Summer Course

Linear System Theory

Control

&

Matrix Computations

Monopoli, Italy

September 8-12, 2008

Exercises, Monday September 8, 2008

SOLUTIONS

Exercise 1 (Discrete-time systems)

The aim of this exercise is

- *to generalize some of the results for differential equations to difference equations*
- *to introduce Laurent polynomials*
- *to illustrate the small difference regarding unimodularity that occurs.*

In the land of difference equations some citizens like to use forward differences, others use backward differences, and some comrades even think that using *z*- transforms mysteriously clarifies matters. The most liberal democratic attitude (when working on the time-axis $\mathbb{Z}$) is to use both forward and backward lags. Of course, when working with $\mathbb{Z}_+$ forward differences are the only possibility.

A (real) *Laurent polynomial* is a 'polynomial' that contains both positive and negative powers of the indeterminate ξ, i.e., an expression of the type

$$p(\xi,\xi^{-1}) = \sum_{k\in\mathbb{Z}} p_k \xi^k$$

with the p_k's $\in \mathbb{R}$, and all but a finite number of them zero. The set of real Laurent polynomials is denoted by $\mathbb{R}[\xi,\xi^{-1}]$. Under the obvious definitions of addition and multiplication, $\mathbb{R}[\xi,\xi^{-1}]$ becomes a commutative ring.

Note this instance of the strange habit of mathematicians to associate the names of their heros with trivialities. It stands to reason that if Pierre Alphonse Laurent (1813 – 1854) would have wanted to be remembered by posterity, it would have been for more that the fact that he introduced polynomials with negative powers.

Laurent polynomials are sometimes called *dipolynomials*.

1. An element u of a ring R with an identity 1 is said to be *unimodular* (or a *unit*) if there exists $v \in R$ such that $uv = vu = 1$. Which elements of $\mathbb{R}[\xi,\xi^{-1}]$ are unimodular? Which elements of $\mathbb{R}^{n\times n}[\xi,\xi^{-1}]$ are unimodular? Contrast this with the unimodular elements of $\mathbb{R}[\xi], \mathbb{R}^{n\times n}[\xi]$.

Assume that (in the obvious notation)

$$(u_\ell \xi^\ell + u_{\ell+1}\xi^{\ell+1} + \cdots + u_L \xi^L)(v_{\ell'}\xi^{\ell'} + v_{\ell'+1}\xi^{\ell'+1} + \cdots + v_{L'}\xi^{L'}) = 1,$$

with u_ℓ, u_L, $v_{\ell'}$, $v_L \neq 0$. Then (equate degrees) $\ell + \ell' = 0$ and $L + L' = 0$. Whence, since $\ell \leq L, \ell' \leq L', \ell = L$ and $\ell' = L'$. Therefore, $u \in \mathbb{R}[\xi,\xi^{-1}]$ is unimodular iff it is of the form $u(\xi,\xi^{-1}) = \alpha\xi^k$ with $0 \neq \alpha \in \mathbb{R}$ and $k \in \mathbb{Z}$.

For $U,V \in \mathbb{R}^{n\times n}[\xi,\xi^{-1}]$, observe that $UV = I \Rightarrow \det(U)\det(V) = 1$. Therefore $U \in \mathbb{R}^{n\times n}[\xi,\xi^{-1}]$unimodular implies $\det(U) \in \mathbb{R}[\xi,\xi-1]$ unimodular. Conversely, if $\det(U) \in \mathbb{R}[\xi,\xi-1]$ is unimodular, then $V = \left(\det(U)\right)^{-1}cof(U)$ (cof(U) denotes the matrix of co-factors, defined as in the case of real matrices) is its inverse.

<u>Conclusion</u>: $U \in \mathbb{R}^{n\times n}[\xi,\xi^{-1}]$ is unimodular iff $\det(U(\xi)) = \alpha\xi^k$ with $0 \neq \alpha \in \mathbb{R}$ and $k \in \mathbb{Z}$. In contrast, $U \in \mathbb{R}^{n\times n}[\xi]$ is unimodular iff $\det(U(\xi)) = \alpha$ with $0 \neq \alpha \in \mathbb{R}$. The ring $\mathbb{R}^{n\times n}[\xi,\xi^{-1}]$ has many more unimodular elements than $\mathbb{R}^{n\times n}[\xi]$.

2. Let σ denote, as usual, the *shift*: $\sigma(f)(t) := f(t+1)$. Let $R \in \mathbb{R}^{\bullet\times \mathtt{w}}[\xi,\xi^{-1}]$ and consider the system of difference equations

$$R(\sigma,\sigma^{-1})w = 0.$$

This defines a dynamical system $\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$. Define $\mathscr{B}$ formally.

$$\mathscr{B} := \{w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} \mid R(\sigma,\sigma^{-1})w = 0\}.$$

Note that $\mathscr{B} = \ker(R(\sigma,\sigma^{-1}))$ with $R(\sigma,\sigma^{-1})$ is viewed as a map from $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ to $(\mathbb{R}^{\mathrm{rowdim}(R)})^{\mathbb{Z}}$.

3. Show that every linear time-invariant complete system $(\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ admits a (minimal) representation of the form (only forward differences (or lags))

$$R(\sigma)w = 0$$

for some $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ and one of the form (only backward differences)

$$R(\sigma^{-1})w = 0$$

for some $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$?

Let $R(\sigma,\sigma^{-1}) = R_\ell\xi^\ell + R_{\ell+1}\xi^{\ell+1} + \cdots + R_L\xi^L$. Define R_+ and R_- by

$$R_+(\sigma,\sigma^{-1}) := \xi^{-\ell}R(\sigma,\sigma^{-1}), \qquad R_-(\sigma,\sigma^{-1}) := \xi^{-L}R(\sigma,\sigma^{-1}).$$

Obviously then $R(\sigma,\sigma^{-1})w = 0$, $R_+(\sigma,\sigma^{-1})w = 0$, and $R_-(\sigma,\sigma^{-1})w = 0$ define the same system but $R_+(\sigma,\sigma^{-1})$ contains only forward differences (in a sense $R_+ \in \mathbb{R}^{\bullet\times\mathtt{w}}(\xi)$) and $R_- \in \mathbb{R}^{\bullet\times\mathtt{w}}(\sigma,\sigma^{-1})$ contains only backward differences (in a sense $R_- \in \mathbb{R}[\xi^{-1}]$).

Exercise 2 (Moving average)

The aim of this exercise is to illustrate that the notion of controllability can shed some light on some common algorithms.

1. Throughout this exercise, the time-axis is $\mathbb{Z}$. Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi, \xi^{-1}]$ and consider the system of difference equations

$$R(\sigma, \sigma^{-1})w = 0.$$

This defines the dynamical system $\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$. It is easy to prove that this system is controllable iff

the rank of the complex matrix $R(\lambda, \lambda^{-1})$ is the same for all $0 \neq \lambda \in \mathbb{C}$.

Prove by an example that you cannot dispense of 'puncturing' 0 from $\mathbb{C}$ in this test.

