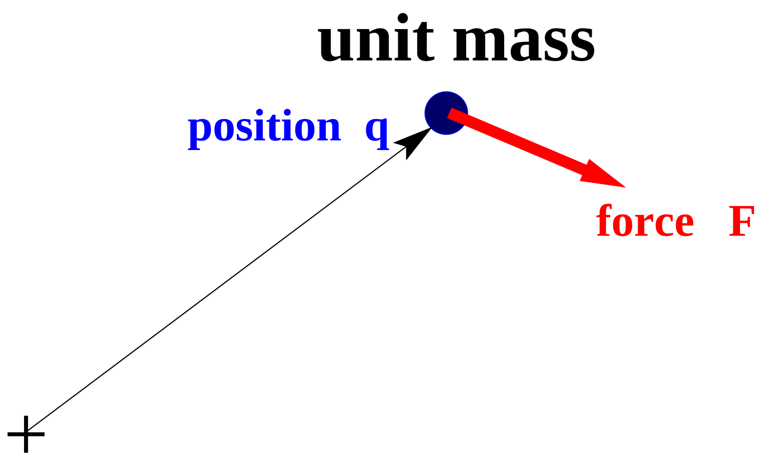
K.1 **periodic, ellipses, with the sun in one of the foci;**

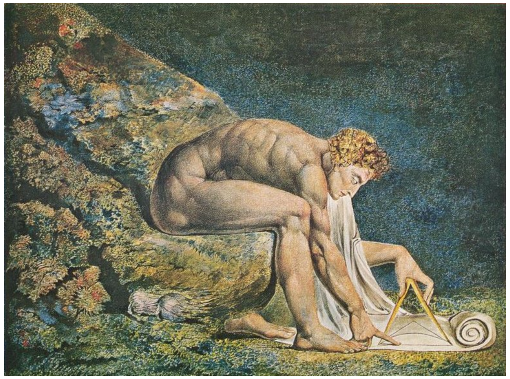
K.2 **the vector from sun to planet sweeps out equal areas in equal time;**

K.3 **the square of the period divided by the third power of the major axis is the same for all the planets**

Dynamical phenomena

- **The second law**





Isaac Newton
by William Blake

$$\mathscr{U} = \left(\mathbb{R}^3 \times \mathbb{R}^3\right)^{\mathbb{R}}$$

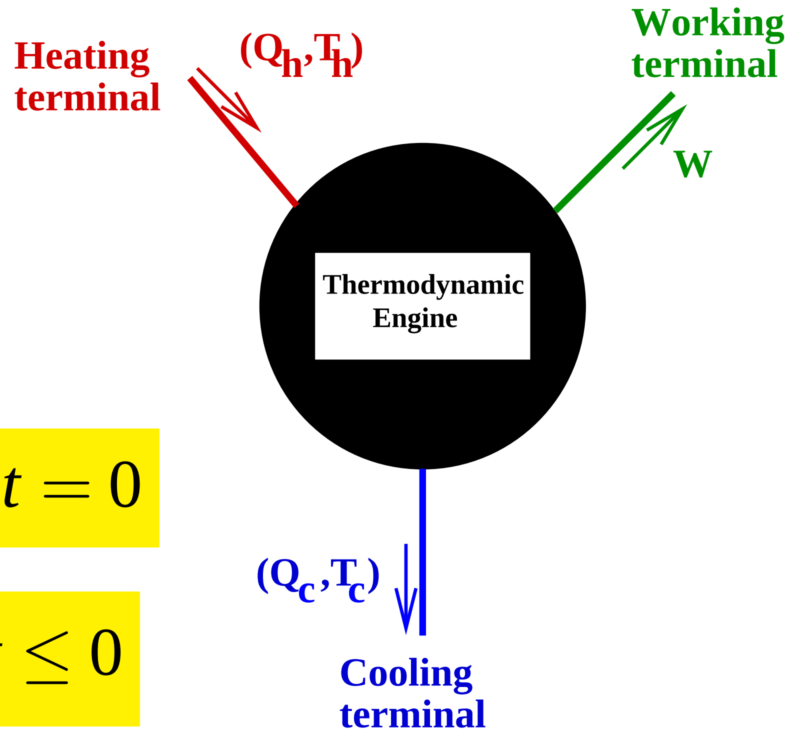
$$\mathscr{B} = \left\{ (F, q) : \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}^3 \mid F = \frac{d^2}{dt^2} q \right\}$$

Dynamical phenomena

- **Heat flows, temperatures, and work**

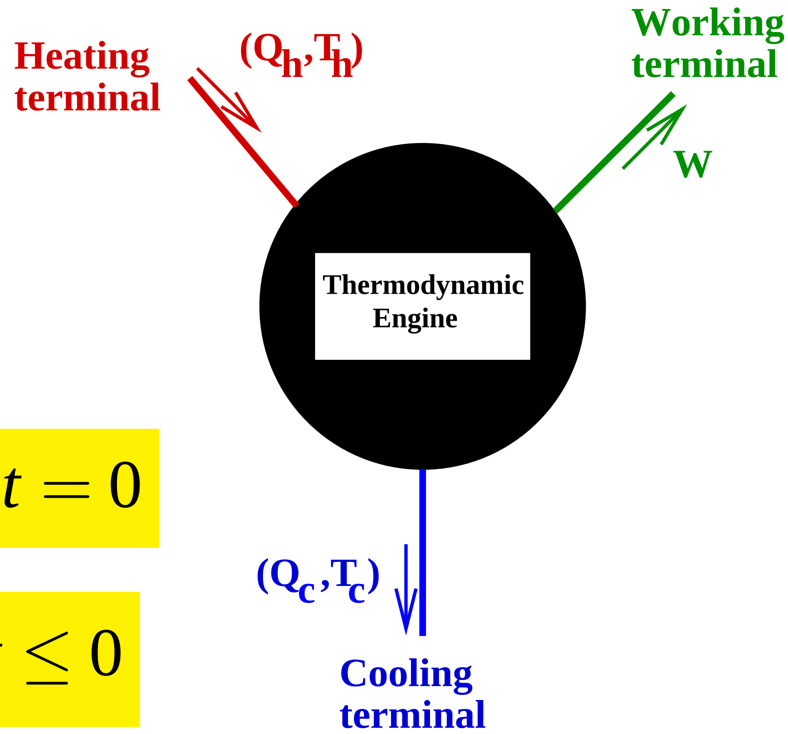
$$\mathscr{B} \rightsquigarrow \int_{-\infty}^{+\infty}(Q_h - Q_c - W)dt = 0$$

and $$\int_{-\infty}^{+\infty}\left(\frac{Q_h}{T_h} - \frac{Q_c}{T_c}\right)dt \leq 0$$



Dynamical phenomena

- **Heat flows, temperatures, and work**



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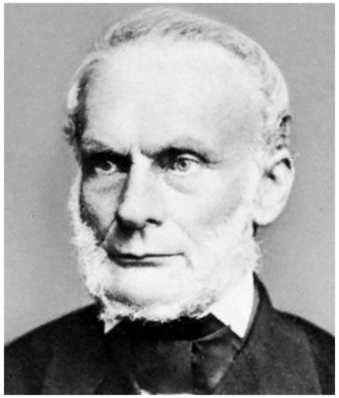
First and second law of thermodynamics

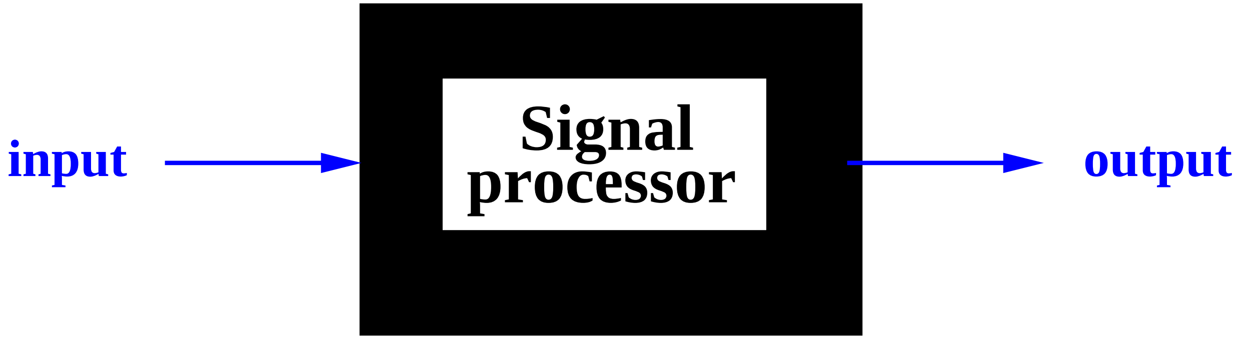
Émilie du Châtelet
1706 – 1749



Rudolf Clausius
1822 – 1888

Dynamical phenomena

- **The input and the output of a signal processor**



Events: maps from $\mathbb{Z}$ **to** $\mathbb{R} \times \mathbb{R} \rightsquigarrow \mathscr{U} = \{(u, y) : \mathbb{Z} \to \mathbb{R}^2\}$

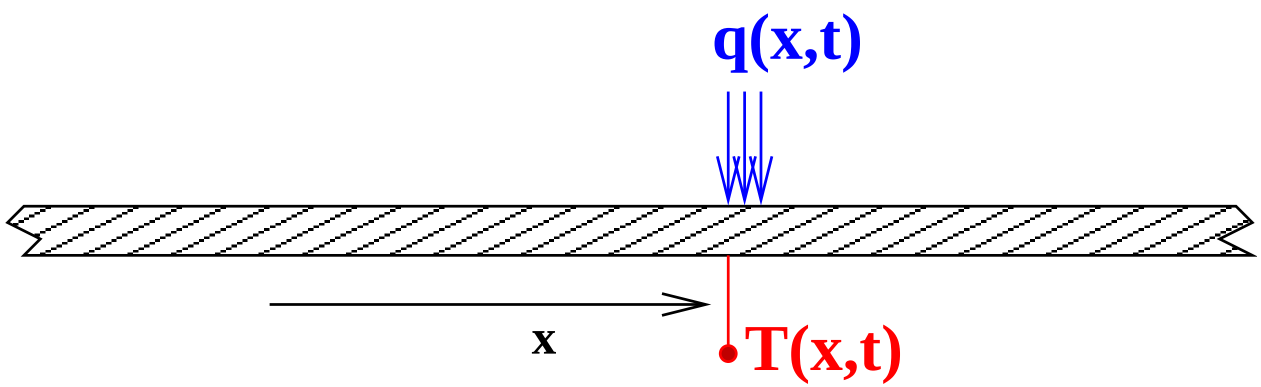
For an MA system

$$\mathscr{B} = \left\{ (u, y) : \mathbb{Z} \to \mathbb{R}^2 \mid y(t) = \frac{1}{2T+1} \sum_{t'=t-T}^{t+T} u(t') \right\}$$

many variations

Distributed phenomena

- **The temperature profile of, and heat absorbed by, a rod**



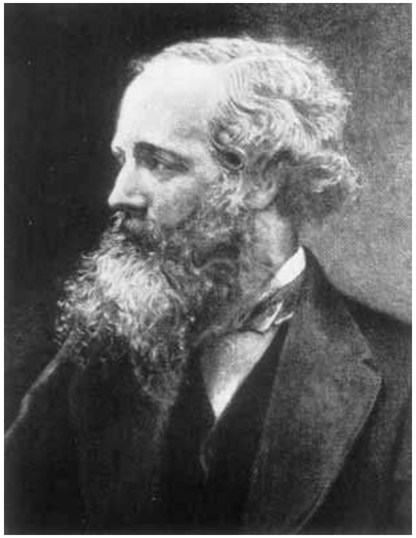
Events: maps from $\mathbb{R} \times \mathbb{R}$ **to** $[0, \infty) \times \mathbb{R}$

$$\mathscr{U} = \{(T, q) : \mathbb{R}^2 \to [0, \infty) \times \mathbb{R}\}$$

$$\mathscr{B} = \left\{(T, q) : \mathbb{R}^2 \to [0, \infty) \times \mathbb{R} \mid \frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q\right\}$$

Distributed phenomena

- **Maxwell's equations for EM fields in free space**



James Clerk Maxwell
1831 – 1879

$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\ , \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\ , \\
c^2\nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

independent variables: (t, x, y, z) **time and space**
dependent variables: $(\vec{E}, \vec{B}, \vec{j}, \rho)$
electric & magnetic field, current & charge density

Stochastic and Fuzzy

Stochastic models

In this lecture, we consider only deterministic models.

Stochastic models : $\Leftrightarrow$ there is a map P (the *'probability'*)

$$P : \mathscr{A} \rightarrow [0,1]$$

with $\mathscr{A}$ a 'σ-algebra'
of subsets of $\mathscr{U}$
& certain axioms on $\mathscr{A}$ and P.

Pierre-Simon Laplace

Andrey Kolmogorov
1903 – 1989

$P(\mathscr{B})$ ='the degree of certainty' (relative frequency, propensity, plausibility, belief) that outcomes (elements from $\mathscr{U}$) are in $\mathscr{B}$; $\cong$ ***'the degree of validity of $\mathscr{B}$ as a model'*.**

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Determinism : P **is a** $\{0,1\}$**-law**

$$\mathscr{A} = \{\varnothing, \mathscr{B}, \mathscr{B}^{\text{complement}}, \mathscr{U}\}, P(\mathscr{B}) = 1.$$

Fuzzy models

Lotfi Zadeh
born 1921

Fuzzy models: **there is a map** μ **(‘*the membership function*’)**

$$\mu : \mathscr{U} \to [0,1]$$

$\mu(x) =$ ***‘the extent to which*** x ***belongs to the model’s behavior’*.**

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$\mu(x) =$ ***‘the extent to which x belongs to the model’s behavior’*.**

Determinism: μ **is ‘*crisp*’:**

$$\text{image}(\mu) = \{0,1\},$$

$$\mathscr{B} = \mu^{-1}(\{1\}) := \{x \in \mathscr{U} \mid \mu(x) = 1\}$$

Behavioral models

Behavioral models **fit the tradition of modeling, but have not been approached as such in a deterministic setting. The behavior captures the essence of a model.**

The behavior is all there is.
Equivalence of models, properties of models,
symmetry, system identification, etc.,
must all refer to the behavior.

Every 'good' scientific theory is prohibition: it forbids certain things to happen...
The more a theory forbids, the better it is.

Replace 'scientific theory' by 'mathematical model' !

Recapitulation

- A model deals with events
- The events belong to an universum, $\mathscr{U}$
- A model is specified by its behavior $\mathscr{B}$, a subset of the event set $\mathscr{U}$
- In dynamical systems, the events are functions of time and the behavior $\mathscr{B}$ is hence a family of time-trajectories.

Dynamical systems

The dynamic behavior

In dynamical systems, ‘events’ are maps, with the time axis as domain, hence functions of time.

