Exercises lectures 12 and 14

August 29, 2008

Exercise 1

The purpose of this exercises is to work out the computations leading to the linear matrix inequality on slide 118 of Lecture 14, and to the algebraic Riccati equation of slide 122 of the same lecture.

1.1 Let $K = K^\top \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, and consider the QDF associated with the two-variable polynomial matrix $X(\zeta)^\top KX(\eta)$. Show that if A and B are the state-, respectively input matrix corresponding to X and U, then

$$\begin{aligned}(\zeta+\eta)X(\zeta)^\top KX(\eta) &= X(\zeta)^T A^T KX(\eta) + U(\zeta)^T B^T KX(\eta)\\ &\quad +X(\zeta)^T KAX(\eta) + X(\zeta)^T KBU(\eta)\end{aligned}$$

(*Hint:* Use the fact that $X(\xi)U(\xi)^{-1} = (\xi I - A)^{-1}B$).

1.2 Define $\Phi(\zeta,\eta)$ as in the slides. Show that K is such that $X(\zeta)^\top KX(\eta)$ induces a storage function for Q_Φ, if and only if the LMI holds.
(*Hint:* Show that the map

$$\begin{aligned}\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{m}}) &\to \mathbb{R}^{\mathtt{n}}\times\mathbb{R}^{\mathtt{m}}\\ \ell &\mapsto \begin{pmatrix}(X(\frac{d}{dt})l)(0)\\ (U(\frac{d}{dt})l)(0)\end{pmatrix}\end{aligned}$$

is surjective. Then use the result proven in 1.1.)

1.3 Prove that the matrix

$$\begin{pmatrix} Q - A^T K - KA & -KB + S^T\\ -B^T K + S & R\end{pmatrix}$$

has rank $\mathtt{m}$.
(*Hint*: Denote with $H \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi]$ a semi-Hurwitz spectral factor of $\Phi(-\xi,\xi)$. Prove that since $H(\xi) = H_0 + H_1\xi + \ldots + H_L\xi^L$ is nonsingular, the coefficient matrix

$$\tilde{H} := \begin{bmatrix} H_0 & H_1 & \ldots & H_L\end{bmatrix}$$

has full row rank.)

1.4 Prove that if $R > 0$ then the algebraic Riccati equation holds.
(*Hint:* Write the Schur complement of R in the matrix of the LMI.)

Exercise 2

The purpose of this exercise is to familiarize the student with several aspects of the theory regarding state space systems.

2.1 In this exercise we prove the "first order in x, zero-th order in w" property for discrete-time state systems.
Let $\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{x}}, \mathfrak{B}_{\text{full}})$ be a discrete-time latent variable system. Assume that it is *complete*, i.e. that

$$w \in \mathfrak{B} \Longleftrightarrow w\mid_{[t_0,t_1]} \in \mathfrak{B}\mid_{[t_0,t_1]} \text{ for all } -\infty < t_0 \geq t_1 < \infty.$$

Prove that Σ is a state system if and only if there exist $E, F, G \in \mathbb{R}^{\bullet\times\bullet}$ such that $\mathfrak{B}_{\text{full}}$ can be described by

$$E\sigma x + Fx + Gw = 0.$$

(*Hint:* For the "only if" part, define

$$\mathcal{V} := \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid \exists (x, w) \in \mathfrak{B}_{\text{full}} \text{ s. t. } \begin{bmatrix} x(1) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\}$$

Prove that $\mathcal{V}$ is a linear space.
The "if" part can be proved by induction, using the state property and the completeness of $\mathfrak{B}$.)

2.2 In this exercise we define the notion of *McMillan degree* of a system, the minimal dimension of any state variable for the system, and we relate the McMillan degree to properties of any polynomial matrix inducing a kernel representation.
We first define the notion of *degree of a polynomial row vector*. Let

$$r = \begin{bmatrix} r_1 & \dots & r_{\mathtt{w}} \end{bmatrix} \in \mathbb{R}^{1\times\mathtt{w}}[\xi];$$

then δ is the *degree of* r if

$$\delta = \max\{d \mid d = \deg(r_i), i = 1, \dots, \mathtt{w}\}.$$

Note that if $\delta = \deg(r)$, then

$$r(\xi) = \xi^{\delta} r_{\text{hc}} + r'(\xi)$$

where $r_{\text{hc}} \in \mathbb{R}^{1\times\mathtt{w}}$ and $\deg(r') < \delta$. We call r_{hc} the *highest coefficient* of r. Given a matrix $R = \text{col}(r_i)_{i=1,\dots,\mathtt{p}}$, we write

$$r_i(\xi) = \xi^{\delta_i} r_{i,\text{hc}} + r_i'(\xi)$$

with $\delta_i = \deg(r_i)$ and $\deg(r') < \delta_i$, $i = 1, \ldots, \mathtt{p}$. We call

$$R_{\rm hc} := {\rm col}(r_{i,{\rm hc}})_{i=1,\ldots,\mathtt{p}}$$

the *highest row coefficient matrix of* R. A matrix R is called *row-reduced* if its highest row coefficient matrix has full rank.