The Smith form for matrices over $\mathbb{R}[\xi, \xi^{-1}]$ reads: Let $M \in \mathbb{R}^{\bullet \times \bullet}[\xi, \xi^{-1}]$. There exist unimodular $U, V \in \mathbb{R}^{\bullet \times \bullet}[\xi, \xi^{-1}]$, such that UMV is of the form

$$UMV = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},$$

with $D = \mathrm{diag}(d_1, d_2, \ldots, d_{\mathtt{r}})$, $d_{\mathtt{k}} \in \mathbb{R}[\xi, \xi^{-1}]$, and $d_{\mathtt{k}+1}$ is a factor of $d_{\mathtt{k}}$, for $\mathtt{k} = 1, \ldots, \mathtt{r} - 1$. In fact, we can take $d_{\mathtt{k}} \in \mathbb{R}[\xi]$ with $d_{\mathtt{k}}(0) = 1$.

Now, proceed exactly as in the continuous-time case: $R(\sigma, \sigma^{-1})w = 0$ defines a controllable system iff $D(\sigma, \sigma^{-1})w = 0$ does. The latter is the case iff each of the systems $d_{\mathtt{k}}(\sigma, \sigma^{-1})w_{\mathtt{k}} = 0$ defines a controllable system. This is the case iff each of the $d_{\mathtt{k}}$'s is unimodular. Expressed in terms of R, this yields the rank condition.

Note finally that the puncturing is indeed necessary. Consider the system described by $\sigma w = 0$. i.e., $\mathscr{B} = 0$. It is obviously controllable. The associated $R(\xi)$ is ξ. $R(\lambda)$ *drops rank at* $\lambda = 0$, but this does not contradict controllability. If you do not like this example, use $\sigma w_1 = \sigma w_2$ instead.

2. Consider the system defined by

$$w_2(t) = \frac{1}{T} \sum_{t'=1}^{T} w_1(t - t'). \tag{MA}$$

This algorithm is called a *moving average (MA)* smoothing. $T \in \mathbb{N}$ is called the *averaging window*. It is very frequently used in order to filter out noise, detecting trends, etc. When T is large, it is tempting to replace this algorithm by

$$w_2(t) = w_2(t-1) + \frac{1}{T}(w_1(t-1) - w_1(t-T-1)). \tag{MA'}$$

(a) Do (MA) and (MA') define the same system (of course, in the behavioral sense, the one and only way ...)?

No, (MA) is controllable and (MA') is not (see part (c)). For example, $w_1(t) = c_1$, $w_2(t) = c_2$ is a solution of (MA'), but not of (MA) if $c_1 \neq c_2$.

Do (MA) and (MA') have the same transfer function?

The transfer function $w_1 \mapsto w_2$ of (MA) is

$$G(\xi) = \frac{1}{T}(\xi^{-1} + \xi^{-2} + \cdots + \xi^{-T}) = \frac{1}{T} \frac{\xi^{-1} - \xi^{-(T+1)}}{1 - \xi^{-1}},$$

and the transfer function of (MA') is

$$G(\xi) = \frac{1}{T} \frac{\xi^{-1} - \xi^{-(T+1)}}{1 - \xi^{-1}}.$$

(MA) and (MA') have the same transfer function.

(b) Compare, by counting the number of additions and multiplications required per time-step, (MA) and (MA') from the computational complexity point of view.

Per "time step" (MA) takes $T-1$ additions and 1 multiplication, while (MA') takes only 2 additions and 1 multiplication. From this point of view, (MA') seems simpler.

(c) Is (MA) controllable? Is (MA') controllable?

Apply the common factor test: (MA) is controllable since

$$R(\xi) = [-\frac{1}{T}(\xi^{-1}+\xi^{-2}+\cdots+\xi^{-T}),1], \qquad \mathrm{rank}(R(\lambda)) = 1, \quad \forall\, 0 \neq \lambda \in \mathbb{C}.$$

For (MA'), we get

$$R(\xi) = [-\frac{1}{T}(1-\xi^{-(T+1)}),1-\xi^{-1}], \qquad \mathrm{rank}(R(1)) = 0,$$

so it is not controllable.

(d) Would you call (MA) stable (meaning $[\![w_1(t)=0,(w_1,w_2)\in\mathscr{B}]\!] \Rightarrow [\![w_2(t)\to 0 \text{ for } t\to\infty]\!]$)? Would you call (MA') stable?

(MA) is stable: if $w_1 = 0$ for $t \geq 0$, then $w_2(t) \xrightarrow{t\to\infty} 0$. (MA') is not stable: if $w_1 = 0$ for $t \geq 0$, then $w_2(t)$ does not necessarily go to zero (it may be a non-zero constant). (MA') should be used cautiously: if an error occurs in the calculations of $w_2(t')$, this error will appear in the results forever after, for all $t > t'$.

3. A close relative of (MA) is *exponential weighting:*

$$w_2(t) = \frac{1-\rho}{\rho}\sum_{t'\in\mathbb{N}}\rho^{t'}w_1(t-t') \tag{EW}$$

with $\rho \in (0,1)$ the *exponential weighting parameter*. Convolutions as (EW) or their continuous time analogs are of course very much related to our linear difference or differential systems, but officially (EW) is not a difference equation because of the infinite number of terms that appear on the right hand side.

For the case at hand the related difference equation is

$$w_2(t) = \rho w_2(t-1) + (1-\rho)\,w_1(t-1) \tag{EW'}$$

(a) (EW) has the drawback that it is hard to give a very concrete characterization of the behavior, since it unclear for which $w_1 : \mathbb{Z} \to \mathbb{R}$ the infinite sum converges. Prove that the infinite sum converges when w_1 is bounded. Prove that (EW) combined with w_1 bounded, and (EW') combined with w_1, w_2 bounded have the same behavior.

Assume that w_1 is bounded and that (w_1,w_2) satisfies (EW). Then

$$\begin{aligned} |w_2(t)| &\leq \frac{1-\rho}{\rho}\left|\sum_{t'\in\mathbb{N}}\rho^{t'}w_1(t-t')\right| \\ &\leq \frac{1-\rho}{\rho}\sum_{t'\in\mathbb{N}}\rho^{t'}\|w_1\|_\infty \\ &\leq \|w_1\|_\infty \end{aligned}$$

Hence $w_1 \in \ell_\infty(\mathbb{Z},\mathbb{R}) \Rightarrow w_2 \in \ell_\infty(\mathbb{Z},\mathbb{R})$, and w_2 is well-defined by (EW). In fact, $\|w_2\|_\infty \leq \|w_1\|_\infty$. Now substitute (EW) in (EW') and verify that (w_1,w_2) satisfy (EW').

To show the converse, assume that $w_1, w_2 \in \ell_\infty(\mathbb{Z},\mathbb{R})$ satisfy (EW'). We need to show that it satisfies (EW). Define w_2' by (EW') (with this w_1). Then, as we have just proven, $w_2' \in \ell_\infty(\mathbb{Z},\mathbb{R})$ and satisfies (EW'). Hence $w_2 - w_2' \in \ell_\infty(\mathbb{Z},\mathbb{R})$ and, since (EW') defines a linear system, $(0, w_2 - w_2')$ satisfies (EW'). Let $\Delta := w_2 - w_2'$. Then Δ satisfies $\Delta(t) = \rho\Delta(t-1)$, i.e., $\Delta(t) = \rho^t\Delta(0)$. Since Δ is bounded (on $\mathbb{Z}$!), this implies $\Delta = 0$. Hence, $w_2 = w_2'$, and hence (w_1,w_2) satisfies (EW).