It is convenient to distinguish in the notation

the domain of the maps, the time set

and the codomain, the signal space

the set where the functions take on their values.

The dynamic behavior

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The behavior of a dynamical system is usually described by a system of ordinary differential equations (ODEs) or difference equations.

In contrast to distributed phenomena

$\rightsquigarrow$ partial differential equations (PDEs)

The dynamic behavior

A ***dynamical system*** $:\Leftrightarrow$ $(\mathbb{T}, \mathbb{W}, \mathscr{B})$

$\mathbb{T} \subseteq \mathbb{R}$ **'time set'**

$\mathbb{W}$ **'signal space'**

$\mathscr{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **the 'behavior'**

a family of trajectories $\mathbb{T} \rightarrow \mathbb{W}$

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mostly, $\mathbb{T} = \mathbb{R}, \mathbb{R}_+, \mathbb{Z},$ **or** $\mathbb{N}$ $(\cong \mathbb{Z}_+)$**,**
and, in this course, $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$**,**
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(finite dimensional) vector-valued time trajectories

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$w : \mathbb{T} \to \mathbb{R}^{\mathtt{w}} \in \mathscr{B} \Leftrightarrow$ **'*w* is compatible with the model'**
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$\mathbb{T} = \mathbb{R}$ **or** $\mathbb{R}_+$ $\rightsquigarrow$ **'continuous-time' systems and ODEs**
$\mathbb{T} = \mathbb{Z}$ **or** $\mathbb{N}$ $\rightsquigarrow$ **'discrete-time' systems and difference eqn's**

We deal extensively with the case $\mathbb{T} = \mathbb{R}$ **first.**

Linear time-invariant differential systems

LTIDSs

LTIDSs

The dynamical system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ $\rightsquigarrow$ $\mathscr{B}$ **is said to be**

$$[\![\ \textbf{linear}\ ]\!] :\Leftrightarrow [\![\ [\![w_1, w_2 \in \mathscr{B}, \alpha \in \mathbb{R}]\!] \Rightarrow [\![\alpha w_1 + w_2 \in \mathscr{B}]\!]\ ]\!]$$

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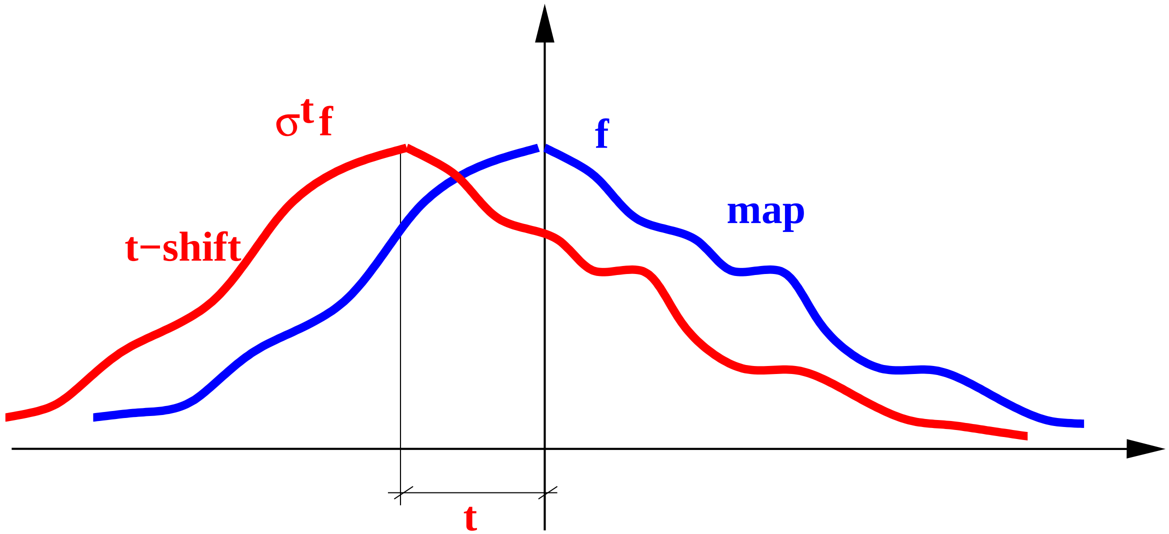
$[\![$ **time-invariant** $]\!] :\Leftrightarrow [\![\; [\![w \in \mathscr{B}$, σ^t **the** t**-shift**$]\!] \Rightarrow [\![\sigma^t w \in \mathscr{B}]\!] \;]\!]$



LTIDSs

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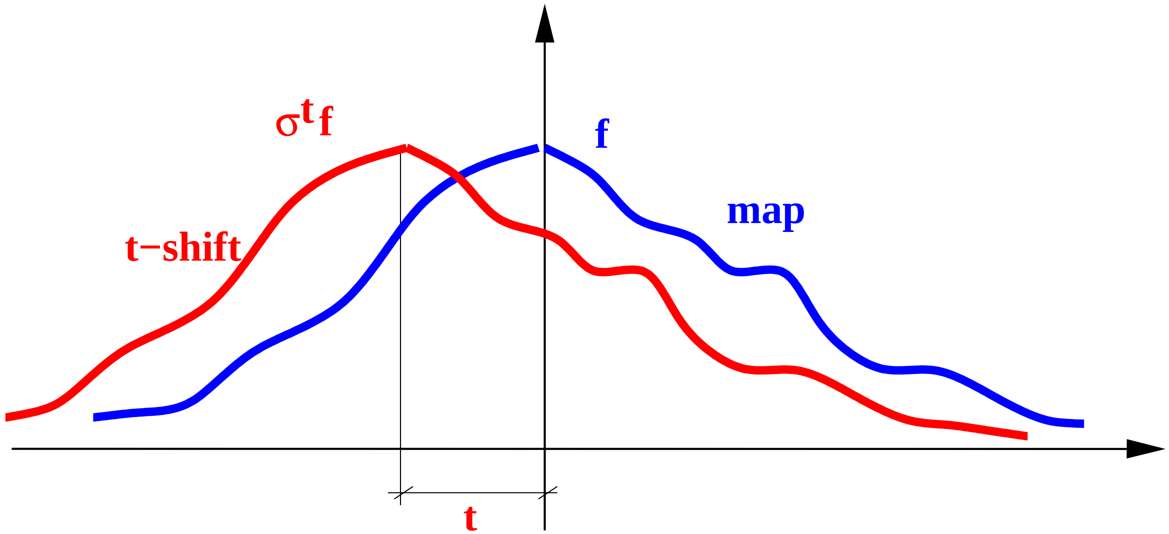

$[\![$ **differential** $]\!] :\Leftrightarrow [\![\mathscr{B}$ **is 'described' by an ODE**$]\!]$**.**

Linearity

This definition of linearity has as a special case

$$u \mapsto y = L(u) \quad L \text{ a linear map}$$

$u \in$ **a space of inputs,** $y \in$ **a space of outputs,** $\quad w = \begin{bmatrix} u \\ y \end{bmatrix}$**.**

$$\mathscr{B} = \left\{ w = \begin{bmatrix} u \\ y \end{bmatrix} \mid y = L(u) \right\} = \text{ the 'graph' of } L$$

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But, a dynamical system, also an input/output system,

is seldom a map !

LTIDSs

The dynamical system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ **is**

a linear time-invariant differential system (LTIDS) $:\Leftrightarrow$ **the behavior consists of the set of solutions of a system of linear, constant coefficient, ODEs**

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0.$$

$R_0, R_1, \cdots, R_{\mathtt{n}} \in \mathbb{R}^{\bullet \times \mathtt{w}}$ **real matrices that parametrize the system, and** $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$**.**

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$$\rightsquigarrow \qquad R\left(\tfrac{d}{dt}\right) w = 0$$

with $R(\xi) = R_0 + R_1 \xi + \cdots + R_{\mathtt{n}} \xi^{\mathtt{n}} \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$
a polynomial matrix, usually 'wide' $\boxed{\phantom{\text{wide}}}$ **or square.**

We should define what we mean by a solution of

$$R\left(\frac{d}{dt}\right)w = 0$$

For ease of exposition, we take $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **solutions. Hence the behavior defined is**

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Hence the behavior defined is

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$\mathscr{B} = \mathtt{kernel}\left(R\left(\frac{d}{dt}\right)\right)$ **‘*kernel representation*’ of this $\mathscr{B}$.**

Notation:

$\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **$\mathscr{L}^{\mathtt{w}}$ = the LTIDSs with $\mathtt{w}$ variables**

$\mathscr{B} \in \mathscr{L}^\bullet$,   **$\mathscr{L}^\bullet$ = the LTIDSs.**

Smoothness of solutions

There are many possibilities for the def'n of the solution set of

$$R\left(\frac{d}{dt}\right)w = 0$$

- $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **solutions — our choice**

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- **Strong solutions: all derivatives appearing in the eqn'ns exist and the ODEs are satisfied. Has very few 'invariance' properties.**
- **Weak solutions:** $w \in \mathscr{L}^{\text{local}}(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$**, solutions interpreted in the sense of distributions.**

  Includes steps, ramps, jumps, jerks, etc.

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  Includes steps, ramps, jumps, jerks, etc.
- **Distributional solutions include impulses and such frivolities.**

Representations of LTIDSs

We will meet numerous representations of LTIDSs

- **As the set of solutions of** $R\left(\frac{d}{dt}\right)w = 0$ $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$ **(our def.)** $R\left(\frac{d}{dt}\right) : \mathscr{C}^\infty\left(\mathbb{R}, \mathbb{R}^{\mathtt{coldim}(R)}\right) \rightarrow \mathscr{C}^\infty\left(\mathbb{R}, \mathbb{R}^{\mathtt{rowdim}(R)}\right)$ **'kernel repr'n'**

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- **With input/output partition**

  $P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u \quad w \cong \begin{bmatrix} u \\ y \end{bmatrix} \quad \det(P) \neq 0, P^{-1}Q$ **proper**

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- **Input/state/output representation**
  in terms of matrices A, B, C, D **such that**
  $\mathscr{B}$ **consists of all** $w's$ **generated by**

$$\frac{d}{dt}x = Ax + Bu, \; y = Cx + Du \quad w \cong \begin{bmatrix} u \\ y \end{bmatrix}$$

Rudolf E. Kalman
born 1930

Representations of LTIDSs

- $w = M\left(\frac{d}{dt}\right)\ell$ **with** $M \in \mathbb{R}[\xi]^{\mathtt{w}\times\bullet}$

  $M\left(\frac{d}{dt}\right) : \mathscr{C}^\infty\left(\mathbb{R}, \mathbb{R}^{\mathtt{coldim}(M)}\right) \to \mathscr{C}^\infty\left(\mathbb{R}, \mathbb{R}^{\mathtt{rowdim}(M)}\right)$ **'image repr'n'**

  $\mathscr{B} = \mathtt{image}\left(M\left(\frac{d}{dt}\right)\right)$

Representations of LTIDSs

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  $\mathscr{B} = \mathtt{image}\left(M\left(\frac{d}{dt}\right)\right)$
- **First principles models often contain 'latent variables' (see later)** $\leadsto$ $R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell$ **'latent variable representation'**

$$\mathscr{B} = \{w \mid \exists\, \ell \text{ such that } \ldots\}$$

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- **Special case:** $\frac{d}{dt}Fx = Ax + Bw$ **DAEs**

$$\mathscr{B} = \{w \mid \exists\, x \text{ such that } \ldots\}$$

Representations of LTIDSs

- **representations with rational symbols**
  $R\left(\frac{d}{dt}\right)w = 0$, $w = M\left(\frac{d}{dt}\right)\ell$, **etc.**
  with $R, M \in \mathbb{R}(\xi)^{\bullet\times\bullet}$**, or proper stable rational, etc. (see lecture 7)**

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- **and then, there are the convolution representations**

$$\int_{-\infty}^{+\infty} H(t')w(t-t')\,dt' = 0$$

(see lecture 4)
with the kernel, input/output, image versions

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  with the kernel, input/output, image versions
- **Rich ... but confusing!**

Injectivity, surjectivity, and bijectivity of differential operators

It is convenient to have the following proposition at hand.

Proposition: Let $P \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$ and consider the map

$$P\left(\frac{d}{dt}\right) : \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}_2}) \to \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}_1})$$

- $P\left(\frac{d}{dt}\right)$ **is injective iff the complex matrix $P(\lambda)$ has rank $\mathtt{n}_2$ for all $\lambda \in \mathbb{C}$. That is, iff $P(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$**
- $P\left(\frac{d}{dt}\right)$ **is surjective iff the polynomial matrix P has rank $\mathtt{n}_1$ (i.e. P is of full row rank). That is, iff there exists a $\mathtt{n}_1 \times \mathtt{n}_1$ submatrix of P with non-zero determinant.**
- $P\left(\frac{d}{dt}\right)$ **is surjective iff P is unimodular. That is iff $\mathtt{n}_1 = \mathtt{n}_2$ and $\mathtt{determinant}(P)$ is a non-zero constant polynomial.**

This proposition will be proven in lecture 2.

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$$P\left(\frac{d}{dt}\right) : \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}_2}) \to \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}_1})$$

- $P\left(\frac{d}{dt}\right)$ **is injective iff the complex matrix** $P(\lambda)$ **has rank** $\mathtt{n}_2$ **for all** $\lambda \in \mathbb{C}$**. That is, iff** $P(\lambda)$ **has full column rank for all** $\lambda \in \mathbb{C}$
- $P\left(\frac{d}{dt}\right)$ **is surjective iff the polynomial matrix** P **has rank** $\mathtt{n}_1$ **(i.e.** P **is of full row rank). That is, iff there exists a** $\mathtt{n}_1 \times \mathtt{n}_1$ **submatrix of** P **with non-zero determinant.**
- $P\left(\frac{d}{dt}\right)$ **is surjective iff** P **is unimodular. That is iff** $\mathtt{n}_1 = \mathtt{n}_2$ **and** $\mathtt{determinant}(P)$ **is a non-zero constant polynomial.**

This proposition will be proven in lecture 2.

The mathematical structure of LTIDSs

What is the mathematical structure of $\mathscr{L}^{\mathtt{W}}$?

What is the mathematical structure of $\mathscr{L}^{\mathtt{W}}$?

In order to cope with this question, we need a few concepts from algebra: *rings and modules*.

**A ring is a mathematical notion that has been introduced in order to capture the structure of the integers, the polynomials, square matrices, etc.,
and modules are like vector spaces over a ring, instead of over a field, as is officially required for a vector space.**

Our interest is mainly is the ring of polynomials and in polynomial modules

These notions are briefly reviewed in the appendix

What is the mathematical structure of $\mathscr{L}^{\mathtt{w}}$?

Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$**, say,** $\mathscr{B} = \text{kernel}\left(R\left(\frac{d}{dt}\right)\right)$

R **determines** $\mathscr{B}$**, but** $\mathscr{B}$ **does not determine** R**. For example, if** U **is unimodular, then** R **and** UR **determine the same behavior!**

What property of R **really determines** $\mathscr{B}$**?**

When do

$$R_1\left(\frac{d}{dt}\right)w = 0 \quad \textbf{and} \quad R_2\left(\frac{d}{dt}\right)w = 0$$

define the same behavior?

$\mathscr{L}^{\mathtt{w}}$ and polynomial submodules

Theorem

There is a one-to-one relation between $\mathscr{L}^{\mathtt{w}}$ and the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{1\times \mathtt{w}}$.

$\mathscr{L}^{\mathtt{w}}$ and polynomial submodules

Theorem

There is a one-to-one relation between $\mathscr{L}^{\mathtt{w}}$ and the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{1\times\mathtt{w}}$.

We now describe this $1 \leftrightarrow 1$ relation.
One direction involves the annihilators of $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.
$\mathtt{n} \in \mathbb{R}[\xi]^{1\times\mathtt{w}}$ **is said to be an annihilator of** $\mathscr{B}$ $:\Leftrightarrow$

$$n(\frac{d}{dt})\mathscr{B} = 0 \qquad \textbf{i.e.} \quad n(\frac{d}{dt})w = 0 \textbf{ for all } w \in \mathscr{B}$$

Denote the annihilators of $\mathscr{B}$ by $\mathscr{N}_{\mathscr{B}}$, a submodule of $\mathbb{R}[\xi]^{1\times\mathtt{w}}$.

The submodule associated with $\mathscr{B}$ by the thm is $\boxed{\mathscr{B} \mapsto \mathscr{N}_{\mathscr{B}}}$

$\mathscr{L}^{\mathtt{w}}$ and polynomial submodules

The other direction is also as expected. The $\mathscr{M}$ be an $\mathbb{R}[\xi]$-submodule of $\mathbb{R}[\xi]^{1\times\mathtt{w}}$. Define

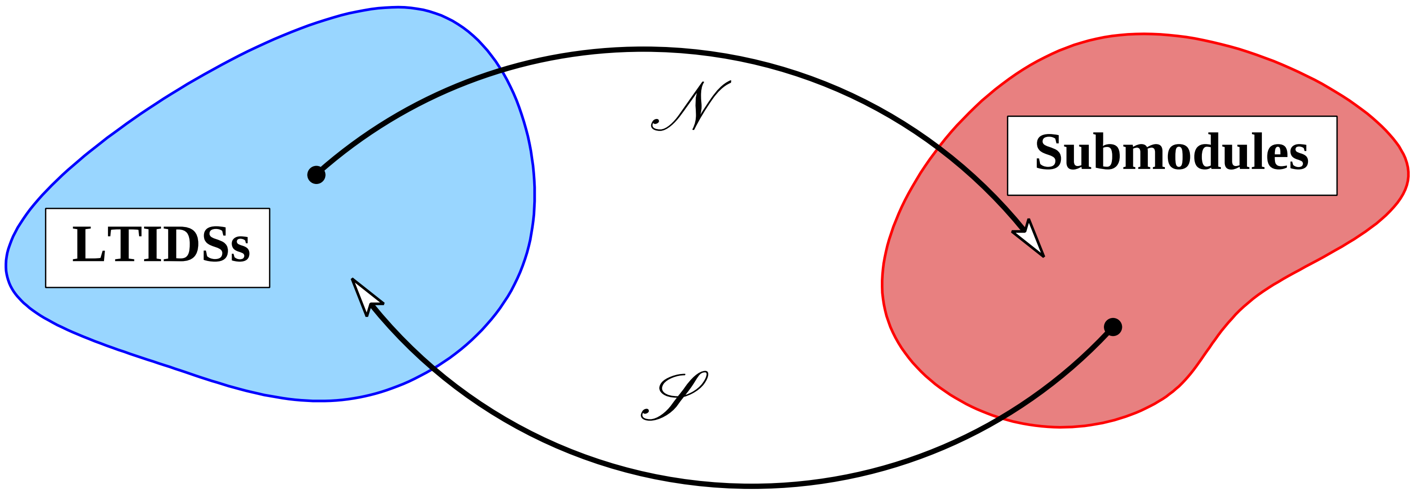
$$\boxed{\mathscr{M} \mapsto \mathscr{S}_{\mathscr{M}} := \left\{ w \in \mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid n\left(\tfrac{d}{dt}\right) w = 0 \textbf{ for all } n \in \mathscr{M} \right\}}$$

The right hand side defines an element of $\mathscr{L}^{\mathtt{w}}$, even though it involves an ∞ number of ODEs.
A kernel representation (with a finite number of ODEs) is easily constructed, by taking for R the polynomial matrix with as rows a basis of $\mathscr{M}$.
sol'n set of ∞-number of linear constant coefficient ODEs $\Leftrightarrow$ ∞-number of linear constant coefficient ODEs!

The behavior associated with $\mathscr{M}$ by the thm is $\boxed{\mathscr{M} \mapsto \mathscr{S}_{\mathscr{M}}}$

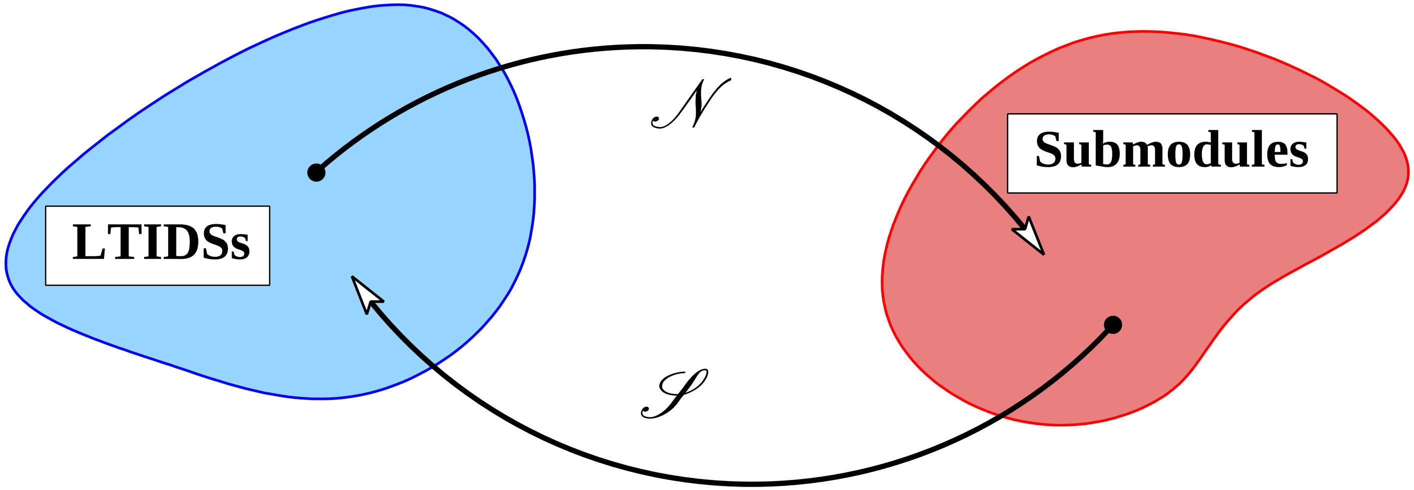
$\mathscr{L}^{\mathtt{w}}$ and polynomial submodules

In pictures:



$\mathscr{L}^{\mathtt{W}}$ and polynomial submodules

In pictures:



We will prove that this association is one-to-one, by showing that the maps $\mathscr{N}$ and $\mathscr{S}$ are inverses of each other.

Notation

- **For $R \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$, denote by $< R >$ the $\mathbb{R}[\xi]$-submodule of $\mathbb{R}[\xi]^{1\times\mathtt{w}}$ generated by its rows.**

Notation

- **For $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$, denote by $< R >$ the $\mathbb{R}[\xi]$-submodule of $\mathbb{R}[\xi]^{1 \times \mathtt{w}}$ generated by its rows.**

The essence of the proof of the thm is the following lemma

Lemma: $[\![n \in \mathscr{N}_{\mathtt{kernel}(R(\frac{d}{dt}))}]\!] \Leftrightarrow [\![n \in < R >]\!]$

The proof is given later.

Therefore

(i) $\mathscr{N}_{\mathtt{kernel}(R(\frac{d}{dt}))} = < R >$ **(by the lemma)**

(ii) $\mathscr{S}_{<R>} = \mathtt{kernel}\left(R\left(\frac{d}{dt}\right)\right)$ **(by the def. of $\mathscr{S}$)**

Submodules & LTIDSs

Corollary: The following are equivalent:

1. $< R_1 > = < R_2 >$
2. $\mathscr{S}_{<R_1>} = \mathscr{S}_{<R_2>}$
3. $\mathtt{kernel}\left(R_1\left(\frac{d}{dt}\right)\right) = \mathtt{kernel}\left(R_2\left(\frac{d}{dt}\right)\right)$
4. $\mathscr{N}_{\mathtt{kernel}\left(R_1\left(\frac{d}{dt}\right)\right)} = \mathscr{N}_{\mathtt{kernel}\left(R_2\left(\frac{d}{dt}\right)\right)}$

Submodules & LTIDSs

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4. $\mathscr{N}_{\texttt{kernel}\left(R_1\left(\frac{d}{dt}\right)\right)} = \mathscr{N}_{\texttt{kernel}\left(R_2\left(\frac{d}{dt}\right)\right)}$

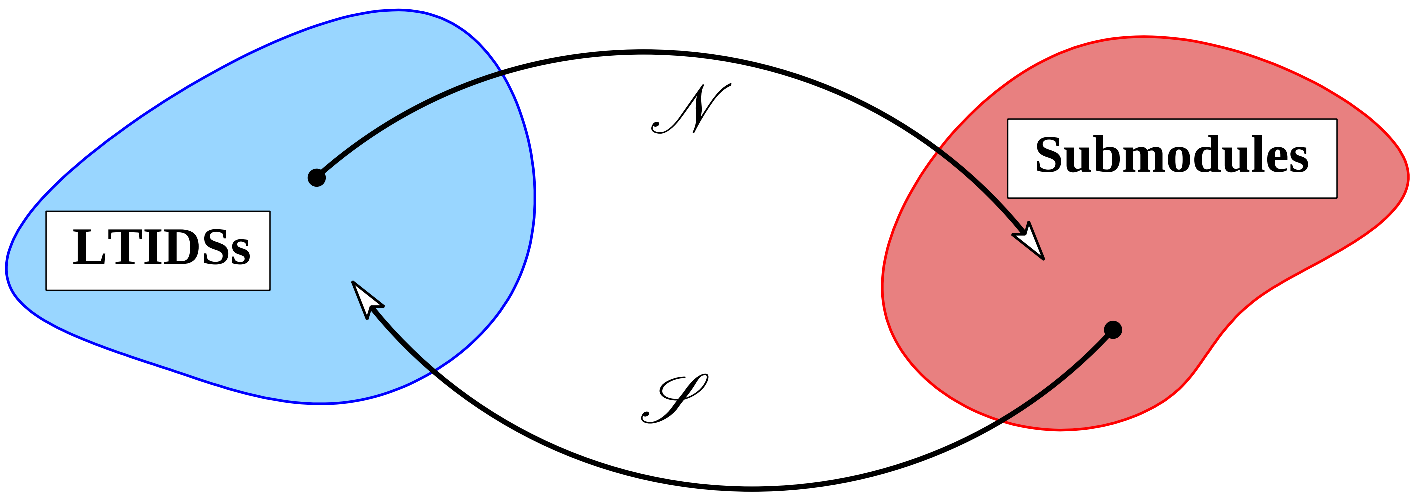
Proof of the Corollary:

1. $\Rightarrow$ 2. is trivial
2. $\Leftrightarrow$ 3. is consequence (ii)
3. $\Rightarrow$ 4. is trivial
4. $\Leftrightarrow$ 1. is consequence (i)

Submodules & LTIDSs

3. $\Leftrightarrow$ 4. implies that $\mathscr{B} \mapsto \mathscr{N}_{\mathscr{B}}$ is injective

1. $\Leftrightarrow$ 2. implies that $\mathscr{M} \mapsto \mathscr{S}_{\mathscr{M}}$ is injective



Hence the maps $\mathscr{B} \mapsto \mathscr{N}_{\mathscr{B}}$ and $\mathscr{M} \mapsto \mathscr{S}_{\mathscr{M}}$ are each other's inverse.

Proof of the lemma

<u>Lemma</u>: $[\![n \in \mathcal{N}_{\texttt{kernel}(R(\frac{d}{dt}))}]\!] \Leftrightarrow [\![n \in < R >]\!]$.

In other words,

$$[\![\ [\![R\left(\frac{d}{dt}\right) w = 0]\!] \Rightarrow [\![n\left(\frac{d}{dt}\right) w = 0]\!]\ ]\!]$$

$$\Leftrightarrow [\![\exists f \in \mathbb{R}[\xi]^{1\times\bullet} \textbf{ such that } n = fR]\!]$$

This lemma states that that the module of annihilators is exactly the module generated by the rows of R.
All annihilators are linear combinations of the rows of R.

No new annihilators sneek in.

Proof of the lemma

Observe first the scalar version of the lemma. Let $d, n \in \mathbb{R}[\xi]$**.**

$$[\![\ [\![d\left(\frac{d}{dt}\right)w=0]\!] \Rightarrow [\![n\left(\frac{d}{dt}\right)w=0]\!]\ ]\!] \Leftrightarrow [\![d \textbf{ is a factor of } n]\!]$$

The proof is an exercise.

Note that even in this special case, the lemma does not hold if we had defined the behavior in terms of compact support solutions, instead of in terms of $\mathscr{C}^\infty$ **solutions.**

<u>Example</u>: Consider $R\left(\frac{d}{dt}\right)w=0$ **with** $0 \neq R \in \mathbb{R}[\xi]$**. With** $\mathscr{B}$ **the** $\mathscr{C}^\infty$ **solutions, the annihilators are the polynomials that have** R **as a factor, indeed the module generated by** R**.**
Take for $\mathscr{B}$ **the compact support solutions instead.**
Then $\mathscr{B}=\{0\}$**. The module of annihilators is then** $\mathbb{R}[\xi]$ **(for all** $R \neq 0$**), while the module generated by** R **consists only of the polynomials that have** R **as a factor.**

Proof of the lemma

We now indicate the proof of the lemma. The proof uses the Smith form. This form implies (prove!) that we can assume without loss of generality that R is of the form

$$R = \begin{bmatrix} \mathrm{diag}(d_1, d_2, \ldots, d_{\mathtt{r}}) & 0 \\ 0 & 0 \end{bmatrix} \text{ **with** } d_1, d_2, \cdots, d_{\mathtt{r}} \neq 0.$$

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With R of this form, we have

$$[\![R\left(\frac{d}{dt}\right)w = 0]\!] \Leftrightarrow [\![d_1\left(\frac{d}{dt}\right)w_1 = d_2\left(\frac{d}{dt}\right)w_2 = \cdots = d_{\mathtt{r}}\left(\frac{d}{dt}\right)w_{\mathtt{r}} = 0]\!].$$

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$$[\![R\left(\frac{d}{dt}\right)w = 0]\!] \Leftrightarrow [\![d_1\left(\frac{d}{dt}\right)w_1 = d_2\left(\frac{d}{dt}\right)w_2 = \cdots = d_{\mathtt{r}}\left(\frac{d}{dt}\right)w_{\mathtt{r}} = 0]\!].$$

Hence, with $n = [n_1 \quad n_2 \quad \cdots \quad n_{\mathtt{r}}]$, we conclude, from the scalar case, that $R\left(\frac{d}{dt}\right)w = 0$ implies $n\left(\frac{d}{dt}\right)w = 0$ iff $d_{\mathtt{k}}$ is a factor of $n_{\mathtt{k}}$ for all $\mathtt{k}$. The lemma follows.

