# Solution to the exercises for lectures 12 and 14

**Exercise 1**

**1.1** From the fact that $X(\xi)U(\xi)^{-1} = (\xi I - A)^{-1}B$ it follows that $\xi X(\xi) = AX(\xi) + BU(\xi)$. Consequently

$$(\zeta+\eta)X(\zeta)^T K X(\eta) = X(\zeta)^T A^T K X(\eta) + U(\zeta)^T B^T K X(\eta) + \\ X(\zeta)^T K A X(\eta) + X(\zeta)^T K B U(\eta),$$

which can be rewritten as

$$\begin{pmatrix} X(\zeta)^T & U(\zeta)^T \end{pmatrix} \begin{pmatrix} A^T K + KA & KB \\ B^T K & 0 \end{pmatrix} \begin{pmatrix} X(\eta) \\ U(\eta) \end{pmatrix}.$$

**1.2** Using the result proven in the previous point, and the definition of $\Phi(\zeta,\eta)$ given in the text in order to arrive at

$$(\zeta+\eta)X(\zeta)^T K X(\eta) = \Phi(\zeta,\eta) \\ - \begin{pmatrix} X(\zeta)^T & U(\zeta)^T \end{pmatrix} \begin{pmatrix} Q - A^T K - KA & -KB + S^T \\ -B^T K + S & R \end{pmatrix} \begin{pmatrix} X(\eta) \\ U(\eta) \end{pmatrix}.$$

We now prove the lemma suggested in the hint. This will lead us immediately to the conclusion.
Let $(x_0, u_0) \in \mathbb{R}^{\mathtt{n}} \times \mathbb{R}^{\mathtt{m}}$. Let $\tilde{u} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$ be such that $\tilde{u}(0) = u_0$. Consider the differential equation $\dot{x} = Ax + B\tilde{u}$, $x(0) = x_0$, and let $\tilde{x} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ be its solution. Evidently, $\mathrm{col}(\tilde{x}, \tilde{u}) \in \mathrm{im} \begin{pmatrix} X(\frac{d}{dt}) \\ U(\frac{d}{dt}) \end{pmatrix}$, so there exists $\ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$ such that

$$\begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} X(\frac{d}{dt}) \\ U(\frac{d}{dt}) \end{pmatrix} \ell.$$

Consequently,

$$\begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} \tilde{x}(0) \\ \tilde{u}(0) \end{pmatrix} = \begin{pmatrix} (X(\frac{d}{dt})\ell)(0) \\ (U(\frac{d}{dt})\ell)(0) \end{pmatrix}.$$

The surjectivity of the map $\ell \mapsto \mathrm{col}((X(\frac{d}{dt})\ell)(0), (U(\frac{d}{dt})\ell)(0))$ shows that the QDF induced by

$$\begin{pmatrix} X(\zeta)^T & U(\zeta)^T \end{pmatrix} \begin{pmatrix} Q - A^T K - KA & -KB + S^T \\ -B^T K + S & R \end{pmatrix} \begin{pmatrix} X(\eta) \\ U(\eta) \end{pmatrix}$$

is nonnegative (and consequently, it is a *bona fide* dissipation function) if and only if

$$\begin{pmatrix} Q - A^T K - KA & -KB + S^T \\ -B^T K + S & R \end{pmatrix}$$

is nonnegative.

**1.3** $H$ is semi-Hurwitz, meaning its determinant is a semi-Hurwitz polynomial, thus nonzero. Now if $H \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi]$ is a nonsingular polynomial matrix, then its rows are linearly independent over the field of rational functions, and *a fortiori* over $\mathbb{R}$. It follows from this that also the rows of $\tilde{H}$ are linearly independent. Consequently, $\tilde{H}^\top \tilde{H}$ has rank $\mathtt{m}$.

By the result suggested in the hint for Question 1.1, the mapping

$$\ell \mapsto \mathrm{col}((X(\frac{d}{dt})\ell)(0), (U(\frac{d}{dt})\ell)(0))$$

is surjective. It follows that the coefficient matrix of $\mathrm{col}(X(\xi), U(\xi))$ has full row rank. Consequently, the matrix

$$\begin{pmatrix} Q - A^T K - KA & -KB + S^T \\ -B^T K + S & R \end{pmatrix}$$

has rank $\mathtt{m}$.