It can be shown that if R is a polynomial matrix of full row rank, then there exists a unimodular matrix U such that UR is row-reduced. Also, it can be shown that if R_1 and R_2 are row-reduced, with row-degrees arranged in e.g. ascending order, and if $R_1 = UR_2$ for some unimodular U, then the row-degrees of R_1 and R_2 are the same.

2.2.a

Prove that if R is row-reduced with row degrees δ_i, $i = 1, \ldots, \mathtt{p}$, then its maximal degree $\mathtt{p} \times \mathtt{p}$ minor has degree equal to $\sum_{i=1}^{\mathtt{p}} \delta_i$.

It follows from the discussion above that $\sum_{i=1}^{\mathtt{p}} \delta_i$ is an invariant of any row-reduced matrix R inducing a kernel representation of $\mathfrak{B} = \ker(R)$. We call this quantity the *McMillan degree of* R or *of* $\mathfrak{B}$, and we denote it with $\mathtt{n}(\mathfrak{B})$.

2.2.b

Assume that R is row-reduced with row degrees δ_i, $i = 1, \ldots, \mathtt{p}$, and denote with Σ_R the polynomial matrix obtained stacking the results of the shift-and-cut map:

$$\Sigma_R := {\rm col}\left(\sigma_+^i(R)\right)_{i=1,\ldots} .$$

Prove that the subspace of $\mathbb{R}^{1\times \mathtt{w}}[\xi]$

$$\Xi_R = \{f \in \mathbb{R}^{1\times \mathtt{w}}[\xi] \mid \exists\ \alpha \in \mathbb{R}^{1\times \bullet} \text{ s.t. } f = \alpha\Sigma_R\}$$

has dimension equal to $\mathtt{n}(\mathfrak{B})$.

2.2.c

Prove that if $\mathfrak{B} = \ker(R)$ and R is row-reduced, then a minimal state map for $\mathfrak{B}$ can be computed selecting the nonzero rows of

$$\Sigma_R := {\rm col}\left(\sigma_+^i(R)\right)_{i=1,\ldots} .$$

Exercise 3

The purpose of this exercise is to familiarize the student with some of the two-variable polynomial algebra concepts used in the analysis of physical systems.

Consider the mechanical system in Figure 1.

The equation for this system is:

$$m\frac{d^2w}{dt^2} + c\frac{d}{dt}w + kw - F \quad = \quad 0$$

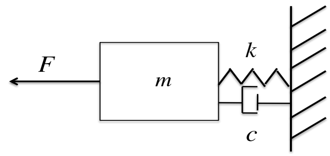

Figure 1: The mechanical system for exercise 3

3.1 Using only the calculus of quadratic differential forms (*not* physical insight!), write down the dissipation equality for this system, corresponding to the supply rate $Q_\Phi(w, F) = F\frac{d}{dt}w$.
(*Hint:* You may find it easier to work with an image representation of the system.)

3.2 A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called *asymptotically stable* if $\lim_{t\to\infty} w(t) = 0$ for all $w \in \mathfrak{B}$. It is easy to see that if $\mathfrak{B}$ is asymptotically stable, then it is autonomous. Assume now that $F = 0$ in the system of Figure 1, i.e. no external influence is exerted on the system. In this exercise we study the asymptotic stability of the system using QDFs.

3.2.a Using your physical insight, write an expression for the total energy of the system. Write also the two-variable polynomial matrix corresponding to the total energy.

3.2.b Using your physical insight, write an expression for the energy dissipated in the system. Write also the two-variable polynomial matrix corresponding to the dissipated energy.

3.2.c Prove that for every trajectory of the system, the derivative of the total energy equals the opposite of the dissipated energy.