Relations between kernel representations

Inclusion

Let $\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{L}^{\mathtt{w}}$. $\mathscr{B}_1 \rightsquigarrow R_1\left(\frac{d}{dt}\right)w = 0, \;\; \mathscr{B}_2 \rightsquigarrow R_2\left(\frac{d}{dt}\right)w = 0.$

$\mathscr{B}_1 \subseteq \mathscr{B}_2$ **iff** $\exists\, F \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ **such that** $R_2 = FR_1$

Proof: $\Rightarrow$**: trivial.** $\Leftarrow$**: takes a bit of work.**

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Proof: $\Rightarrow$**: trivial.** $\Leftarrow$**: takes a bit of work.**

$\mathscr{B}_1 = \mathscr{B}_2$ **iff** $\exists\, F_1, F_2 \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ **such that** $R_1 = F_2R_2, R_2 = F_1R_1$

In particular, $\mathscr{B}_1 = \mathscr{B}_2$ **if** $R_1 = UR_2, U$ **unimodular.**

Equations specify behavior, but not the other way around

Minimal kernel representations

The kernel representation $R\left(\frac{d}{dt}\right)w = 0$ of $\mathscr{B}$ is said to be *minimal* if among all kernel representations of $\mathscr{B}$, R has a minimal number of rows.

Minimal kernel representations

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Proposition: The following are equivalent.

- $R\left(\frac{d}{dt}\right)w = 0$ **is minimal.**
- **The rows of R are linearly independent. They form a basis for the $\mathbb{R}[\xi]$-module generated by the rows of R.**
- R **has full row rank.**
- $R\left(\frac{d}{dt}\right)$ **is surjective.**

Minimal kernel representations

The kernel representation $R\left(\frac{d}{dt}\right)w = 0$ of $\mathscr{B}$ is said to be *minimal* if among all kernel representations of $\mathscr{B}$, R has a minimal number of rows.

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- $R\left(\frac{d}{dt}\right)w = 0$ **is minimal.**
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- R **has full row rank.**
- $R\left(\frac{d}{dt}\right)$ **is surjective.**

All minimal kernel representations of $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ are generated from a minimal one, $R\left(\frac{d}{dt}\right)w = 0$, by the transformation group

$$R \xrightarrow{U \text{ unimodular}} UR$$

$\rightsquigarrow$ **canonical forms, invariants, etc.**

Recapitulation

- **Dynamical systems** $\rightsquigarrow \Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$ **with behavior** $\mathscr{B} \subseteq (\mathbb{W})^{\mathbb{T}}$ **a family of time trajectories**
- **LTIDSs:** $\mathscr{B}$ **is the sol'n set of a system of linear constant coefficient ODEs**
- **LTIDSs** $1 \leftrightarrow 1$ $\mathbb{R}[\xi]$**-modules**
- **A minimal kernel repr. of a LTIDS is uniquely defined up to unimodular premultiplication**

Latent variables

Kernels, images, and projections

A model is a subset . There are many ways to specify a subset. For example,

- **as the solution set of equations**

- **as an image of a map**

- **as a projection**

Kernels, images, and projections

A model $\mathscr{B}$ is a subset of $\mathscr{U}$. There are many ways to specify a subset. For example,

- **as the solution set of equations**

$$f : \mathscr{U} \rightarrow \bullet; \qquad \mathscr{B} = \{w \mid f(w) = 0\}$$

- **as an image of a map**

$$f : \bullet \rightarrow \mathscr{U}; \qquad \mathscr{B} = \{w \mid \exists\, \ell \text{ such that } w = f(\ell)\}$$

- **as a projection**

$$\mathscr{B}_{\text{extended}} \subseteq \mathscr{U} \times \mathscr{L}; \qquad \mathscr{B} = \{w \mid \exists\, \ell \text{ such that } (w, \ell) \in \mathscr{B}_{\text{extended}}\}$$

Kernels, images, and projections

A model $\mathscr{B}$ is a subset of $\mathscr{U}$. There are many ways to specify a subset. For example,

- **as the solution set of equations** **'kernel representation'**

$$f : \mathscr{U} \to \bullet; \qquad \mathscr{B} = \{w \mid f(w) = 0\}$$

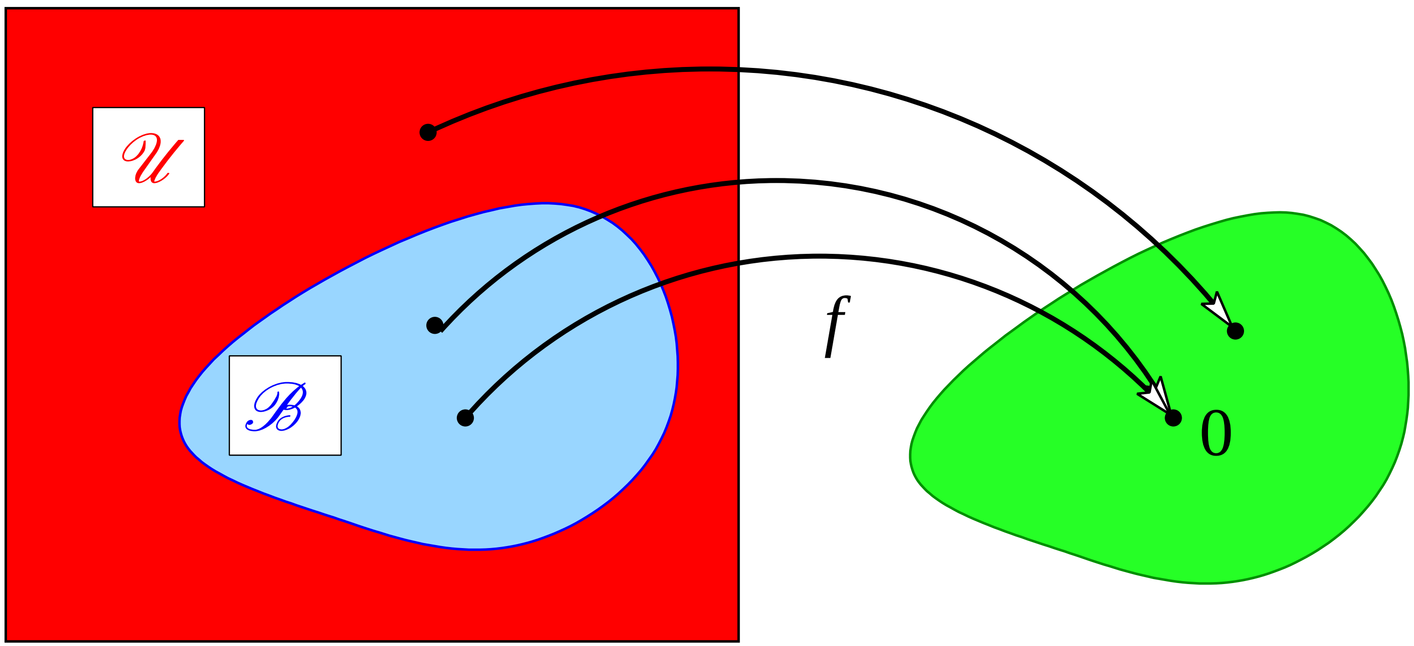
- **as an image of a map** **'image representation'**

$$f : \bullet \to \mathscr{U}; \qquad \mathscr{B} = \{w \mid \exists\, \ell \text{ such that } w = f(\ell)\}$$

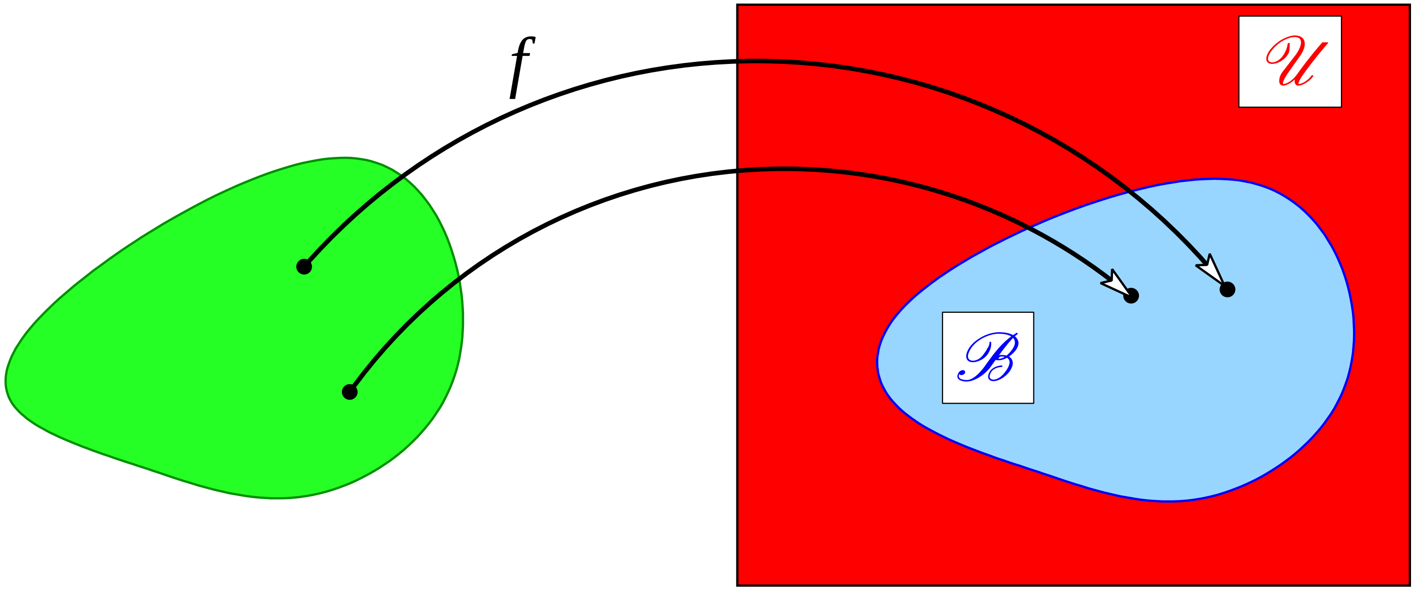
- **as a projection** **'latent variable representation'**

$$\mathscr{B}_{\text{extended}} \subseteq \mathscr{U} \times \mathscr{L}; \qquad \mathscr{B} = \{w \mid \exists\, \ell \text{ such that } (w, \ell) \in \mathscr{B}_{\text{extended}}\}$$