**1.4** The answer follows immediately from the hint given.

**Exercise 2**

**2.1** We prove only the "only if" part, since the other implication follows easily from the hint.

That $\mathcal{V}$ is a linear space follows easily from the fact that $\mathfrak{B}$ is a linear subspace of the set of sequences $(x, w)$. Now if $\mathcal{V}$ is a linear space, then it admits a representation as the kernel of a matrix. In order to conclude the proof, partition the matrix according to the partition of the vectors of $\mathcal{V}$ as

$$\begin{bmatrix} E & F & G \end{bmatrix}.$$

**2.2** **2.2.a** The easiest way is to write the matrix $R(\xi)$ as

$$R(\xi) = \mathrm{diag}(\xi^{\delta_i})_{i=1,\ldots,\mathtt{p}} R_{\mathrm{hc}} + R'(\xi)$$

where $R'(\xi)$ is a matrix whose row-degrees are strictly lower than those of $R(\xi)$. Since $R_{\mathrm{hc}}$ is full row rank, it has a nonsingular submatrix $P$. It is readily seen that the submatrix of $R(\xi)$ consisting of the columns corresponding to those of $P$ has determinant

$$\det(P)\xi^{\sum_{i=1}^{\mathtt{p}} \delta_i} + \text{terms of lower order}$$

Observe also that any other selection of a square submatrix of $R(\xi)$ (and consequently of a minor of $R(\xi)$) corresponds to a selection of a square submatrix of $R_{\text{hc}}$. If this latter submatrix is nonsingular, it follows from the argument used before that the determinant of the corresponding square submatrix of $R(\xi)$ has degree $\sum_{i=1}^{\mathtt{p}} \delta_i$; otherwise, this determinant will have lower degree.

**2.2.b** Use the representation

$$R(\xi) = \text{diag}(\xi^{\delta_i})_{i=1,\ldots,\mathtt{p}} R_{\text{hc}} + R'(\xi)$$

and apply the shift-and-cut operation to $R(\xi)$. After $k$ iterations of the shift-and-cut operation, the only nonzero rows of $\sigma_+^k(R)$ are those corresponding to row degrees higher than or equal to $k$. Also, because of the fact that $R_{\text{hc}}$ has full row rank, the nonzero rows of $\sigma_+^k(R)$ are linearly independent. This yields the claim, since there are exactly $\delta_i$ nonzero shift-and-cuts of the $i$-th row of $R$.

**2.2.c** The claim follows straightforwardly from the result proved in 2.2.b.

**Exercise 3**

**3.1** From the dissipation equality we know that $\Phi(-\xi,\xi) = \Delta(-\xi,\xi)$, with $\Delta$ the two-variable $1 \times 1$ polynomial matrix inducing the dissipation rate. The supply rate $Q_\Phi(w,F) = F\frac{d}{dt}w$ is represented by the two-variable polynomial matrix

$$\Phi(\zeta,\eta) = \begin{bmatrix} 0 & \frac{1}{2}\zeta \\ \frac{1}{2}\eta & 0 \end{bmatrix}.$$

Working with an image representation of the system allows us to avoid studying the QDFs along the behavior of the system, and to consider them acting on the whole $\mathfrak{C}^\infty$ instead. An observable image representation of the system is induced by

$$M(\xi) = \begin{bmatrix} 1 \\ m\xi^2 + c\xi + k \end{bmatrix}$$

The QDF $Q_{\Phi'}$ acting on the latent variable $\ell$, such that $Q_{\Phi'}(\ell) = Q_\Phi(w)$ for all $w$ and $\ell$ related by $w = M\left(\frac{d}{dt}\right)\ell$ is induced by the two-variable polynomial

$$\begin{aligned} \Phi'(\zeta,\eta) &= M(\zeta)^\top \Phi(\zeta,\eta) M(\eta) \\ &= \eta\frac{k}{2} + c\eta\zeta + \zeta\frac{k}{2} + \frac{1}{2}m\eta^2\zeta + \frac{1}{2}m\zeta^2\eta \end{aligned}$$