What we have used here is that while (EW') has many solutions for each w_1, it has only one bounded solution if w_1 is bounded. It is this solution that is given by (EW).

(b) Compare the computational complexity of (MA), (EW), and (EW').

Per time step, (MA) requires $T-1$ additions and one multiplication, (EW) requires in principle an infinite number of multiplications, and (EW') requires one addition and two multiplications. Exponential weighting implemented by (EW') is hence for several reasons to be preferred above (MA) systems.

(c) Is (EW') controllable? (You may use the result of part 1).

Then R corresponding to (EW') is

$$\begin{bmatrix} -\frac{1-\rho}{\rho}\xi^{-1} & 1-\rho\xi^{-1} \end{bmatrix}.$$

There is no common factor, so that the system is controllable.

Obviously, these are plenty of good reasons to prefer exponential weighting implemented by (EW') over Moving Average for data smoothing.

Exercise 3 (Time-reversibility)

The aim of this exercise is

- *to let you think of the nature of differential systems*
- *use the powerful theorem on the structure of minimal kernel representations in a simple but meaningful application*

The time-invariant dynamical system $\Sigma = (\mathbb{R}, \mathbb{W}, \mathscr{B})$ is said to be *time-reversible* if $w \in \mathscr{B}$ implies $\texttt{reverse}(w) \in \mathscr{B}$, where $\texttt{reverse}(w)$ is defined by $\texttt{reverse}(w)(t) := w(-t)$.

1. Do Kepler's laws define a time-reversible system?

Kepler's laws define the system $(\mathbb{R}, \mathbb{R}^3, \mathscr{B})$ with $w \in \mathscr{B}$ iff w is periodic and satisfies:

K.1 the set $\{ v \in \mathbb{R}^3 \mid \exists\, t \in \mathbb{R} : v = w(t) \}$ is an ellipse with the sun (at a fixed point, say $0 \in \mathbb{R}^3$), in one of the foci,

K.2 the vector $w(t) \in \mathbb{R}^3$ from the sun to the planet sweeps out equal areas in equal times,

K.3 $\dfrac{\text{(period)}^2}{\text{(major axis of the ellipse)}^3}$ is a universal constant (i.e., the same for all the planets).

<u>Comment</u>: It is hard not to become filled with awe every time one writes this down: Kepler deduced these laws - highly accurate and exact under very reasonable idealizations - from the mere observation of about half a dozen cases!

Consider $\texttt{reverse}(w)$. Obviously $w(-t)$ also sweeps out the same ellipse, but in opposite direction, with equal areas in equal times, and with the same period. Whence, $\texttt{reverse}(w)$ satisfies K.1, K.2, and K.3. Therefore Kepler's laws define a time-reversible system.

Let $f : \mathbb{R}^{\mathtt{w}(\mathtt{n}+1)} \to \mathbb{R}^{\mathtt{m}}$ and consider the behavioral differential equation

$$f \circ (w, \frac{d}{dt}w, \ldots, \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}}w) = 0.$$

Precisely,

$$\mathscr{B} = \{ w \in \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid f(w(t), \frac{d}{dt}w(t), \ldots, \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}}w(t)) = 0 \ \forall\, t \in \mathbb{R} \}.$$

2. Prove that this defines a time-reversible system if "f contains only even derivatives". Make precise what "*contains only even derivatives*" means. Use mathematical language, not colorful prose.

The question occurs: *Is this condition, only even order derivatives, also necessary for time-reversibility?* But this is asking the impossible, even for LIDSs, in view of the highly non-uniqueness of behavioral equations. A better question is therefore:

Can a time-reversible system always be represented by
a system of differential equations which contains only even order derivatives?

It turns out that this a very good question, and that the answer is in the affirmative for controllable LTIDSs. In this exercise, we tip the curtain for systems described by *only one* linear differential equation.

What does it mean that the differential equations defined by

$$f : (a_0, a_1, \ldots, a_{\mathtt{n}}) \in \mathbb{R}^{\mathtt{w}(\mathtt{n}+1)} \rightarrow \mathbb{R}^{\mathtt{w}}$$

contains only even derivatives? The easiest way to answer is: there must exist

$$g : (a_0, a_1, \ldots, a_{\mathbf{int}(\mathtt{n}/2)}) \in \mathbb{R}^{\mathtt{w}(\mathbf{ent}(\mathtt{n}/2)+1)} \rightarrow \mathbb{R}^{\mathtt{w}}$$

(ent(x):= the largest integer less than or equal to x), such that

$$f(a_0, a_1, \ldots, a_{\mathtt{n}}) = g(a_0, a_1, \ldots, a_{\mathbf{ent}(\mathtt{n}/2)})$$

for all $(a_0, a_1, \ldots, a_{\mathtt{n}}) \in \mathbb{R}^{\mathtt{w}}$.

Let $\mathscr{B}$ be governed by

$$g \circ \left(w, \frac{d^2}{dt^2}w, \ldots, \frac{d^{2\mathtt{n}'}}{dt^{2\mathtt{n}'}}w\right) = 0.$$

Assume that $w \in \mathscr{B}$. Observe that $\frac{d}{dt}\,\texttt{reverse}(w) = -\texttt{reverse}(\frac{d}{dt}w)$, whence $\frac{d^{2\mathtt{k}}}{dt^{2\mathtt{k}}}\,\texttt{reverse}(w) = \texttt{reverse}(\frac{d^{2\mathtt{k}}}{dt^{2\mathtt{k}}}w)$. Therefore

$$\begin{aligned} w \in \mathscr{B} :\; & \Leftrightarrow g \circ \left(w, \tfrac{d^2}{dt^2}w, \ldots, \tfrac{d^{2\mathtt{n}'}}{dt^{2\mathtt{n}'}}w\right) = 0 \\ & \Leftrightarrow g\left(w(t), \tfrac{d^2}{dt^2}w(t), \ldots, \tfrac{d^{2\mathtt{n}'}}{dt^{2\mathtt{n}'}}w(t)\right) = 0 \quad \forall t \in \mathbb{R} \\ & \Leftrightarrow g\left(w(-t), \tfrac{d^2}{dt^2}w(-t), \ldots, \tfrac{d^{2\mathtt{n}'}}{dt^{2\mathtt{n}'}}w(-t)\right) = 0 \quad \forall t \in \mathbb{R} \\ & \Leftrightarrow g \circ \left(\texttt{reverse}(w), \texttt{reverse}(\tfrac{d^2}{dt^2}w), \ldots, \texttt{reverse}(\tfrac{d^{2\mathtt{n}'}}{dt^{2\mathtt{n}'}}w)\right) = 0 \\ & \Leftrightarrow: \texttt{reverse}(w) \in \mathscr{B}. \end{aligned}$$

Whence, $w \in \mathscr{B}$ iff $\texttt{reverse}(w) \in \mathscr{B}$. Hence the system defined by $\mathscr{B}$ is time-reversible.