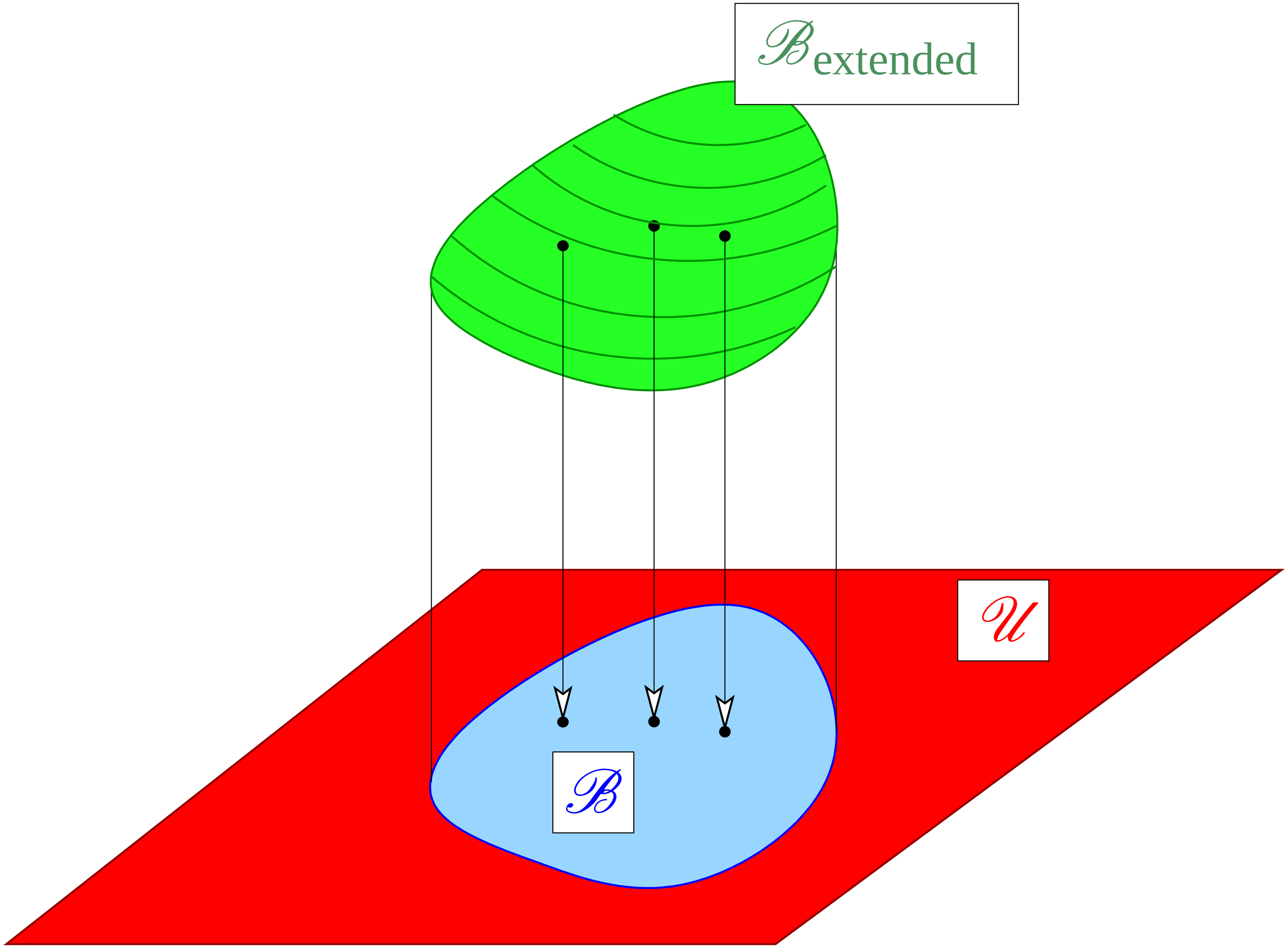
Kernel



Image



Projection



Latent variable representations

Combining equations with latent variables $\rightsquigarrow$

$\mathscr{B}_{\text{extended}}$ **specified by**

$$\mathscr{B}_{\text{extended}} = \{(w, \ell) \mid f(w, \ell) = 0 = 0\}$$

$$\mathscr{B} = \{w \mid \exists\, \ell \text{ such that } f(w, \ell) = 0\}$$

Latent variable representations

Combining equations with latent variables $\rightsquigarrow$

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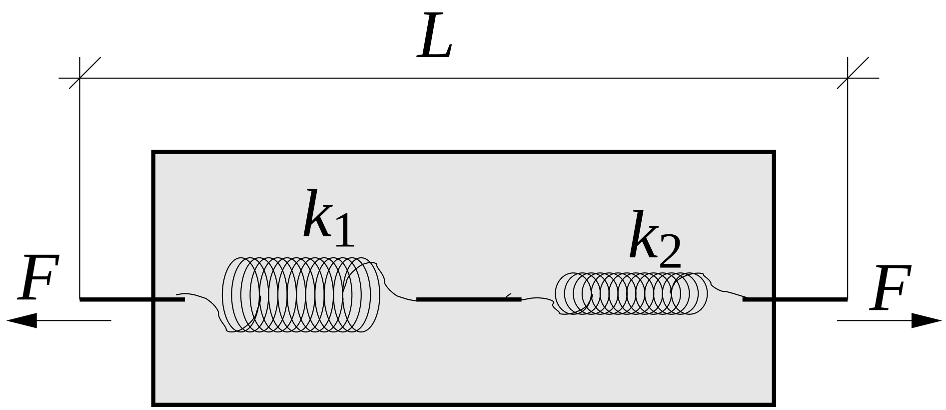
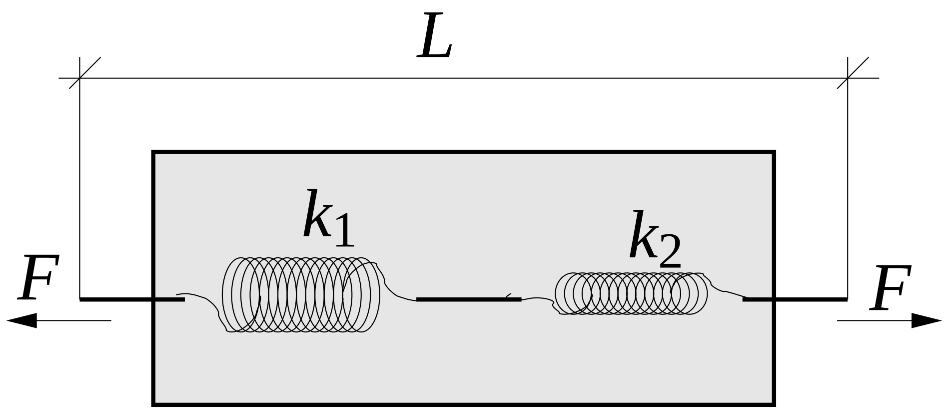
First principles models usually come in this form.
Latent variables naturally emerge from interconnections.

Two springs interconnected in series



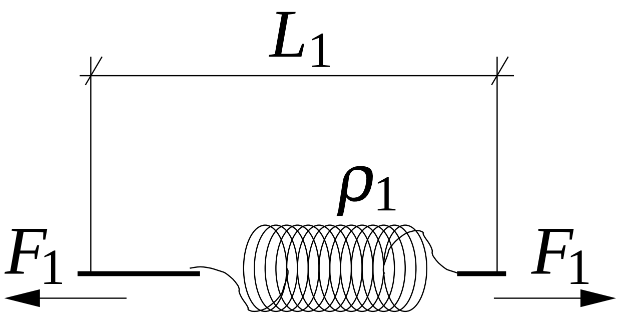
’!’! Model relation between *L* and *F* !!

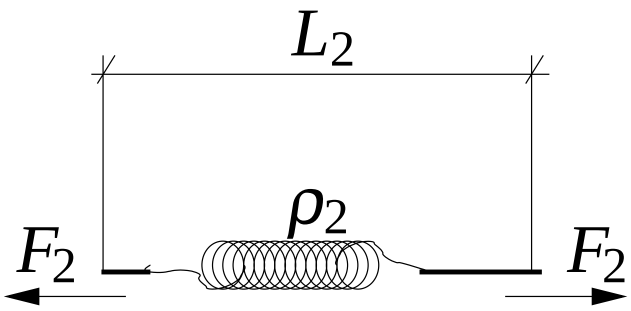
Two springs interconnected in series



’!’! Model relation between L and F !!

View as interconnection of two springs





Two springs interconnected in series

Model for (L, F) (assume that for the individual springs the length is a function of the force exerted).

$$\begin{aligned} L_1 &= \rho_1(F_1) & L_2 &= \rho_1(F_2) \\ F &= F_1 = F_2 & L &= L_1 + L_2 \end{aligned}$$

Two springs interconnected in series

Model for (L, F) (assume that for the individual springs the length is a function of the force exerted).

$$L_1 = \rho_1(F_1) \qquad L_2 = \rho_1(F_2)$$
$$F = F_1 = F_2 \qquad L = L_1 + L_2$$

L, F**: ‘manifest variables’** L_1, F_1, L_2, F_2**: ‘latent variables’**

$\leadsto$ $$\boxed{L = \rho_1(F) + \rho_2(F)}$$

Latent variables are easily eliminated, for this example.

Two springs interconnected in series

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L, F: **'manifest variables'** L_1, F_1, L_2, F_2: **'latent variables'**

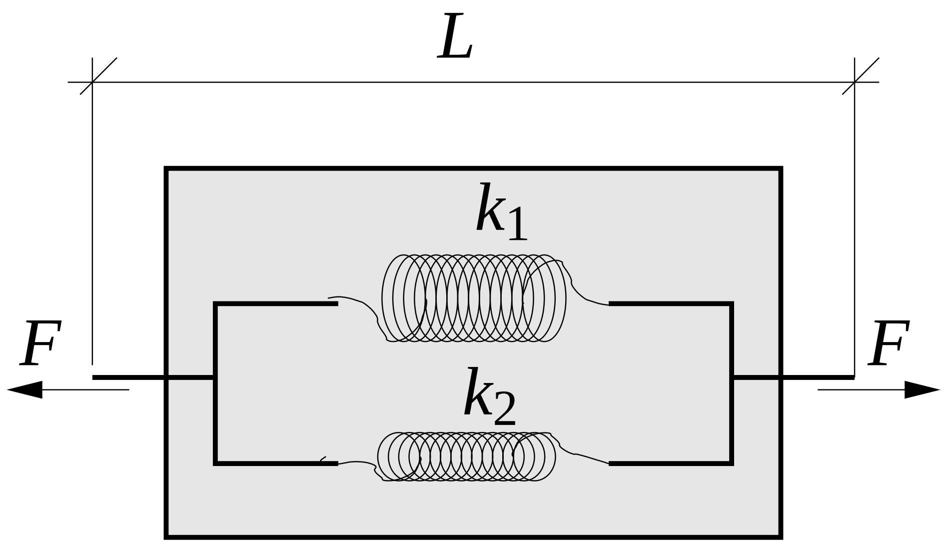
$$\rightsquigarrow \qquad \boxed{L = \rho_1(F) + \rho_2(F)}$$

Latent variables are easily eliminated, for this example.

In the linear case: $L_1 = L_1^* + \rho_1 F_1 \qquad L_2 = L_2^* + \rho_2 F_2$

After elimination $\rightsquigarrow L = L_1^* + L_2^* + (\rho_1 + \rho_2) F$

Two springs interconnected in parallel



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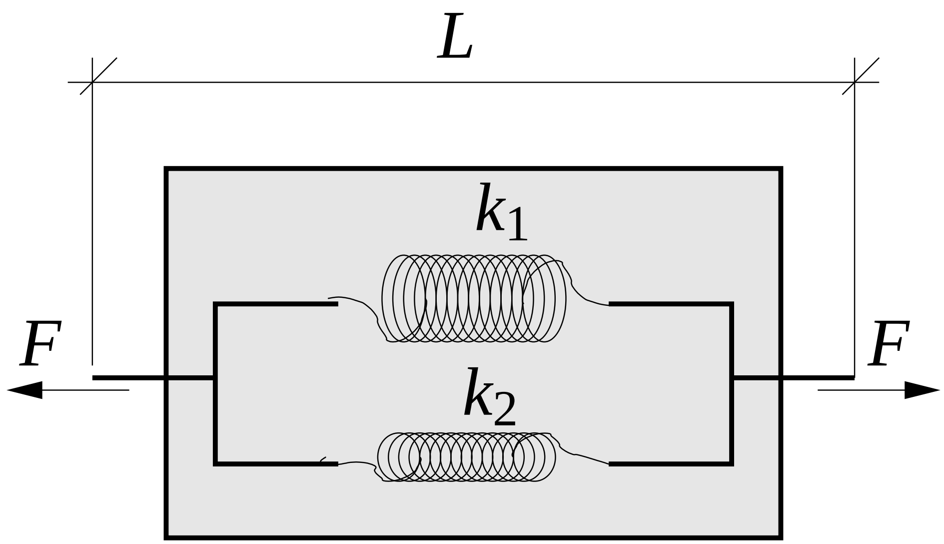
Two springs interconnected in parallel



'!'! Model relation between *L* and *F* !!

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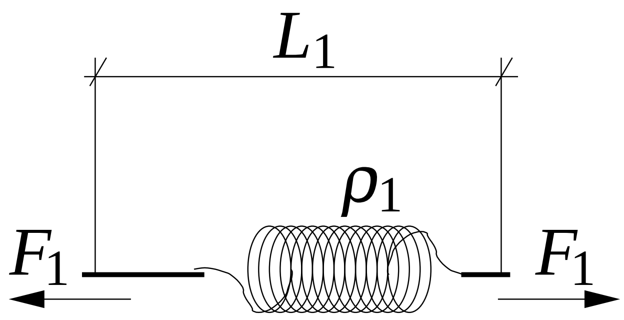

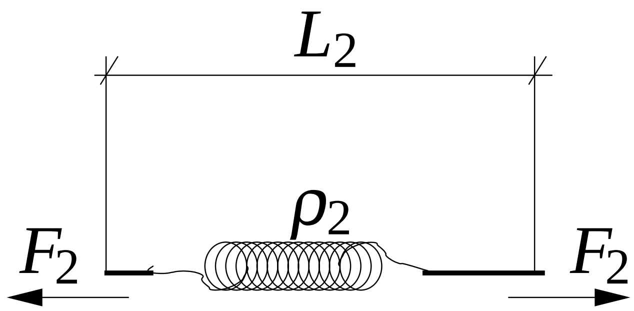

Two springs interconnected in parallel

Model for (L, F) (assume that for the individual springs the length is a function of the force exerted, and neglect the dimensions of the interconnecting mechanism).

$$
\begin{array}{rclcrcl}
L_1 & = & \rho_1(F_1) & \qquad & L_2 & = & \rho_1(F_2) \\
F & = & F_1 + F_2 & \qquad & L & = & L_1 = L_2
\end{array}
$$

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L, F: **'manifest variables'** L_1, F_1, L_2, F_2: **'latent variables'**

$\rightsquigarrow$ $$\boxed{\mathscr{B} = \{(L, F) \mid \exists\, \alpha : L = \rho_1(\alpha), \quad \rho_1(\alpha) = \rho_2(F - \alpha)\}}$$

Latent variables are not easily eliminated, for this example,

Two springs interconnected in parallel

Model for (L,F) (assume that for the individual springs the length is a function of the force exerted, and neglect the dimensions of the interconnecting mechanism).

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L,F**: ‘manifest variables’** L_1,F_1,L_2,F_2**: ‘latent variables’**

$$\rightsquigarrow \quad \boxed{\mathscr{B} = \{(L,F) \mid \exists\, \alpha : L = \rho_1(\alpha), \quad \rho_1(\alpha) = \rho_2(F-\alpha)\}}$$

Latent variables are not easily eliminated, for this example, unless we are in the linear case: $L_1 = L_1^* + \rho_1 F_1, L_2 = L_2^* + \rho_2 F_2$

After elimination $\rightsquigarrow L = \frac{\rho_2}{\rho_1+\rho_2} L_1^* + \frac{\rho_1}{\rho_1+\rho_2} L_2^* + \frac{\rho_1\rho_2}{\rho_1+\rho_2} F$

A dynamic example

RLC circuit

First principles models invariably contain (many) auxiliary variables in addition to the variables whose behavior we wish to model.

RLC circuit

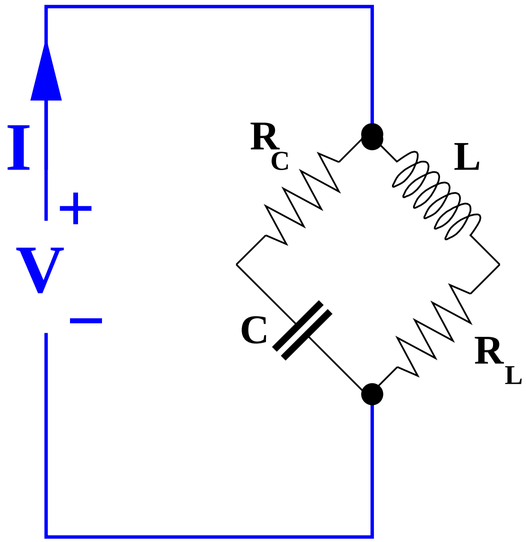
First principles models invariably contain (many) auxiliary variables in addition to the variables whose behavior we wish to model.

Can these latent variables be eliminated?

We illustrate the emergence of latent variables and the elimination question by means of an extensive example in the dynamic systems case.