In order to find the dissipation function, we first compute

$$\Phi'(-\xi, \xi) = -c\xi^2$$

Notice that $\Phi'(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$, and consequently that a spectral factorization of this polynomial exists. It is straightforward to compute it as

$$\Phi'(-\xi, \xi) = \left(-\sqrt{c}\xi\right)\left(\sqrt{c}\xi\right)$$

Note that this is the only possible factorization of the polynomial as $p(-\xi)p(\xi)$; conclude then that the (only) dissipation function is

$$\sqrt{c}\zeta\sqrt{c}\eta = c\zeta\eta$$

In order to see what the physical interpretation of this dissipation function is, notice that the latent variable $\ell$ in the image representation equals the displacement $w$ of the mass- consequently this dissipation function corresponds to

$$c\left(\frac{d}{dt}w\right)^2.$$

Now the dissipation equality states that

$$\Phi'(\zeta, \eta) - \Delta(\zeta, \eta) = (\zeta + \eta)\Psi(\zeta, \eta)$$

with $Q_\Psi$ being the storage function corresponding to $Q_\Delta$. In the case at hand

$$\left(\eta\frac{k}{2} + c\eta\zeta + \zeta\frac{k}{2} + \frac{1}{2}m\eta^2\zeta + \frac{1}{2}m\zeta^2\eta\right) - (c\zeta\eta) = (\zeta+\eta)\left(\frac{k}{2} + \frac{1}{2}m\eta\zeta\right)$$

**3.2 3.2.a,3.2.b** The two-variable polynomials corresponding to the kinetic, respectively potential energy are:

$$\begin{aligned} E_{kin}(\zeta, \eta) &= \frac{1}{2}m\eta\zeta \\ E_{pot}(\zeta, \eta) &= \frac{k}{2} \end{aligned}$$

These two-variable polynomials act on the external variable $w$ of the system described by

$$m\frac{d^2}{dt^2}w + c\frac{d}{dt}w + cw = 0$$

The total energy is induced by $E(\zeta, \eta) = E_{kin}(\zeta, \eta) + E_{pot}(\zeta, \eta) = \frac{1}{2}m\eta\zeta + \frac{k}{2}$. The dissipated energy is $c\left(\frac{d}{dt}w\right)^2$, corresponding to the two-variable polynomial $c\zeta\eta$.

**3.2.c** The derivative of the total energy is

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}m\left(\frac{d}{dt}w\right)^2+\frac{k}{2}w^2\right) &= m\frac{d}{dt}w\frac{d^2}{dt^2}w+kw\frac{d}{dt}w\\ &= \left(m\frac{d^2}{dt^2}w+kw\right)\frac{d}{dt}w\\ &= \left(m\frac{d^2}{dt^2}w+kw+c\frac{d}{dt}w\right)\frac{d}{dt}w\\ &- c\left(\frac{d}{dt}w\right)^2\end{aligned}$$

Note that since the equation governing the behavior of the system is $m\frac{d^2}{dt^2}w+kw+c\frac{d}{dt}w=0$, it follows that along the trajectories of the system the derivative of the total energy is indeed equal to $-c\left(\frac{d}{dt}w\right)^2$.

**3.2.d** In polynomial terms: the derivative of the total energy is induced by

$$\begin{aligned}&(\zeta+\eta)\left(\frac{1}{2}m\eta\zeta+\frac{k}{2}\right)=\left(\frac{1}{2}m\eta\zeta^2+\frac{k}{2}\eta\right)+\left(\frac{1}{2}m\eta^2\zeta+\frac{k}{2}\zeta\right)\\ =\;&\frac{1}{2}\left(m\zeta^2+\frac{k}{2}\right)\eta+\zeta\frac{1}{2}\left(m\eta^2+\frac{k}{2}\right)\\ =\;&\frac{1}{2}\left(m\zeta^2+\frac{k}{2}+c\zeta\right)\eta+\zeta\frac{1}{2}\left(m\eta^2+\frac{k}{2}+c\eta\right)-c\zeta\eta\end{aligned}$$

and in the first two terms of the last expression we recognize the "tail" corresponding to the kernel representation of the system

$$m\xi^2+c\xi+k.$$

In general, the computations involved are not easy to carry out with pen and paper. However, Gröbner bases techniques can be used on software packages for symbolic computation for this purpose.

**3.2e** Integrating:

$$\int_0^T-\left(D\left(\frac{d}{dt}\right)w\right)^\top\left(D\left(\frac{d}{dt}\right)w\right)dt=Q_\Psi(w)(T)-Q_\Psi(w)(0)$$

Since $Q_\Psi(w)\geq 0$ for all $w\in\mathfrak{B}$, it follows from this equality that

$$\int_0^T-\left(D\left(\frac{d}{dt}\right)w\right)^\top\left(D\left(\frac{d}{dt}\right)w\right)dt\geq -Q_\Psi(w)(0)$$