3. Let $p \in \mathbb{R}[\xi]$. Prove that the system (in $\mathscr{L}^1$) described by

$$p(\frac{d}{dt})w = 0 \tag{DE}$$

is time-reversible if and only if p is either an even or an odd polynomial. A polynomial $p \in \mathbb{R}[\xi]$ is called *even* if it contains only even powers of ξ, i.e. if $p(\xi) = p(-\xi)$, and *odd* if it contains only odd powers of ξ, i.e. if $p(\xi) = -p(-\xi)$,

Hint: in the time-reversible case, $p(-\frac{d}{dt})w = 0$ is also a kernel representation of the behavior defined by the kernel representation (DE), and two minimal kernel representations are related by pre-multiplication by an $\mathbb{R}[\xi]$-unimodular element.

Let $\mathscr{B}$ be described by $p(\frac{d}{dt})w = 0$ with $p \neq 0$ (treat the trivial case $p = 0$ separately). Then (see the proof of 2) rev($\mathscr{B}$) is described by $p(-\frac{d}{dt})w = 0$. By the structure theorem for kernel representations, $\mathscr{B} = \text{rev}(\mathscr{B})$ iff there exists a unimodular $U \in \mathbb{R}[\xi]$, such that $p(-\xi) = U(\xi)p(\xi)$. But $U \in \mathbb{R}[\xi]$ is unimodular iff U is a nonzero constant, say α. Hence $\mathscr{B} = \text{rev}(\mathscr{B})$ iff there exists $\alpha \neq 0$ such that

$$p(-\xi) = \alpha p(\xi).$$

But this can only be the case if $\alpha = +1$ (if p has even degree), or $\alpha = -1$ (if p has odd degree). If $\alpha = 1$, p is hence even, and if $\alpha = 1$, p is hence odd.

4. Let $p, q \in \mathbb{R}^{\mathtt{w}}[\xi]$. Prove that the system (in $\mathscr{L}^2$) described by

$$p(\frac{d}{dt})w_1 = q(\frac{d}{dt})w_2 \tag{$*$}$$

is time-reversible if and only if p and q are either (i) both even or (ii) both odd polynomials.

Repeat, *mutatis mutandis*, the proof of 3.

5. Assume in addition that p and q are co-prime (we have seen in lecture 2 that this means that $(*)$ defines a controllable system). Prove that time-reversibility then implies that p and q are both even.

This is true in general, indeed

A controllable LTIDS is time-reversible iff it can described by
a differential equation that contains only even order derivatives

If p and q are co-prime, then they can not be both odd (since ξ is then a common factor of p and q). Therefore, a controllable linear system

$$p(\frac{d}{dt})w_1 = q(\frac{d}{dt})w_2$$

is time-reversible iff p and q are both even. Therefore iff this differential equation contains only even derivatives.

<u>Comment</u>: For the general multivariable case, this result becomes: $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ is time-reversible iff it admits a (minimal) kernel representation of the form

$$R_+(\frac{d}{dt})w = 0, \qquad R_-(\frac{d}{dt})w = 0,$$

with R_+ even and R_- odd. $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ controllable is time-reversible iff it allows a (minimal) kernel representation of the form

$$R(\frac{d}{dt})w = 0,$$

with R even.

Exercise 4 (Controllability and interconnections)

The aim of this exercise is

- *to illustrate the behavioral concept of controllability*
- *to show its fragility under system operations*

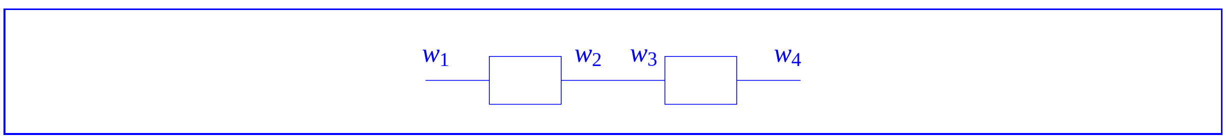
1. Let $\mathscr{B}, \mathscr{B}' \in \mathscr{L}^\bullet$ be described by

$$\begin{aligned} R_1(\frac{d}{dt})w_1 &= R_2(\tfrac{d}{dt})w_2 \\ R_3(\frac{d}{dt})w_3 &= R_4(\tfrac{d}{dt})w_4 \end{aligned}$$

Define their *series interconnection* (also called a *cascade interconnection*) by these behavioral equations, combined with

$$w_2 = w_3.$$

Of course, we assume that the dimensions are such that this makes sense. In the resulting behavior, consider (w_1, w_4) as the manifest variables and (w_2, w_3) as latent variables.



Consider the system with transfer function $\frac{1}{s}$, i.e. the integrator,

$$\frac{d}{dt}y_1 = u_1$$

and the system with transfer function s, i.e. the differentiator,

$$y_2 = \frac{d}{dt}u_2.$$

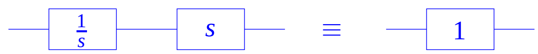
Are these systems controllable? Compute behavioral equations for the manifest behavior of the series connection defined by $u_2 = y_1$. Is this system controllable? What is its transfer function? Now consider the series connection in opposite order, i.e. the interconnection defined by $u_1 = y_2$. Compute behavioral equations for the manifest behavior of this series connection. Is this system controllable? What is its transfer function?

Are the two resulting series connections equivalent? If not, give a signal that belongs to the manifest behavior of one, but not the other. Does series connection of single-input/single-output connections 'commute'?

(1) $\Leftrightarrow R(\frac{d}{dt})\begin{bmatrix}u_1\\y_1\end{bmatrix} = 0$ **with** $R(\xi) = [-1 \ \ \xi]$; $\text{rank}(R(\lambda)) = 1$ **for all** $\lambda \in \mathbb{C} \Rightarrow$ **controllable.**

(2) $\Leftrightarrow R(\frac{d}{dt})\begin{bmatrix}u_2\\y_2\end{bmatrix} = 0$ **with** $R(\xi) = [-\xi \ \ 1]$; $\text{rank}(R(\lambda)) = 1$ **for all** $\lambda \in \mathbb{C} \Rightarrow$ **controllable.**

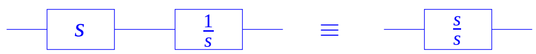
series connection:



$$\frac{d}{dt}y_1 = u_1, \qquad y_2 = \frac{d}{dt}u_2, \qquad u_2 = y_1$$

Eliminating y_1 **and** u_2**, yields the system** $u_1 = y_2$**, which is obviously controllable. The transfer function is 1.**

Consider now the series connection in the opposite order:



$$\frac{d}{dt}y_1 = u_1, \qquad y_2 = \frac{d}{dt}u_2, \qquad u_1 = y_2$$

Eliminating y_2 **and** u_1**, yields the system**

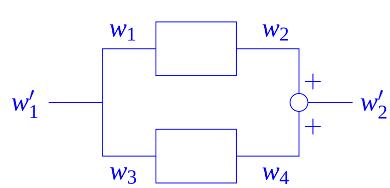
$$\frac{d}{dt}y_1 = \frac{d}{dt}u_2 \quad \textbf{i.e.} \quad R(\frac{d}{dt})\begin{bmatrix}u_2\\y_1\end{bmatrix} = 0, \qquad \textbf{with } R(\xi) = [-\xi \ \ \xi]$$

$\text{rank}(R(\lambda)) = 1$ **for** $0 \neq \lambda \in \mathbb{C}$**, and** $\text{rank}(R(\lambda)) = 0$ **for** $\lambda = 0$**, so the system is not controllable. The transfer function is** $\xi^{-1}\xi = 1$**.**

The two series connections are not equivalent, even though they have the same transfer function. Any non-zero constant input-output belongs to the second series connection, but not to the first. Hence series connection does not commute. It does commute, though, for the transfer functions, i.e., for the controllable part.