RLC circuit

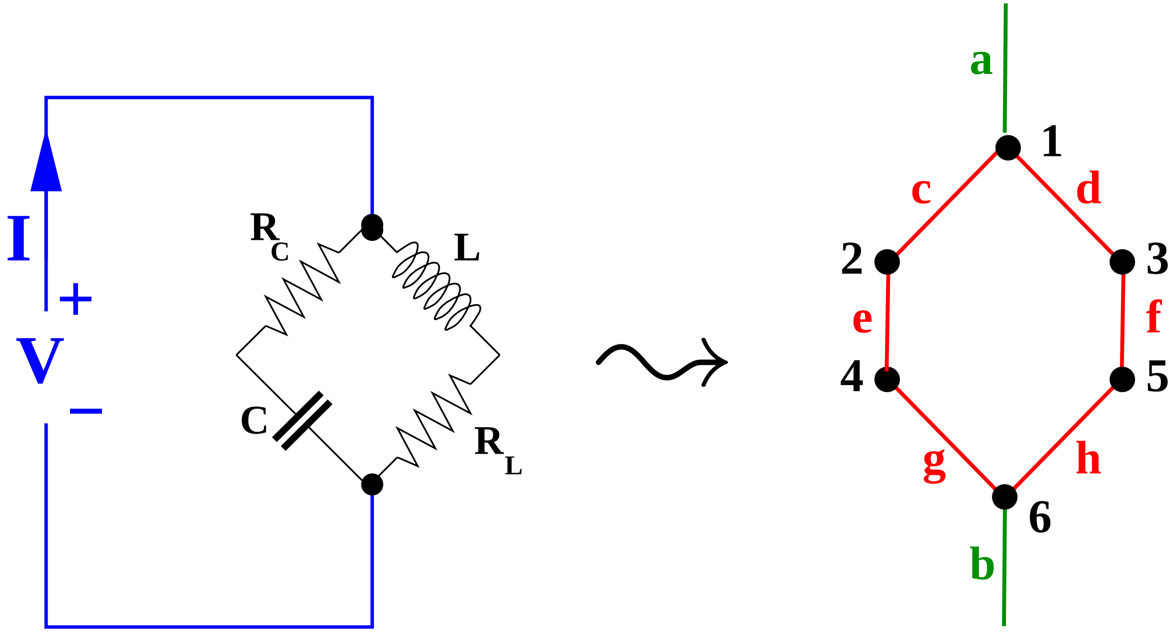
Model the port behavior of



by tearing, zooming, and linking (see lecture 13).

RLC circuit

Model the **port behavior** of

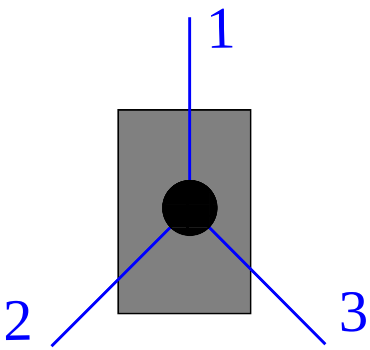


by tearing, zooming, and linking (see lecture 13).

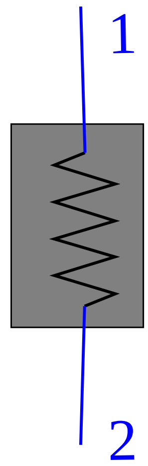
In each node there is an element $\rightsquigarrow$ module equations involving 2 variables (potential, current) for each terminal,

In each branch a connection $\rightsquigarrow$ interconnection equations

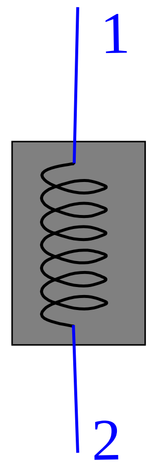
Modules



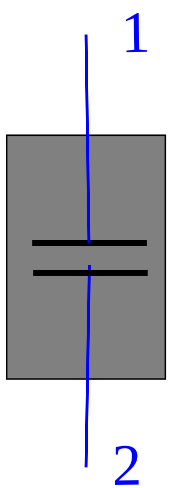
connector 1



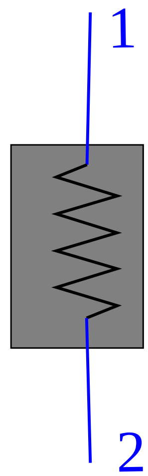
resistor R_C



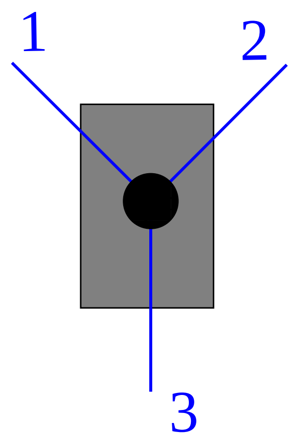
inductor L



capacitor C



resistor R_L



connector 2

Module equations

vertex 1 : $V_{\text{connector}_1,1} = V_{\text{connector}_1,2} = V_{\text{connector}_1,3}$

$I_{\text{connector}_1,1} + I_{\text{connector}_1,2} + I_{\text{connector}_1,3} = 0$

vertex 2 : $V_{R_C,1} - V_{R_C,2} = R_C I_{R_C,1},\ I_{R_C,1} + I_{R_C,2} = 0$

vertex 3 : $L\frac{d}{dt} I_{L,1} = V_{L,1} - V_{L,2},\ I_{L,1} + I_{L,2} = 0$

vertex 4 : $C\frac{d}{dt}\left(V_{C,1} - V_{C,2}\right) = I_{C,1},\ I_{C,1} + I_{C,2} = 0$

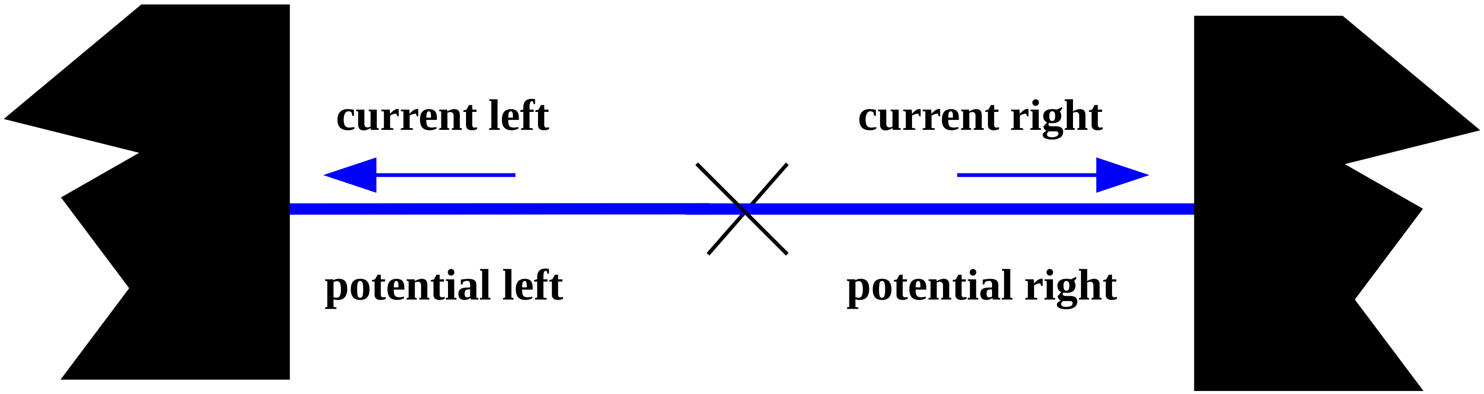
vertex 5 : $V_{R_L,1} - V_{R_L,2} = R_L I_{R_L,1}$

$I_{R_L,1} + I_{R_L,2} = 0$

vertex 6 : $V_{\text{connector}_2,1} = V_{\text{connector}_2,2} = V_{\text{connector}_2,3}$

$I_{\text{connector}_2,1} + I_{\text{connector}_2,2} + I_{\text{connector}_2,3} = 0$

Interconnection



Interconnection of two electrical terminals

Interconnection equations:

potential left = potential right

current left + current right = 0

Interconnection equations

$$\begin{array}{llll}
\textbf{edge c}: & V_{R_{C,1}} = V_{\text{connector1}_2} & I_{R_{C,1}} + I_{\text{connector1},2} = 0 \\
\textbf{edge d}: & V_{L_1} = V_{\text{connector1}_3} & I_{L_1} + I_{\text{connector1}_3} = 0 \\
\textbf{edge e}: & V_{R_{C,2}} = V_{C_1} & I_{R_{C,2}} + I_{C_1} = 0 \\
\textbf{edge f}: & V_{L_2} = V_{R_{C,1}} & I_{L_2} + I_{R_{L,1}} = 0 \\
\textbf{edge g}: & V_{C_2} = V_{\text{connector2}_1} & I_{C_2} + I_{\text{connector2}_1} = 0 \\
\textbf{edge h}: & V_{R_{L,2}} = V_{\text{connector2}_2} & I_{R_{L,2}} + I_{\text{connector2}_2} = 0
\end{array}$$

Manifest variable assignment

$$V_{\text{externalport}} = V_{\text{connector}_1,1} - V_{\text{connector}_2,3}$$

$$I_{\text{externalport}} = I_{\text{connector1}_1}$$

Manifest variable assignment

In total 28 latent variables $V_{\text{connector}_1,1}, \ldots, V_{R_{C,1}}, I_{R_{C,1}}, \ldots, I_{\text{connector}_2,3}$
2 manifest variables, $\left(V_{\text{externalport}}, I_{\text{externalport}}\right)$
26 equations.

Which equation(s) govern(s) $\left(V_{\text{externalport}}, I_{\text{externalport}}\right)$

A constant-coefficient linear differential equation?
One that does not contain the branch variables?

Does the fact that all the equations before elimination of the latent (auxiliary) variables are constant-coefficient linear differential equations imply the same after elimination?

The port equation

The port $\Sigma = (\mathbb{R}, \mathbb{R}^2, \mathscr{B})$ **behavior** $\mathscr{B}$ **is specified by:**

Case 1: $CR_C \neq \frac{L}{R_L}$

$$\left(\frac{R_C}{R_L} + \left(1 + \frac{R_C}{R_L}\right) CR_C \frac{d}{dt} + CR_C \frac{L}{R_L} \frac{d^2}{dt^2}\right) V_{\text{externalport}}$$
$$= \left(1 + CR_C \frac{d}{dt}\right) \left(1 + \frac{L}{R_L} \frac{d}{dt}\right) R_C I_{\text{externalport}}$$

Case 2: $CR_C = \frac{L}{R_L}$

$$\left(\frac{R_C}{R_L} + CR_C \frac{d}{dt}\right) V_{\text{externalport}} = (1 + CR_C) \frac{d}{dt} R_C I_{\text{externalport}}$$

The elimination theorem

Elimination theorem

Theorem

$\mathscr{L}^\bullet$ **is closed under projection**

Elimination theorem

Theorem

$\mathscr{L}^\bullet$ is closed under projection

Consider

$$\mathscr{B} = \{(w_1, w_2) : \mathbb{R} \to \mathbb{R}^{\mathtt{w}_1} \times \mathbb{R}^{\mathtt{w}_2} \mid (w_1, w_2) \in \mathscr{B}\}$$

Define the projection

$$\mathscr{B}_1 = \{w_1 : \mathbb{R} \to \mathbb{R}^{\mathtt{w}_1} \mid \exists\, w_2 : \mathbb{R} \to \mathbb{R}^{\mathtt{w}_1} \textbf{ such that } (w_1, w_2) \in \mathscr{B}\}$$

The theorem states that $\boxed{[\![\mathscr{B} \in \mathscr{L}^{\mathtt{w}_1+\mathtt{w}_2}]\!] \Rightarrow [\![\mathscr{B}_1 \in \mathscr{L}^{\mathtt{w}_1}]\!]}$

This is, as seen, important in modeling.

Proof of the elimination theorem

We indicate the proof. Consider

$$R_1\left(\frac{d}{dt}\right)w_1 + R_2\left(\frac{d}{dt}\right)w_2 = 0 \quad \rightsquigarrow \quad \textbf{behavior } \mathscr{B}$$

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Pre-multiply by a unimodular polynomial matrix U**. Then**

$$U\left(\frac{d}{dt}\right)R_1\left(\frac{d}{dt}\right)w_1+U\left(\frac{d}{dt}\right)R_2\left(\frac{d}{dt}\right)w_2=0 \rightsquigarrow \textbf{also behavior } \mathscr{B}$$

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Define $\quad \mathscr{B}_1 := \{w_1 \mid \exists\, w_2 \textbf{ such that } (w_1, w_2) \in \mathscr{B}\}$

Let V **be unimodular.**

$$U\left(\frac{d}{dt}\right)R_1\left(\frac{d}{dt}\right)w_1 + U\left(\frac{d}{dt}\right)R_2\left(\frac{d}{dt}\right)V\left(\frac{d}{dt}\right)\tilde{w}_2 = 0 \rightsquigarrow \textbf{behavior } \tilde{\mathscr{B}}$$

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$$R_1\left(\frac{d}{dt}\right)w_1 + R_2\left(\frac{d}{dt}\right)w_2 = 0 \quad \rightsquigarrow \quad \textbf{behavior } \mathscr{B}$$

Pre-multiply by a unimodular polynomial matrix U. Then

$$U\left(\frac{d}{dt}\right)R_1\left(\frac{d}{dt}\right)w_1 + U\left(\frac{d}{dt}\right)R_2\left(\frac{d}{dt}\right)w_2 = 0 \rightsquigarrow \textbf{ also behavior } \mathscr{B}$$

Define $\quad \mathscr{B}_1 := \{w_1 \mid \exists\, w_2 \textbf{ such that } (w_1, w_2) \in \mathscr{B}\}$

Let V be unimodular.

$$U\left(\frac{d}{dt}\right)R_1\left(\frac{d}{dt}\right)w_1 + U\left(\frac{d}{dt}\right)R_2\left(\frac{d}{dt}\right)V\left(\frac{d}{dt}\right)\tilde{w}_2 = 0 \rightsquigarrow \textbf{behavior } \tilde{\mathscr{B}}$$

Then $\quad \mathscr{B}_1 := \{w_1 \mid \exists\, \tilde{w}_2 \textbf{ such that } (w_1, \tilde{w}_2) \in \tilde{\mathscr{B}}\}$

Proof of the elimination theorem

The *Smith form* implies that we can choose U and V such that

$$UR_2V = \begin{bmatrix} \mathrm{diag}(d_1, d_2, \ldots, d_{\mathtt{r}}) & 0 \\ 0 & 0 \end{bmatrix}$$

with $d_1, d_2, \cdots, d_{\mathtt{r}} \neq 0$.

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with $d_1, d_2, \cdots, d_{\mathtt{r}} \neq 0$. Partition UR_1 conformably as $\begin{bmatrix} R_1' \\ R_1'' \end{bmatrix}$.

Note that $\mathrm{diag}(d_1, d_2, \ldots, d_{\mathtt{r}}) \left(\frac{d}{dt}\right)$ is surjective. Conclude that $R_1''\left(\frac{d}{dt}\right) w_1 = 0 \rightsquigarrow$ behavior $\mathscr{B}_1$. QED.

