Exercises Lecture 6

Exercise 1

Let $\boldsymbol{A} \subset \mathbb{R}^{n\times n}$ be compact and quadratically stable, and suppose that $A : \mathbb{R} \to \boldsymbol{A}$ is continuous. Show that $\dot{x}(t) = A(t)x(t)$ is exponentially stable.

Exercise 2

a) For any real vector space $\mathcal{X}$ let F be a symmetric-valued convex function defined on $\mathcal{X}$, and let $\mathcal{S}$ be the convex hull of $\{x_1, \ldots, x_N\} \subset \mathcal{X}$. For $\lambda_1, \ldots, \lambda_N \in \mathbb{R}_+$ that sum up to one, show that

$$F\left(\sum_{k=1}^{N} \lambda_k x_k\right) \preccurlyeq \sum_{k=1}^{N} \lambda_k F(x_k).$$

Use this property to prove that

$$F(s) \prec 0 \ \text{ for all } \ s \in \mathcal{S} \quad \iff \quad F(x_k) \prec 0 \ \text{ for all } \ k = 1, \ldots, N.$$

b) Let $\boldsymbol{A}$ be the convex hull of $\{A_1, \ldots, A_N\} \subset \mathbb{R}^{n\times n}$. Show that $\boldsymbol{A}$ is quadratically stable iff there exists some $K \in \mathbb{R}^{n\times n}$ with

$$K \succ 0 \ \text{ and } \ A_i^T K + K A_i \prec 0 \ \text{ for all } \ i = 1, \ldots, N.$$

Exercise 3

Verify quadratic stability of the family of systems $\dot{x} = A(\delta_1, \delta_2, \delta_3)x$ with

$$A(\delta_1, \delta_2, \delta_3) = \begin{pmatrix} -1 & 2\delta_1 & 2 \\ \delta_2 & -2 & 1 \\ 3 & -1 & \delta_3 - 12 \end{pmatrix}, \quad (\delta_1, \delta_2, \delta_3) \in [-0.2, 0.4] \times [-0.8, 0.4] \times [-3, 3].$$

Exercise 4

For real $m \times m$-matrices $Q = Q^T$, $R = R^T$, S define

$$L := \left\{ z \in \mathbb{C} : \begin{pmatrix} I_m \\ zI_m \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} I_m \\ zI_m \end{pmatrix} \prec 0 \right\}.$$

a) If $A \in \mathbb{R}^{n\times n}$, show that $\lambda(A) \subset L$ iff there exists $n \times n$ matrix K with

$$K \succ 0 \ \text{ and } \ \begin{pmatrix} I \\ A \otimes I_m \end{pmatrix}^T \begin{pmatrix} K \otimes Q & K \otimes S \\ K \otimes S^T & K \otimes R \end{pmatrix} \begin{pmatrix} I \\ A \otimes I_m \end{pmatrix} \prec 0.$$

Hint: You just need to mimic the proof on the slides for the open left-half plane and use the usual rules for the Kronecker product. You can assume that A can be diagonalized by similarity.

b) Consider the LMI-region defined by

$$Q = \begin{pmatrix} -1 & 1 \\ 1 & 10 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & -2 \\ -2 & -10 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Test whether

$$\begin{pmatrix} -0.42 & 1 & 0.45 \\ -0.48 & -1.9 & -0.45 \\ -0.22 & -0.38 & -1.1 \end{pmatrix}$$

has all its eigenvalues in the LMI region by solving an LMI problem.