<u>Comment</u>: When we write $\frac{s}{s}$ for a transfer function, or, generally, a transfer function with a common factor in the numerator and denominator, we mean exactly the same as $\frac{1}{1}$, with the common factor cancelled. Indeed, in rational functions one can, *by definition of a rational function*, cancel (or add) common factors *ad libitum*. So, when you read or hear: *assume that there are no common factors in the numerator and denominator of this or that transfer function*, smile, and muse '*innocence is bliss*'. What this assumption usually means is that people actually have a kernel representation, in which lack of common factors means controllability. But since they have been brought up without the notion of kernel representation, but with the thought that a system IS a transfer function, they have to resort to convoluted meaningless statements involving common factors.

2. Define, in the above spirit of series connection, *parallel connection*.



Parallel connection:

$$w_1' = w_1 = w_3, \qquad w_2' = w_2 + w_4,$$

with w_1' **and** w_2' **the manifest variables and** w_1**,** w_2**,** w_3**, and** w_4**, the latent variables.**

3. Decide, by means of a proof or a counterexample, which of the above operations preserve controllability. Of course, we assume that we deal with systems in $\mathscr{L}^\bullet$, and that the dimensions are appropriate:

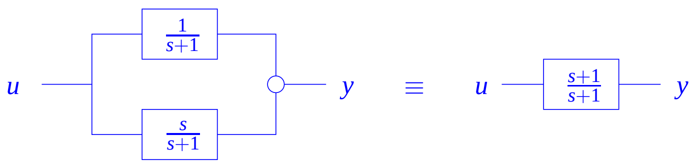
   (a) series connection

   Series connection does not preserve controllability, see part 1.

   (b) parallel connection

   Parallel connection connection does not preserve controllability.

   Example:



$$\left(\frac{d}{dt}+1\right)y_1 = u_1, \qquad \left(\frac{d}{dt}+1\right)y_2 = \frac{d}{dt}u_2, \qquad u = u_1 = u_2, \qquad y = y_1 + y_2$$

   After elimination: $(\frac{d}{dt}+1)y = (\frac{d}{dt}+1)u$**, so** $R(\frac{d}{dt})\begin{bmatrix}u\\y\end{bmatrix} = 0$**, with** $R(\xi) = [-\xi-1 \quad \xi+1]$**, which drops rank for** $\lambda = -1$**.**

   (c) addition, i.e., does $\mathscr{B}_1, \mathscr{B}_2$ controllable imply $\mathscr{B}_1 + \mathscr{B}_2$ controllable?

   Define $\mathscr{B}_1 + \mathscr{B}_2$ **by,**

$$\mathscr{B}_1 : \; R_1(\frac{d}{dt})w_1 = 0, \qquad \mathscr{B}_2 : \; R_2(\frac{d}{dt})w_2 = 0, \qquad \mathscr{B}_1 + \mathscr{B}_2 : \; w = w_1 + w_2,$$

   with w_1**,** w_2 **latent variables, and** w **the manifest one.** $\mathscr{B}_1$ **and** $\mathscr{B}_2$ **are controllable iff the full behavior is controllable, which implies that** $\mathscr{B}_1 + \mathscr{B}_2$ **is controllable (elimination preserves controllability, see part (e)).**

   (d) intersection

   Let

$$\mathscr{B}_1 : \; R_1(\frac{d}{dt})w_1 = 0, \qquad \mathscr{B}_2 : \; R_2(\frac{d}{dt})w_2 = 0.$$

   The intersection of $\mathscr{B}_1$ **and** $\mathscr{B}_2$ **does not preserve controllability.**

   Take, for example, $R_1 = [p_1 \;\; q_1]$**,** $R_2 = [p_2 \;\; q_2]$**,** $\begin{bmatrix}R_1\\R_2\end{bmatrix} = \begin{bmatrix}p_1 & q_1\\p_2 & q_2\end{bmatrix}$ **drops rank at the roots of** $p_1q_2 - q_1p_2$**.**

   (e) elimination

   Elimination preserves controllability. Go back to the basic definition of controllability for a straightforward proof, that is also valid for nonlinear systems. Or consider the representation $R(\frac{d}{dt})w = M\left(\frac{d}{dt}\right)\ell$**, use a unimodular pre-multiplication, if necessary, to write this as** $R'(\frac{d}{dt})w = 0, R"(\frac{d}{dt})w = M"(\frac{d}{dt})\ell$**, with** $M"$ **of full row rank. Note that** $R'(\frac{d}{dt})w = 0$ **is a kernel representation of the manifest behavior. Finally, observe that rank constancy of** $[R(\lambda) \quad M(\lambda)]$ **implies rank constancy of** $R'(\lambda)$**. Hence controllability of the full behavior implies controllability of the manifest behavior.**

Exercise 5 (Elimination)

The aim of this exercise is

- *to make you appreciate sensitivity issues related to the elimination theorem*
- *to show that in order to obtain equations for the (manifest) behavior with nice structure, latent variables are indispensable and need not, and should not, be eliminated*

1. Let $S \subseteq \mathbb{R}^{\mathtt{n}_1} \times \mathbb{R}^{\mathtt{n}_2}$ and define
$$S_1 := \{x_1 \in \mathbb{R}^{\mathtt{n}_1} \mid \exists\, x_2 \in \mathbb{R}^{\mathtt{n}_2} : (x_1, x_2) \in S\},$$
i.e., S_1 is the projection of S on the first $\mathtt{n}_1$ components. Which of the following implications hold?

   (a) $[\![S \text{ open}]\!] \Rightarrow [\![S_1 \text{ open}]\!]$?

   $[\![S$ open $]\!] \Rightarrow [\![S_1$ open$]\!]$. Let $x_1 \in S_1$. We want to show that there is an $\varepsilon > 0$ such that $\|x_1' - x_1\| < \varepsilon$ implies $x_1' \in S_1$. There is $x_2 \in \mathbb{R}^{\mathtt{n}_2}$ such that $(x_1, x_2) \in S$. Since S is open, there is $\varepsilon > 0$, such that $\|x_1' - x_1\| + \|x_2' - x_2\| < \varepsilon$ implies $(x_1', x_2') \in S$. Hence if $\|x_1' - x_1\| < \varepsilon$, there is x_2' such that $\|x_1' - x_1\| + \|x_2' - x_2\| < \varepsilon$ which implies $(x_1', x_2') \in S$. This proves that $x_1' \in S_1$, hence S_1 is open.