Applications of the elimination theorem

$$[\![\frac{d}{dt}x = Ax + Bu, y = Cx + Du]\!] \;\Rightarrow\; [\![P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u]\!]$$

$$[\![E\frac{d}{dt}x = Ax + Bw]\!] \;\Rightarrow\; [\![R\left(\frac{d}{dt}\right)w = 0]\!]$$

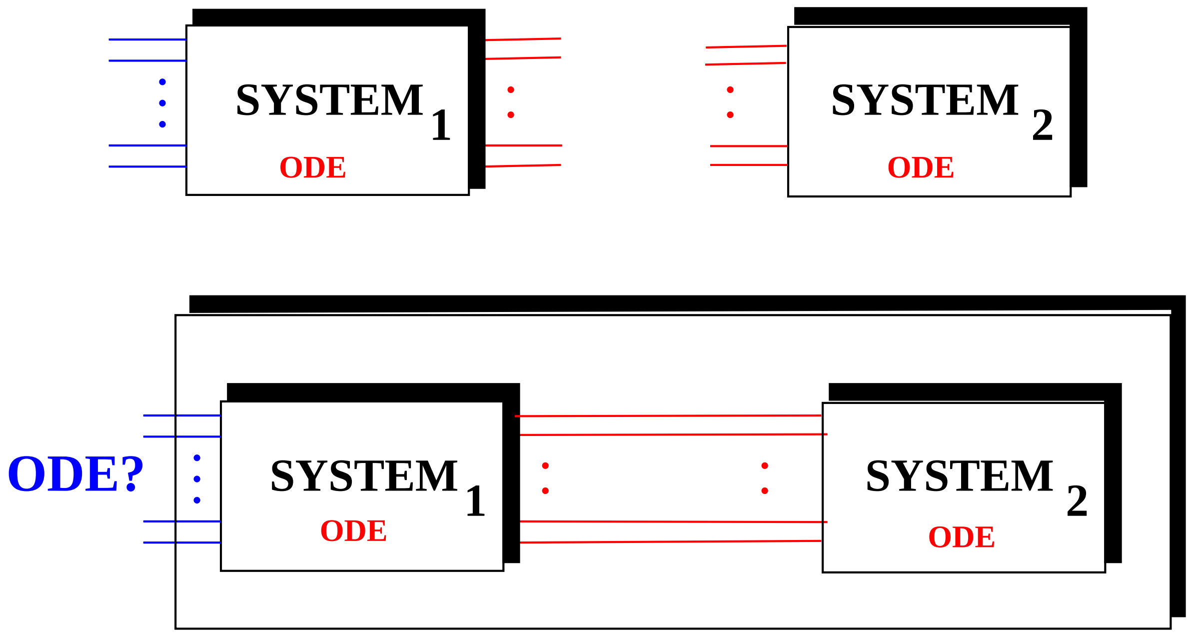
linear DAE's allow elimination of nuisance variables

$$[\![R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell]\!] \;\Rightarrow\; [\![R'\left(\frac{d}{dt}\right)w = 0]\!]$$

elimination of latent variables in LTIDSs is always possible.

$$[\![w = M\left(\frac{d}{dt}\right)\ell]\!] \;\Rightarrow\; [\![R'\left(\frac{d}{dt}\right)w = 0]\!]$$

There is no nonlinear elimination theorem



The interconnection is described by an ODE if systems 1 and 2 are LTIDSs.

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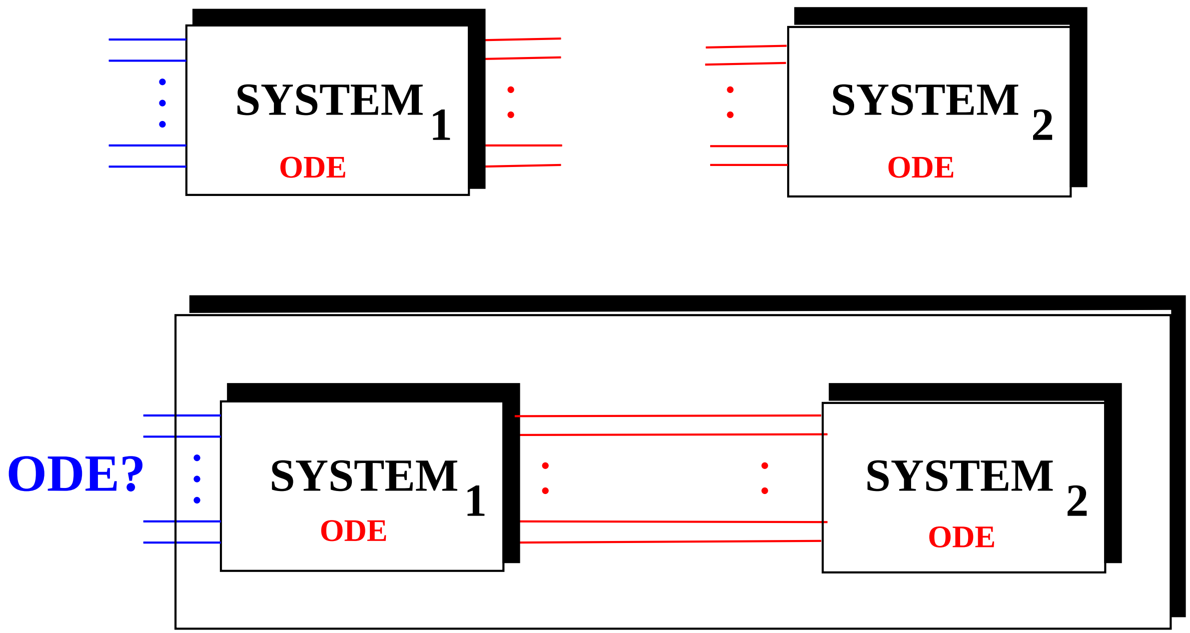

The interconnection is described by an ODE if systems 1 and 2 are LTIDSs.

In the nonlinear case, very unlikely that the interconnection is described by an ODE, even if systems 1 and 2 are!

Why are ODE's so common?

Recapitulation

- Models are usually given as equations
- First principles models invariantly contain (many) latent variables
- In LTIDSs, latent variables can be completely eliminated
- There is no nonlinear elimination theorem

Other time sets

Continuous time

The theory is identical for LTIDSs with time set

$$[0,\infty),(-\infty,0] \qquad \textbf{or} \;\; [t_1,t_2]\,.$$

The appropriate ring is still $\mathbb{R}[\xi]$

Discrete time

For discrete time systems with time axis $\mathbb{N}$ or $\mathbb{Z}_+$, the appropriate ring is still $\mathbb{R}[\xi]$.

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For discrete time systems with time axis $\mathbb{N}$ or $\mathbb{Z}_+$, the appropriate ring is still $\mathbb{R}[\xi]$.

For discrete time systems with time axis $\mathbb{Z}$, however, the appropriate ring is $\mathbb{R}[\xi,\xi^{-1}]$.

Elements of this ring are called *'Laurent polynomials'*. An element of $\mathbb{R}[\xi,\xi^{-1}]^{\mathtt{n}\times\mathtt{n}}$ is unimodular iff its determinant is a non-zero monomial.

Discrete time

Example:

$$
\begin{aligned}
w(t+1) &= w(t) & \rightsquigarrow \quad & \xi - 1 \\
w(t) &= w(t-1) & \rightsquigarrow \quad & 1 - \xi^{-1} \\
w(t+2) &= w(t+1) & \rightsquigarrow \quad & \xi^2 - \xi
\end{aligned}
$$

Discrete time

Example:

$$\begin{array}{rcllcl} w(t+1) & = & w(t) & \rightsquigarrow & \xi - 1 \\ w(t) & = & w(t-1) & \rightsquigarrow & 1-\xi^{-1} \\ w(t+2) & = & w(t+1) & \rightsquigarrow & \xi^2 - \xi \end{array}$$

All these equations are equivalent for $\mathbb{T} = \mathbb{Z}$.
Transformations:

second equation = $\xi^{-1}*$ first;
third equation = $\xi *$ first

None of these equations are equivalent for $\mathbb{T} = \mathbb{Z}_+$.
The 2nd equation does not really make sense. What is $w(0)$?

Summary of Lecture 1

The main points

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  $\mathscr{B}$ is the behavior of the model.

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- **The elimination theorem:** $\mathscr{L}^{\bullet}$ **is closed under projection.**

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End of lecture 1

Mathematical Appendix

Rings and modules

Groups

A group is a set G, with

- **a binary operation $G \times G \to G$, called *multiplication*.** **Multiplication is usually written as juxtaposition of the multiplied elements.**
- **a unary operation $^{-1} : G \to G$, called *inversion*.** **The inverse of g is written as g^{-1}.**
- **an *identity* $e \in G$ (often denoted as 1).**

These operations satisfy, for all $g, g_1, g_2, g_3 \in G$:

- $(g_1 g_2) g_3 = g_1 (g_2 g_3)$ **(multiplication is associative);** $ge = eg = g$; $gg^{-1} = g^{-1} g = e$.

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A group is called *abelian (or commutative)* if $g_1 g_2 = g_2 g_1$ for all $g_1, g_2 \in G$. For an abelian group multiplication is usually denoted as $g_1 + g_2$ (instead of $g_1 g_2$), and the identity as 0 (instead of e or 1).

Rings

A ring is a set R equipped with two binary operations

$$+ : R \times R \to R \quad \textbf{and} \quad * : R \times R \to R$$

called *addition* and *multiplication*. Multiplication is usually just written as juxtaposition of the multiplied elements, rather than with a $*$.

These operations satisfy:

- $(R,+)$ **is an abelian group with identity element 0,**
- **multiplication is associative, with identity element 1,**
- **multiplication distributes over addition.**

So, for all $a,b,c \in R$, there holds:

$(ab)c = a(bc)$ **denoted as abc, $a1 = 1a = a$,**
$a(b+c) = ab+ac, (a+b)c = ac+bc$**.**

Commutative rings

Multipication need not be commutative.
If it is, we call the ring ***commutative*****.**

Examples: $\mathbb{Z}$ **(commutative)**, $\mathbb{R}[\xi]$ **(commutative)**, $\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$,
$\mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}}$, $\mathbb{R}[\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}]$ **(commutative).**

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$\mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}}$, $\mathbb{R}[\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}]$ **(commutative).**

Every element $r \in R$ **has an additive inverse** $-r$**.**

But it need not have a muliplicative inverse. For example, in $\mathbb{Z}$ **only** 1 **and** -1 **have a multiplicative inverse.**

Units

An element $a \in R$ is called a *unit* if it is invertible with respect to multiplication: if $\exists\, b \in R$ such that $ab = ba = 1$;
b is then uniquely determined by a and is writtes as a^{-1}.
The set of all units in R forms a group under multiplication.

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The term *unimodular* for (polynomial) matrices is used as a synonym for 'unit'.

$M \in \mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}}$ is unimodular iff $\det(M)$ is a non-zero polynomial of zero degree. That is, iff $\det(M)$ is a unit in $\mathbb{R}[\xi]$.

Modules

Let R be a *commutative* ring.

A *module $\mathscr{M}$ over R* (also called an R*-module*) is abelian group $(,\mathscr{M},+)$ with an operation, called *scalar multiplication*, mapping $R\times\mathscr{M}\to\mathscr{M}$. Multiplication is usually written by juxtaposition, i.e. as rx for $r\in R$ and $x\in\mathscr{M}$.

These operations satisfy, for all $r,s\in R,$ and $x,y\in\mathscr{M}$,

- $r(x+y)=rx+ry$**,**
- $(r+s)x=rx+sx$**,**
- $(rs)x=r(sx)$ **(therefore written as rsx),**
- $1x=x$**.**

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- $(rs)x = r(sx)$ **(therefore written as rsx),**
- $1x = x$**.**

The following example is especially important for us:

$\mathbb{R}[\xi]^{\mathtt{n}}$ is a module over $\mathbb{R}[\xi]$. So is, of course, $\mathbb{R}[\xi]^{1\times\mathtt{n}}$.

Modules

There is an enormous variety of rings and modules:

principal ideal domain, finitely generated, cyclic, free, projective, injective, simple, semisimple, indecomposable, Euclidean, Noetherian, Artinian, Bézoutian, Hermitian, etc.

Like visiting the zoo.

Free modules

An R**-module** $\mathscr{M}$ **is said to be** ***finitely generated*** **if there exist a set** $\{g_1, g_2, \cdots, g_{\mathtt{r}}\}$ **of elements of** $\mathscr{M}$ **(called** ***generators*** **of** $\mathscr{M}$**) such that each element** m **of** $\mathscr{M}$ **is of the form**

$$m = c_1 g_1 + c_2 g_2 + \cdots + c_{\mathtt{r}} g_{\mathtt{r}} \text{ with } c_1, c_2, \ldots, c_{\mathtt{r}} \in R.$$

An R**-module** $\mathscr{M}$ **is said to be** ***free*** **if there exist a set of generators** $\{e_1, e_2, \cdots, e_{\mathtt{r}}\}$ **of** $\mathscr{M}$ **(called a** ***basis*** **of** $\mathscr{M}$**) such that the** $e_{\mathtt{k}}$**'s are independent, that is,**

$$c_1 e_1 + c_2 e_2 + \cdots + c_{\mathtt{r}} e_{\mathtt{r}} = 0 \text{ implies } c_1 = c_2 = \cdots = c_{\mathtt{r}} = 0$$

The cardinality of the basis is uniquely defined, and is called the ***rank***, ***order***, **or** ***dimension*** **of** $\mathscr{M}$**.**

Submodules of $\mathbb{R}[\xi]^n$

Clearly $\mathbb{R}[\xi]^n$ is an $\mathbb{R}[\xi]$-module. We are especially interested in this module and its submodules. The $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^n$ are tame animals of our zoo: they are free, have a basis, and behave very much like vector spaces.

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Let $\mathscr{M}$ be an $\mathbb{R}[\xi]$-submodule of $\mathbb{R}[\xi]^{\mathtt{n}}$. It has a basis, say, $\{e_1, e_2, \cdots, e_{\mathtt{r}}\}$. Any other basis $\{e'_1, e'_2, \cdots, e'_{\mathtt{r}}\}$ of $\mathscr{M}$ is generated by the matrix multiplication

$$\begin{bmatrix} e'_1 \\ e'_2 \\ \cdots \\ e'_{\mathtt{r}} \end{bmatrix} = U \begin{bmatrix} e_1 \\ e_2 \\ \cdots \\ e_{\mathtt{r}} \end{bmatrix}$$

with $U \in \mathbb{R}[\xi]^{\mathtt{r}\times\mathtt{r}}$ unimodular.

The Smith form

Elements of $\mathbb{R}[\xi]^{n_1 \times n_2}$ can be brought into a simple canonical form by pre- and postmultiplication by a unimodular matrix. This canonical form is called the *Smith form*.