3.2.d We now translate the conclusion of the previous point in terms of calculus of QDFs. Let $E, \Delta \in \mathbb{R}[\zeta, \eta]$ be the two-variable polynomials inducing the total energy and the dissipated energy for the system at hand. Let $R \in \mathbb{R}[\xi]$ be the polynomial corresponding to the kernel representation of the system obtained by letting $F = 0$. Prove that there exists $Y \in \mathbb{R}[\zeta, \eta]$ such that

$$(\zeta + \eta)E(\zeta, \eta) = -\Delta(\zeta, \eta) + R(\zeta)^\top Y(\zeta, \eta) + Y(\eta, \zeta)^\top R(\eta)$$

3.2e Prove the following statement: let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, and assume that there exists $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ such that

(i) $Q_\Psi(w) \geq 0$ for all $w \in \mathfrak{B}$;

(ii) there exists $D \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ such that

$$\frac{d}{dt}Q_\Psi(w) = -\left(D\left(\frac{d}{dt}\right)w\right)^\top D\left(\frac{d}{dt}\right)w$$

for all $w \in \mathfrak{B}$, and $\operatorname{rank}(\operatorname{col}(D(\lambda), R(\lambda))) = \mathtt{w}$. Then $\mathfrak{B}$ is asymptotically stable.

(*Hint*: Integrate the relation $\frac{d}{dt}Q_\Psi(w) = -\left(D\left(\frac{d}{dt}\right)w\right)^\top D\left(\frac{d}{dt}\right)w$ between 0 and T.)

Exercise 4

In this exercise we consider single-input, single-output systems, and we study how two classical "canonical realizations" (characterized by specially structured matrices A, B, C, and D) arise in a natural way by choosing a suitable state map.

Consider the behavior described in kernel form by the equation

$$p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u$$

where

$$\begin{aligned} p(\xi) &= p_0 + p_1\xi + \ldots + p_n\xi^n \\ q(\xi) &= q_0 + q_1\xi + \ldots + q_n\xi^n \end{aligned}$$

where we assume that $p_n \neq 0$.

4.1 We begin by applying the shift-and-cut operation and deriving the state equations for the corresponding state map.

4.1.a Write the polynomial matrix $X \in \mathbb{R}^{n\times 2}[\xi]$ obtained by applying the shift-and-cut map to the matrix

$$\begin{bmatrix} p(\xi) & -q(\xi) \end{bmatrix}$$

4.1.b Is $X(\xi)$ obtained in this way a minimal state map? Explain.

4.1.c Verify that the matrices A, B, C, and D corresponding to this state map are

$$A = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & -\frac{p_0}{p_n} \\ 1 & 0 & 0 & \ldots & 0 & -\frac{p_1}{p_n} \\ \vdots & \ddots & \ddots & \ddots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -\frac{p_{n-2}}{p_n} \\ 0 & 0 & 0 & \ldots & 1 & -\frac{p_{n-1}}{p_n} \end{bmatrix} \qquad B = \begin{bmatrix} q_0 - \frac{p_0 q_n}{p_n} \\ q_1 - \frac{p_1 q_n}{p_n} \\ \vdots \\ q_{n-2} - \frac{p_{n-2} q_n}{p_n} \\ q_{n-1} - \frac{p_{n-1} q_n}{p_n} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & \frac{1}{p_n} \end{bmatrix} \qquad D = \frac{q_n}{p_n}$$

4.1.d If the shift-and-cut matrix is multiplied on the left by the permutation matrix

$$\Pi := \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

or equivalently, if we reverse the order of the rows of $X(\xi)$, a different set of matrices (A', B', C', D') describes the system in input-state-output form. How do these matrices look like?
The matrices A', B', C', D' form the so-called *observer canonical form.*

4.2 Let

$$\frac{q(\xi)}{p(\xi)} = h_0 + h_1\xi^{-1} + \ldots + h_n\xi^{-n} + \ldots$$

be the power series expansion at infinity of the rational function $\frac{q(\xi)}{p(\xi)}$. The numbers h_i, $i = 0, \ldots$, are called the *Markov parameters* of the "transfer function" $\frac{q(\xi)}{p(\xi)}$.

4.2.a Define the polynomial matrix

$$X(\xi) := \begin{bmatrix} 1 & -h_0 \\ \xi & -h_1 - h_0\xi \\ \vdots & \vdots \\ \xi^{n-1} & -h_{n-1} - h_{n-2}\xi - \ldots - h_0\xi^{n-1} \end{bmatrix}$$

Prove that this matrix is a minimal state map for the system.

4.2.b Find the matrices A, B, C, D corresponding to the state map X.