   (b) $[\![S \text{ closed}]\!] \Rightarrow [\![S_1 \text{ closed}]\!]$?

   $[\![S$ closed $]\!] \not\Rightarrow [\![S_1$ closed $]\!]$. Take $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 = 1\}$. Then $S_1 = \{x_1 \in \mathbb{R} \mid x_1 \neq 0\}$. S is closed, but S_1 is not.

   (c) $[\![S \text{ linear}]\!] \Rightarrow [\![S_1 \text{ linear}]\!]$?

   $[\![S$ linear $]\!] \Rightarrow [\![S_1$ linear $]\!]$. Let $x_1', x_1'' \in S_1$ and $\alpha, \beta \in \mathbb{R}$. Then there are x_2' and x_2'', such that $(x_1', x_2'), (x_1'', x_2'') \in S$. Since S is linear, $(\alpha x_1' + \beta x_1'', \alpha x_2' + \beta x_2'') \in S$. This implies $(\alpha x_1' + \beta x_1'') \in S_1$.

   (d) $[\![S \text{ a polytope}]\!] \Rightarrow [\![S_1 \text{ a polytope}]\!]$?

   A *polytope* in $\mathbb{R}^{\mathtt{n}}$ is a set specified by a finite set of linear inequalities, i.e. a set specified by $a_1, a_2, \ldots, a_{\mathtt{m}} \in \mathbb{R}^{\mathtt{n}}$ and $b_1, b_2, \ldots, b_{\mathtt{m}} \in \mathbb{R}$ as follows
$$\{x \in \mathbb{R}^{\mathtt{n}} \mid a_{\mathtt{k}}^\top x \geq b_{\mathtt{k}} \text{ for } \mathtt{k} = 1, 2, \ldots, \mathtt{m}\}.$$

   At last, something that requires some thinking: $[\![S$ a polytope $]\!] \Rightarrow [\![S_1$ a polytope $]\!]$.
   We first prove the case $\mathtt{n}_2 = 1$. Write the inequalities that define S as
$$a_{\mathtt{k},1}^T x_1 + a_{\mathtt{k},2} x_2 \geq b_{\mathtt{k}}, \quad \textbf{for } \mathtt{k} = 1, 2, \ldots, \mathtt{m}.$$
   Note that we may as well assume that $a_{\mathtt{k},2} = 0$, 1, or -1. Hence these inequalities become
$$\begin{aligned} a_{\mathtt{k},1}^T x_1 &\geq b_{\mathtt{k}}, && \textbf{for } \mathtt{k} = 1, 2, \ldots, \mathtt{m}_0,\\ x_2 &\geq -a_{\mathtt{k},1}^T x_1 + b_{\mathtt{k}}, && \textbf{for } \mathtt{k} = \mathtt{m}_0 + 1, \mathtt{m}_0 + 2, \ldots, \mathtt{m}_0 + \mathtt{m}_1,\\ x_2 &\leq -a_{\mathtt{k},1}^T x_1 - b_{\mathtt{k}}, && \textbf{for } \mathtt{k} = \mathtt{m}_0 + \mathtt{m}_1 + 2, \mathtt{m}_0 + \mathtt{m}_1 + 2, \ldots, \mathtt{m}. \end{aligned}$$
   Hence for a given x_1, there exists x_2 such that these inequalities are satisfied iff
$$\begin{aligned} a_{\mathtt{k},1}^T x_1 &\geq b_{\mathtt{k}}, && \textbf{for } \mathtt{k} = 1, 2, \ldots, \mathtt{m}_0,\\ -a_{\mathtt{k}',2}^T x_1 + b_{\mathtt{k}'} &\leq a_{\mathtt{k}'',2}^T x_1 - b_{\mathtt{k}'}, && \textbf{for } \mathtt{k}' = \mathtt{m}_0 + 1, \mathtt{m}_0 + 2, \ldots, \mathtt{m}_0 + \mathtt{m}_1,\\ & && \mathtt{k}'' = \mathtt{m}_0 + \mathtt{m}_1 + 1, \mathtt{m}_0 + \mathtt{m}_1 + 2, \ldots, \mathtt{m}. \end{aligned}$$
   These inequalities clearly define a polytope for the variables x_1.
   When $\mathtt{n}_2 > 1$, use induction on $\mathtt{n}_2$, i.e., eliminate the variables $x_{2,1}, x_{2,2}, \ldots, x_{2,\mathtt{n}_2}$, one at a time.

(e) $[\![S$ an algebraic variety $]\!] \Rightarrow [\![S_1$ an algebraic variety $]\!]$?

An *algebraic variety* in $\mathbb{R}^{\mathtt{n}}$ is the zero set of a polynomial, i.e. a set specified by a $p \in \mathbb{R}^{\mathtt{n}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ as follows

$$\{(x_1, x_2, \ldots, x_{\mathtt{n}}) \in \mathbb{R}^{\mathtt{n}} \mid p(x_1, x_2, \ldots, x_{\mathtt{n}}) = 0\}.$$

[S an algebraic variety] $\not\Rightarrow$ [S_1 an algebraic variety]. Take $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$. Then $S_1 = [-1, 1]$. This is not an algebraic variety since a subset of $\mathbb{R}$ is an algebraic variety iff it is a finite set, or all of $\mathbb{R}$. Note that another counterexample has been given in our answer to (b).

There holds (we refer to the literature for definitions of the terms used, and for proofs): $[\![S$ a semi-algebraic variety $]\!] \Rightarrow [\![S_1$ an semialgebraic variety$]\!]$. So with polynomial equalities, inequalities, and inequations, variable elimination is possible. This is the content of the *Tarski-Seidenberg* theorem.

2. Let $f : \mathbb{R}^{\mathtt{n}} \to \mathbb{R}^{\mathtt{n}}, h : \mathbb{R}^{\mathtt{n}} \to \mathbb{R}^{\mathtt{p}}$, both 'nice', even 'very nice'. Consider the system

$$\frac{d}{dt}x = f(x), y = h(x).$$

Is

$$\mathscr{B} := \{y : \mathbb{R} \to \mathbb{R}^{\mathtt{m}} \mid \exists x : \mathbb{R} \to \mathbb{R}^{\mathtt{n}} : \frac{d}{dt}x(t) = f(x(t)), y(t) = h(x(t)) \;\; \forall t \in \mathbb{R}\}$$

the solution set of a system of differential equations?

Discuss only the system $\mathtt{n} = 2, \mathtt{m} = 1, f(x_1, x_2) = (x_2, 0), h(x_1, x_2) = x_1^2$.

Warning: As many research questions, this one is a bit ambiguous and unfinished. What sort of differential equation for y are we looking for? A polynomial expression in y and its derivatives? One with smooth functions? Or anything?

The question is: do there exist $k,q \in \mathbb{N}$ and a map $F : \mathbb{R}^{(k+1)p} \to \mathbb{R}^q$ such that $\mathscr{B}$ consists of the set of solutions of

$$F \circ \left(y, \frac{d}{dt}y, \ldots, \frac{d^k}{dt^k}y \right) = 0? \tag{A}$$

The answer very much depends on the required smoothness (of solutions of the differential equations but, more to the point, of F).