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<u>Theorem</u> The Smith form Let $M \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$. There exist unimodular $U \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_1}$ and $V \in \mathbb{R}[\xi]^{\mathtt{n}_2 \times \mathtt{n}_2}$ such that

$$UMV = \begin{bmatrix} \operatorname{diag}(d_1, d_2, \ldots, d_{\mathtt{r}}) & 0_{\mathtt{r} \times (\mathtt{n}_2 - \mathtt{r})} \\ 0_{(\mathtt{n}_1 - \mathtt{r}) \times \mathtt{r}} & 0_{(\mathtt{n}_1 - \mathtt{r}) \times (\mathtt{n}_2 - \mathtt{r})} \end{bmatrix}$$

with $d_1, d_2, \ldots, d_{\mathtt{r}} \in \mathbb{R}[\xi]$, monic, and $d_{\mathtt{k}}$ a factor of $d_{\mathtt{k}+1}$ for $\mathtt{k} = 1, 2, \cdots, \mathtt{r} - 1$. They are called the invariant factors of M.

Canonical forms and invariants

Another bit of nice-to-know mathematics

Relations

A relation on an indexed family of sets, $S_\alpha, \alpha \in A$, is a subset of $S = \Pi_{\alpha \in A} S_\alpha$.

Think of the elements of $s_\alpha, \alpha \in A$, as being 'related' if $\Pi_{\alpha \in A} s_\alpha \in \Pi_{\alpha \in A} S_\alpha$.

'Relation' captures the notion of 'model' much better than'map'.

Note that we view a dynamical system basically as a relation among the values $w(t)$ for $t \in \$$. The behavior of a dynamical system is a relation on $\Pi_{t \in \mathbb{T}} \mathbb{W}_t$, with all the $\mathbb{W}_t$'s equal to $\mathbb{W}$.

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A binary relation involves only two sets, the cardinality of A equals 2. An $\mathtt{n}$-ary relation, $\mathtt{n}$ sets, the cardinality of $A = \mathtt{n}$.

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A common example of a binary relation is obtained from a map $f : X \to Y$, the relation being the *graph* of f:

$$\boxed{\mathtt{graph}(f) = \{(x,y) \in X \times Y | y = f(x)\}}$$

Binary relations

If in a $\mathtt{n}$-ary relation all the A_α's are equal, $A_1 = A_2 = \cdots = A_{\mathtt{n}} = A$, we call the relation an $\mathtt{n}$-ary relation on A (the term *endorelation* is also used).

A binary relation on A is thus a subset of A^2.

The notation $a_1 R a_2$, $a_1 \overset{R}{\sim} a_2$, is often used if (a_1, a_2) belongs to the binary relation $R \subseteq A^2$.
$a_1 {\sim} a_2$ is also used when it is clear what R is.

Many important binary relations are obtained from additional structure on the subset that defines the relation.

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- $a \sim a$ **(*reflexivity*)**
- $[\![a \sim b]\!] \Rightarrow [\![b \sim a]\!]$ **(*symmetry*)**
- $[\![a \sim b$ **and** $b \sim c]\!] \Rightarrow [\![a \sim c]\!]$ **(*transitivity*)**

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$\sim$ partitions A into disjoint subsets, called equivalence classes. All elements in a given equivalence class are equivalent among themselves, and no element is equivalent with any element from a different class. Think therefore of an equivalence relation on A as a partition of A into disjoint subsets and declaring elements of the same subset equivalent.

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The equivalence class of elements equivalent with a is denoted as $[a]$, $[a]_\sim$, or $[a]_R$. $a_1 = a_2 (\texttt{modulo} \ \sim)$ and $a_1 \overset{\sim}{=} a_2$ mean that a_1 and a_2 belong to the same equivalence class.

Canonical form

Let $\sim$ be an equivalence relation on A.

A canonical form (also called normal form, standard form) for $\sim$ is a subset C of A such that

$$C \cap [a]_\sim \neq \emptyset \quad \textbf{for all } a \in A.$$

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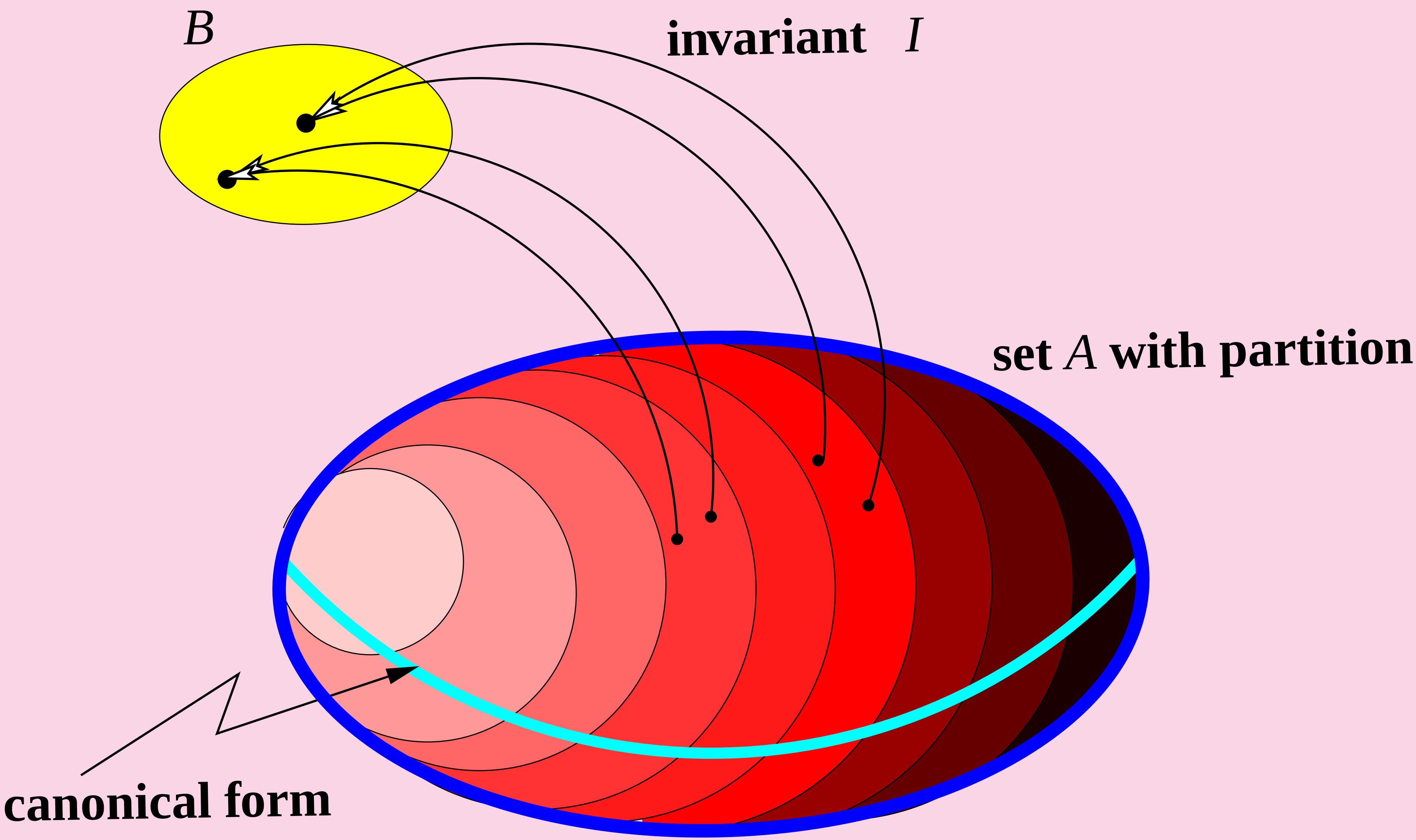
An invariant for $\sim$ is a map I from A to a set B such that

$$[\![a_1 \sim a_2]\!] \Rightarrow [\![I(a_1) = I(a_2)]\!].$$

It is said to be a complete invariant if

$$[\![a_1 \sim a_2]\!] \Leftrightarrow [\![I(a_1) = I(a_2)]\!].$$

Canonical form & invariant



Transformation group

A transformation group on A is a set of maps that form a subgroup of the bijections on A. In other words, there is a group G and a map T from G to the bijections on A, such that for all $g, g_1, g_2 \in G$, there holds:

- $T_1 = \mathtt{id}_A$ (**$\mathtt{id}_A$ denotes the identity map on A**)
- $T_{g^{-1}} = T_g^{-1}$
- $T_{g_1 g_2} = T_{g_2} \circ T_{g_1}$ (**$\circ$ denotes composition of maps**)

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The set

$$\mathscr{O}_a := \{a' \in A \mid \exists\, g \in G \text{ such that } a' = T_g(a)\}$$

is called the *orbit* of a under the transformation group T_G.

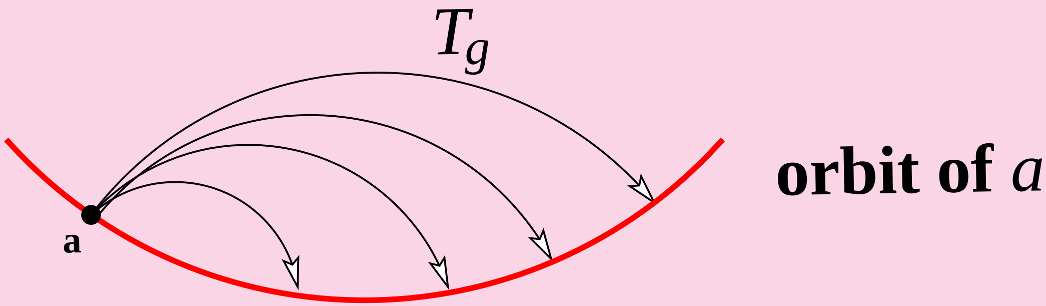
Transformation group

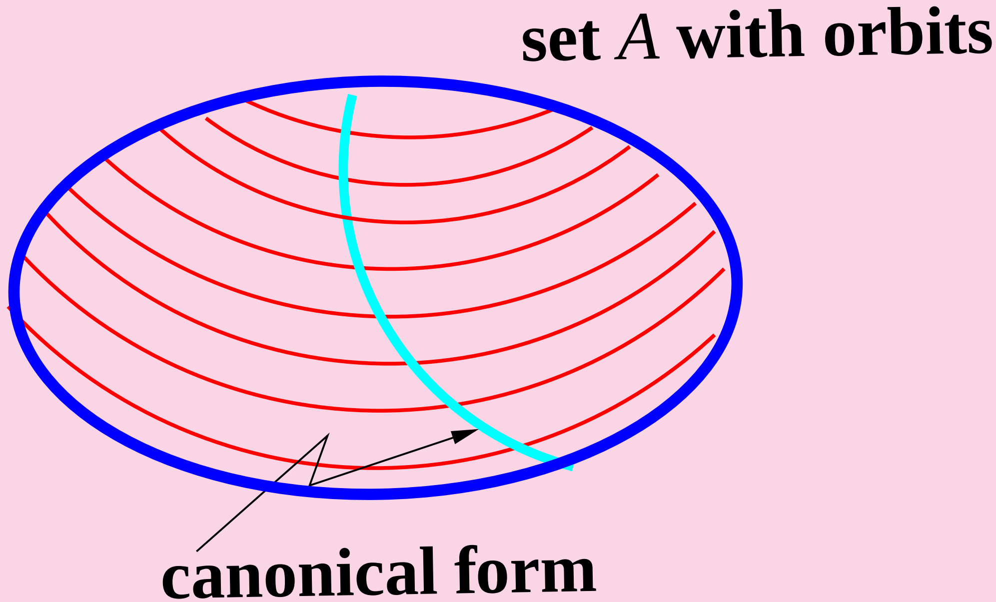
A transformation group on A induces an equivalence relation on A by declaring

$$[\![a_1 \sim a_2]\!] :\Leftrightarrow [\![a_2 \in \mathscr{O}_{a_1}]\!]$$

In other words, the partition consists of the orbits.

Transformation group





Example: the Jordan form

Denote by $\mathscr{G}\ell(\mathtt{n})$ the invertible elements of $\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.
$\mathscr{G}\ell(\mathtt{n})$ defines a transformation group on $\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ by

$$M \xrightarrow{S\in\mathscr{G}\ell(\mathtt{n})} SMS^{-1}.$$

This transformation corresponds to choosing a new basis on $\mathbb{R}^{\mathtt{n}}$, and looking how the linear transformation $M : \mathbb{R}^{\mathtt{n}} \circlearrowleft$ acts in this new basis on $\mathbb{R}^{\mathtt{n}}$.

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Invariant: $I:\mathbb{R}^{\mathtt{n}\times\mathtt{n}} \to \mathbb{R}[\xi], \quad I(M) := \mathtt{det}(I\xi - M)$
Other invariants

$I : M \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}} \mapsto$ **the set of eigenvalues of M.**
$I : M \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}} \mapsto$ **the minimal polynomial of M**
the rank, the trace, the determinant, etc.

Example: the rank as a complete invariant

Consider $\mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_2}$. Define the transformation group

$$M \quad \xrightarrow{S_1 \in \mathscr{G}\ell(\mathtt{n}_1),\ S_2 \in \mathscr{G}\ell(\mathtt{n}_2)} \quad S_1 M S_2.$$

This corresponds to looking how the linear transformation $M : \mathbb{R}^{\mathtt{n}_2} \rightarrow \mathbb{R}^{\mathtt{n}_1}$ acts like in new basis on $\mathbb{R}^{\mathtt{n}_1}$ and $\mathbb{R}^{\mathtt{n}_2}$.

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Complete invariant: the rank

Canonical form: the set of matrices of the form

$$\begin{bmatrix} I_{\mathtt{r} \times \mathtt{r}} & 0_{\mathtt{r} \times (\mathtt{n}_2 - \mathtt{r})} \\ 0_{(\mathtt{n}_1 - \mathtt{r}) \times \mathtt{r}} & 0_{(\mathtt{n}_1 - \mathtt{r}) \times (\mathtt{n}_2 - \mathtt{r})} \end{bmatrix}$$

Example: the SVD

Consider now $\mathbb{R}^{\mathtt{n}_1}$ and $\mathbb{R}^{\mathtt{n}_2}$ as *Euclidean spaces*, that is, with distances fixed by the usual Euclidean norm. Hence basis choices must respect distances. An element $M \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is said to be *orthogonal* if $M^\top M = I_{\mathtt{n}\times\mathtt{n}}$.

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Complete invariant:

$M \in \mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_2} \mapsto$ **the singular values** $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\mathtt{r}} > 0$ **of** M.

Invariants: the rank, the induced norm, the Frobenius norm, the Schatten and Ky-Fan norms, etc.

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$$\begin{bmatrix} \Sigma & 0_{\mathtt{r} \times (\mathtt{n}_2 - \mathtt{r})} \\ 0_{(\mathtt{n}_1 - \mathtt{r}) \times \mathtt{r}} & 0_{(\mathtt{n}_1 - \mathtt{r}) \times (\mathtt{n}_2 - \mathtt{r})} \end{bmatrix},$$

with Σ diagonal, with positive elements on the diagonal in non-increasing order.

See lecture 3 for more details.

Example: the Smith form