MORE on DISSIPATIVE SYSTEMS

Lectures
by

Jan C. Willems

K.U. Leuven

INTERCONNECTION

&

DISSIPATIVITY

BEHAVIORAL STATE SYSTEMS

Let $\Sigma = (\mathbb{R}, \mathbb{W}, \mathbb{X}, \mathfrak{B})$ be a continuous time state system.

This means: $\mathbb{R}$ = time-axis, $\mathbb{W}$ = space of manifest variables, $\mathbb{X}$ = state space, $\mathfrak{B}$ = behavior, $\quad \mathfrak{B} \subseteq (\mathbb{W} \times \mathbb{X})^{\mathbb{R}}$.

External behavior :

$$\mathfrak{B}_{\text{ext}} := \{w \mid \exists\, x \text{ such that } (w, x) \in \mathfrak{B}\}$$

$$\rightsquigarrow \quad \Sigma_{\text{ext}} := (\mathbb{R}, \mathbb{W}, \mathfrak{B}_{\text{ext}}).$$

In the (limited) classical input/output setting, $(u, y) = w$.

Assume that Σ is time-invariant, i.e. $\sigma^t \mathfrak{B} = \mathfrak{B}$ for all $t \in \mathbb{R}$, where σ^t denotes the t-shift, $(\sigma^t f)(t') := f(t' + t)$.

BEHAVIORAL STATE SYSTEMS

The **state property** is expressed by the requirement:

$$(w_1, x_1), (w_2, x_2) \in \mathfrak{B}, t_0 \in \mathbb{R}, \quad \text{and} \quad x_1(t_0) = x_2(t_0)$$

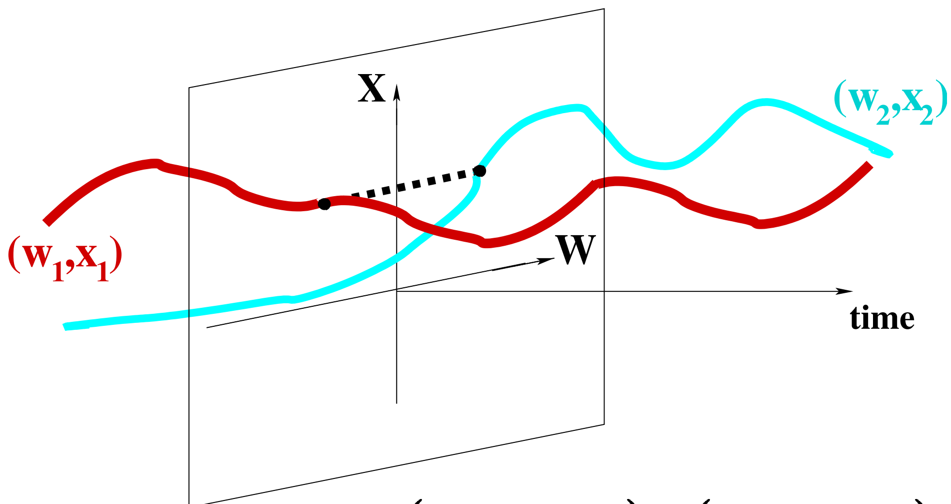
$$\Rightarrow \qquad (w_1, x_1) \wedge_{t_0} (w_2, x_2) \in \mathfrak{B}.$$