We discuss this elimination problem through the example

$$\frac{d}{dt}x_1 = x_2, \qquad \frac{d}{dt}x_2 = 0, \qquad y = x_1^2. \tag{B}$$

For the case at hand,

$$\mathscr{B} = \{ y : \mathbb{R} \to \mathbb{R} \mid \exists x_1, x_2 : \mathbb{R} \to \mathbb{R} \text{ satisfying (B)} \}.$$

$\mathscr{B}$ hence consists of the polynomials of degree less than or equal to 2 that are squares of polynomials of degree less than or equal to one. Hence $\frac{d^3}{dt^3}y = 0$, and so it suffices to look at differential equations of second order in order to find one that defines the manifest behavior.

Consider the differential equation

$$\left(\frac{d}{dt}y \right)^2 - 2y\frac{d^2}{dt^2}y = 0. \tag{C}$$

Define $\mathscr{B}' = \{ y : \mathbb{R} \to \mathbb{R} \mid y \text{ satisfies (C)} \}$.

It is easy to see that $\mathscr{B} = \mathscr{B}' \cup (-\mathscr{B}')$, with $-\mathscr{B}' = \{ y : \mathbb{R} \to \mathbb{R} \mid -y \in \mathscr{B}' \}$. It follows that the behavior of (C) combined with

$$y \geq 0 \tag{D}$$

is exactly $\mathscr{B}$. Note that we can make (C) combined with (D) appear as if it were a differential equation by introducing a function $P : \mathbb{R} \to \mathbb{R}$ (P is "flat") such that $P(y) > 0$ for $y < 0$ and $P(y) = 0$ for $y \geq 0$ and consider the differential equation

$$P(y) + \left(\left(\frac{d}{dt}y \right)^2 - 2y\frac{d^2}{dt^2}y \right)^2 = 0. \tag{E}$$

Obviously the solution set of (E) is exactly $\mathscr{B}$.

Note that there exist such P's that are C^∞ but there are no such P's that are polynomials.

Further, it can be shown that among the behaviors that contain $\mathscr{B}$ and come from an F that is polynomial in its arguments, (C) yields the smallest possible behavior.

<u>Conclusion</u>: For the example at hand, if elimination is done in the class of differential equations that are polynomial in their arguments, then exact elimination is not possible. If elimination is done in the class of differential equations that are C^∞ in their arguments, then exact elimination is possible. However, in general we suspect that there are examples of systems with f,h polynomial that allow no representation of the from (A) with $F \in \mathscr{C}^\infty$.

Obviously, except for linear differential systems, elimination is awkward and inconvenient. This has rather important implications in practice. Indeed, assume that one wants to describe a phenomenon that involves certain variables, say y. Then to start the analysis by making the seemingly innocent assumption that these variables are described by a system of differential equations is much less generic than one could intuitively expect, based on the typical models that are studied in courses. If the system is an interconnection of systems (and which system is not?) that individually are described by smooth differential equations, there is no guarantee that this will hold also for y.

Exercise 6 (The image and the kernel of a differential operator)

The aim of this exercise is

- *to lead you through a derivation of the expression of the set of solutions of a scalar linear constant coefficient differential equation*
- *to illustrate what are nice 'proofs'*

What is a 'nice' proof? In the words of the mathematician Paul Halmos, a proof is nice when it is structured so that it reduces the question at hand to 'thinking', not grinding away.

To count or to think, that is the question.

Counting is dirty work, thinking is chic. There was a time that great value was put on thinking out nice proofs. Nowadays, a proof should merely 'work'. Fashions change.

In this exercise, we lead you through nice proofs of things that you undoubtedly know: to compute the image and kernel of a scalar differential operator.

Let $0 \neq p \in \mathbb{C}[\xi]$,

$$p(\xi) = p_0 + p_1\xi + p_2\xi^2 + \cdots + p_{\mathtt{n}}\xi^{\mathtt{n}},$$

with $p_{\mathtt{n}} \neq 0$, whence $\mathtt{degree}(p) = \mathtt{n}$. In factored form:

$$p(\xi) = p_{\mathtt{n}}\Pi_{\mathtt{k}=1,2,\ldots,\mathtt{m}}(\xi - \lambda_{\mathtt{k}})^{\mathtt{n}_{\mathtt{k}}},$$

with the $\lambda_{\mathtt{k}}$'s the distinct roots of p and the $\mathtt{n}_{\mathtt{k}}$'s their multiplicities. Of course, by the fundamental theorem of algebra, $\sum_{\mathtt{k}=1}^{\mathtt{m}} \mathtt{n}_{\mathtt{k}} = \mathtt{degree}$.

Consider $p(\frac{d}{dt})$ as an operator from $\mathscr{C}^\infty(\mathbb{R},\mathbb{C})$ to $\mathscr{C}^\infty(\mathbb{R},\mathbb{C})$. A central question in the theory of differential equations and in systems theory (stability analysis, elimination) is:

What is its image, what is its kernel?

Prove first the preliminary **lemma 1**: Let $\lambda \in \mathbb{C}$. Denote by $\exp_\lambda$ the *exponential map with parameter* $\lambda \in \mathbb{C}$, defined as the map $t \in \mathbb{R} \mapsto e^{\lambda t} \in \mathbb{C}$. Then $\frac{d}{dt}(\exp_\lambda f) = \exp_\lambda((\frac{d}{dt} + \lambda)f)$, whence $p(\frac{d}{dt})(\exp_\lambda f) = \exp_\lambda(p(\frac{d}{dt} + \lambda)f)$.

The proof of the preliminary lemma is obvious for $\frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}}$ by induction on $\mathtt{n}$. By expanding $p(\frac{d}{dt})$ as a sum of powers of $\frac{d}{dt}$, the lemma follows for general p's.

1. We start with the image.

   Prove that $p(\frac{d}{dt})$ is surjective.

   Proceed as follows.

   (a) First prove that $\frac{d}{dt}$ is surjective.

   Let $y \in C^\infty(\mathbb{R},\mathbb{C})$. Define x by $x(t) = \int_0^t y(t')\,dt'$. Then $\frac{d}{dt}x = y$, showing that $\frac{d}{dt} : C^\infty(\mathbb{R},\mathbb{C}) \to C^\infty(\mathbb{R},\mathbb{C})$ is indeed surjective.

   (b) Use this and lemma 1 to show that $\frac{d}{dt} + \lambda$ is surjective.

   Obviously $f \mapsto \exp_\lambda f$ is a bijective map on $C^\infty(\mathbb{R},\mathbb{C})$. Whence, $f \mapsto (\frac{d}{dt} + \lambda)f = \exp_{-\lambda} \frac{d}{dt}(\exp_\lambda f)$ is also surjective.

   (c) Observe that the composition of surjective operators is surjective.

   Obvious.

(d) Use the factored form of p to conclude that p is surjective.

Obvious from (b) and (c).