Since this inequality hols for all $T\in\mathbb{R}_+$, it follows that

$$\int_0^\infty\left(D\left(\frac{d}{dt}\right)w\right)^\top\left(D\left(\frac{d}{dt}\right)w\right)dt$$

is finite. Now an argument by contradiction will lead us to prove the claim. Indeed, assume that $ve^{\lambda t}$ is (the value at $t$ of) a trajectory in $\mathfrak{B}$, with $v \in \mathbb{C}^{\mathtt{w}}$, $v \neq 0$. Observe that then $R(\lambda)ve^{\lambda t} = 0$ for all $t$, equivalently $R(\lambda)v = 0$.
Now $D\left(\frac{d}{dt}\right) ve^{\lambda t} = D(\lambda)ve^{\lambda t}$, and the integral

$$\int_0^\infty \left(D\left(\frac{d}{dt}\right)w\right)^\top \left(D\left(\frac{d}{dt}\right)w\right) dt$$

can only be finite if $\lambda \in \mathbb{R}_-$, or if $D(\lambda)v = 0$. In the latter case though, it would hold

$$\begin{bmatrix} R(\lambda) \\ D(\lambda) \end{bmatrix} v = 0$$

a contradiction.

**Exercise 4**

Consider the behavior described in kernel form by the equation

$$p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u$$

where

$$\begin{aligned} p(\xi) &= p_0 + p_1\xi + \ldots + p_n\xi^n \\ q(\xi) &= q_0 + q_1\xi + \ldots + q_n\xi^n \end{aligned}$$

**4.1 4.1.a**

$$X(\xi) = \begin{bmatrix} p_1 + \ldots + p_n\xi^{n-1} & -q_1 - \ldots - q_n\xi^{n-1} \\ \vdots & \vdots \\ p_n & -q_n \end{bmatrix}$$

**4.1.b** The matrix induces a minimal state map since its rows are linearly independent (and of course generate the space $\Xi_R$ (mod $R$)).

**4.1.c** You can check that the equations

$$\begin{aligned} \xi X(\xi) &= AX(\xi) + B\begin{bmatrix}0 & 1\end{bmatrix} + F(\xi)R(\xi) \\ \begin{bmatrix}1 & 0\end{bmatrix} &= CX(\xi) + D\begin{bmatrix}0 & 1\end{bmatrix} + G(\xi)R(\xi) \end{aligned}$$

hold, with

$$F(\xi) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and } G(\xi) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

**4.1.d** The easiest way to obtain the answer is to proceed as follows. If we perform a nonsingular transformation $x \rightsquigarrow Tx$ of the state variable, and if the state equations are

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

then the equations in the new state space basis are

$$\begin{aligned} \frac{d}{dt}Tx &= TAT^{-1}Tx + TBu \\ y &= CT^{-1}Tx + Du \end{aligned}$$

and consequently the new matrices are $A' = TAT^{-1}$, $B' = TB$, $C' = CT^{-1}$, $D' = D$. Apply this now with $T = \Pi$.

**4.2 4.2.a** The rows of $X(\xi)$ are evidently linearly independent. If we prove that they span the space $\Xi_R \pmod{R}$, we will have shown that the matrix $X(\xi)$ is a minimal state map. In order to do this, compare the coefficients of equal powers of $\xi$ in

$$q(\xi) = (h_0 + h_1\xi^{-1} + \ldots)p(\xi)$$

to conclude that

$$q_k = \sum_{i=0}^{n-k} h_i p_{k+i}$$

$k = 0, \ldots, n$. From these equalities it follows that if we call $X'(\xi)$ the state map discussed in 4.1.a, then the following relation exists between it and $X(\xi)$:

$$X'(\xi) = \begin{bmatrix} p_1 & \cdots & \cdots & \cdots & p_{n-1} & p_n \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ p_{n-1} & p_n & 0 & 0 & \cdots & 0 \\ p_n & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} X(\xi)$$

Observe that this matrix is nonsingular since $p_n \neq 0$. Now use the fact that the rows of the matrix $X(\xi)$ span $\Xi_R \pmod{R}$ in order to conclude that also the rows of $X'(\xi)$ have the same property.

**4.2.b** Proceeding as described in 4.1.d, we conclude that the new matrices are $A' = TAT^{-1}$, $B' = TB$, $C' = CT^{-1}$, $D' = D$, where

$$T = \begin{bmatrix} p_1 & \cdots & \cdots & \cdots & p_{n-1} & p_n \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ p_{n-1} & p_n & 0 & 0 & \cdots & 0 \\ p_n & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$