You may wish to contrast this proof with the following, perhaps more common, one. We need to prove that for each 'input' $u \in \mathscr{C}^\infty(\mathbb{R},\mathbb{C})$ there exists an 'output' $y \in \mathscr{C}^\infty(\mathbb{R},\mathbb{C})$ such that $p(\frac{d}{dt})y = u$. Introduce the linear system:

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{p_0}{p_{\mathrm{n}}} & -\frac{p_1}{p_{\mathrm{n}}} & -\frac{p_2}{p_{\mathrm{n}}} & \cdots & -\frac{p_{\mathrm{n}-1}}{p_{\mathrm{n}}} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ p_{\mathrm{n}} \end{bmatrix} u, \;\; y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x,$$

and use the well-known formula involving e^{At} to obtain more that we bargained for: an $\mathtt{n}$-dimensional family of solutions.

The alternative proof constructs an $\mathtt{n}$-dimensional family of solutions y to $p(\frac{d}{dt})y = u$ as (in the obvious notation)

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-t')}Bu(t')\,dt'.$$

2. We proceed with the kernel and will prove that

$$\mathtt{kernel}(p(\frac{d}{dt})) = \{w : \mathbb{R} \to \mathbb{C} \mid \exists p_k \in \mathbb{C}(\xi) \text{ with degree}(p_{\mathrm{k}}) < \mathrm{n}_{\mathrm{k}} \text{ such that } w = \Sigma_{\mathrm{k}=1,2,\ldots,\mathrm{m}} p_{\mathrm{k}} \exp_{\lambda_{\mathrm{k}}}\}$$

First prove **lemma 2**. Let $\mathbb{V}$ be a vector space and $L_1, L_2 : \mathbb{V} \to \mathbb{V}$ linear maps. Prove that $\dim(\mathtt{kernel}(L_1L_2)) \leq \dim(\mathtt{kernel}(L_1)) + \dim(\mathtt{kernel}(L_2))$. Conclude from this that if L_1 and L_2 commute, if both have a finite-dimensional kernel, and if $\mathtt{kernel}(L_1) \cap \mathtt{kernel}(L_2) = 0$, then

$$\mathtt{kernel}(L_1L_2) = \mathtt{kernel}(L_1) + \mathtt{kernel}(L_2).$$

The result is trivial if $\dim(\mathtt{kernel}(L_1))$ or $\dim(\mathtt{kernel}(L_2))$ are infinite. Assume that they are finite. Note that $\mathtt{kernel}(L_1L_2) = L_2^{-1}\mathtt{kernel}(L_1)$. The result follows from

$$\dim(L^{-1}\mathfrak{L}) \leq \dim(\mathfrak{L}) + \dim(\mathtt{kernel}(L))$$

for $\mathfrak{L}$ a linear space and L a linear map.

Obviously if L_1 and L_2 commute $\mathtt{kernel}(L_1) + \mathtt{kernel}(L_2) \subset \mathtt{kernel}(L_1L_2)$. Combined with $\mathtt{kernel}(L_1) \cap \mathtt{kernel}(L_2) = \{0\}$, this yields $\dim(\mathtt{kernel}(L_1)) + \dim(\mathtt{kernel}(L_2)) \leq \dim(\mathtt{kernel}(L_1L_2))$. Together with $\dim(\mathtt{kernel}(L_1L_2)) \leq \dim(\mathtt{kernel}(L_1)) + \dim(\mathtt{kernel}(L_2))$, we obtain $\dim(\mathtt{kernel}(L_1)) + \dim(\mathtt{kernel}(L_2)) = \dim(\mathtt{kernel}(L_1L_2))$.
Hence $\mathtt{kernel}(L_1L_2) = \mathtt{kernel}(L_1) + \mathtt{kernel}(L_2)$.

Now prove the expression for the kernel as follows:

(a) What is the kernel of $\frac{d}{dt}$?

$\frac{d}{dt}f = 0$ iff f is a constant.

(b) Prove by induction what $\mathtt{kernel}(\frac{d^k}{dt^k})$ is.

$\mathtt{kernel}(\frac{d^n}{dt^n})$ consists of the polynomials of degree less than n. The proof is by induction on n. It is obviously true for $n = 0$. Assume that it holds for less than n. Then

$$\begin{aligned} \frac{d^n}{dt^n} f = 0 \Leftrightarrow & \frac{d^{n-1}}{dt^{n-1}} \frac{d}{dt} f = 0 \\ \Leftrightarrow & \frac{d}{dt} f \text{ is any polynomial, say } p \text{ of degree less than } n-1 \\ \Leftrightarrow & f(t) = f(0) + \int_0^t p(t')\,dt'. \end{aligned}$$

This shows that f is any polynomial of degree less tan n.

(c) Use this and the lemma 1 to obtain $\mathtt{kernel}((\frac{d}{dt} - \lambda)^n)$.

$$\begin{aligned} (\frac{d}{dt} - \lambda)^n f = 0 \quad \Leftrightarrow & \quad \exp_\lambda \frac{d^n}{dt^n} \exp_{-\lambda} f = 0 \\ \Leftrightarrow & \quad \frac{d^n}{dt^n} \exp_{-\lambda} f = 0 \\ \Leftrightarrow & \quad \exp_{-\lambda} f \text{ is any polynomial of degree less than } n \\ \Leftrightarrow & \quad f = \exp_\lambda p \text{ with any pol. } p \text{ of degree less than } n. \end{aligned}$$

(d) Prove that the maps

$$t \mapsto t^j exp_{\lambda_k}(t), k = 1, \ldots, m, j = 0, \ldots, n_k - 1$$

are linearly independent.

Assume to the contrary that the map $t \mapsto t^k e^{\lambda_j t}$ depends on the maps that precede it in this list. Now prove, using (c) that

$$\left(\frac{d}{dt} - \lambda_1\right)^{n_1}, \ldots, \left(\frac{d}{dt} - \lambda_{j-1}\right)^{n_{j-1}} \left(\frac{d}{dt} - \lambda_j\right)^k$$

annihilates the predecessors but not $t \mapsto t^k e^{\lambda_j t}$. This contradiction shows linear independence.

(e) Use lemma 2 to obtain the expression for $\mathtt{kernel}(p(\frac{d}{dt}))$.

Note that the operators $(\frac{d}{dt} - \lambda_i)^{n_i}$ and $(\frac{d}{dt} - \lambda_j)^{n_j}$ commute, and, by (d), they have non-empty intersection if $i \neq j$.
Repeatedly using the lemma of this section and the linear independence from part (d), show that

$$\mathtt{kernel}(p(\frac{d}{dt})) = \sum_{k=1}^{m} \mathtt{kernel}\left(\left(\frac{d}{dt} - \lambda_k\right)^{n_k}\right).$$

Now use(c) to obtain the expression for $\mathtt{kernel}(p(\frac{d}{dt}))$.

3. The expression of $\mathtt{kernel}(p(\frac{d}{dt}))$ suggests that $\dim(\mathtt{kernel}(p(\frac{d}{dt}))) = \text{degree}(p)$. Prove this.

The independence proven in (d) shows that indeed
$\dim(\mathtt{kernel}(p(\frac{d}{dt}))) = \sum_{k=1}^{m} n_k = \text{degree}(p)$.

With thx to Ivan Markovsky for his help in working out these solutions.