$\wedge_{t_0}$ denotes *concatenation* at t_0, defined as

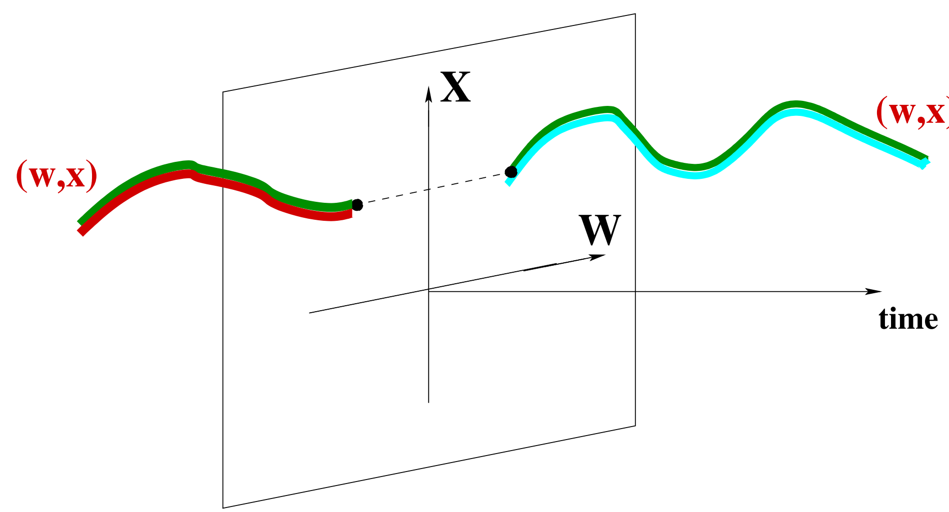
$$f_1 \underset{t_0}{\wedge} f_2(t) := \begin{cases} f_1(t) \text{ for } t < t_0 \\ f_2(t) \text{ for } t \geq t_0 \end{cases}$$

BEHAVIORAL STATE SYSTEMS

In pictures:



$$(w_1, x_1), (w_2, x_2) \in \mathfrak{B} \Rightarrow (w, x) \in \mathfrak{B}$$



BEHAVIORAL STATE SYSTEMS

This state definition is the implementation of the idea:

The state at time t, $x(t)$, contains all the information (about (w, x)!) that is relevant for the future behavior.

The state = the memory.

The past and the future are 'independent', conditioned on (given) the present state.

Example: $\Sigma : \quad \dot{\mathrm{x}} = f(\mathrm{x}, \mathrm{u}), \mathrm{y} = h(\mathrm{x}, \mathrm{u}), \mathrm{w} = (\mathrm{u}, \mathrm{y}).$

GENERAL DISSIPATIVE SYSTEMS

Let

$$s : \mathbb{W} \to \mathbb{R}$$

be a function, called the ***supply rate***, and assume that

$$w \in \mathfrak{B}_{\mathrm{ext}} \;\Rightarrow s(w) \in \mathcal{L}^{\mathrm{loc}}$$

Σ is said to be ***dissipative w.r.t.*** s if $\exists$

$$V : \mathbb{X} \to \mathbb{R},$$

called the ***storage function,*** such that

$$V(x(t_2)) \leq V(x(t_1)) + \int_{t_1}^{t_2} s(w(t))\, dt$$

$\forall\, (w, x) \in \mathfrak{B}$, and $t_2 \geq t_1$.

GENERAL DISSIPATIVE SYSTEMS

The basic theory is easily generalized to this setting. Assume:

1. **State space $\mathbb{X}$ of Σ connected:**
   every state reachable from every other state;

2. **Observability: given u, y,**
   $\exists$ at most one x such that $(w, x) \in \mathfrak{B}$.

Let $\mathrm{x}^* \in \mathbb{X}$ be an element of $\mathbb{X}$, a 'normalization' point for the storage functions, since these are only defined by an additive constant.

The def'ns of $V_{\mathrm{available}}$ and V_{required} remain unchanged (with, of course, $s(u, y)$ replaced by $s(w)$).

GENERAL DISSIPATIVE SYSTEMS

Basic theorem (general version): Let Σ and s be given. The following are equivalent:

1. Σ is dissipative w.r.t. s (i.e. $\exists$ a storage f'n V)
2. $$\oint s(w) \ dt \geq 0$$
   for all periodic $(w, x) \in \mathfrak{B}$.
3. $V_{\text{available}} < \infty$
4. $V_{\text{required}} > -\infty$

GENERAL DISSIPATIVE SYSTEMS

Moreover, assuming that any of these conditions are satisfied, then

$$\boldsymbol{V}_{\mathbf{available}} \quad \text{and} \quad \boldsymbol{V}_{\mathbf{required}}$$

are both storage functions, the set of storage f'ns is convex, and

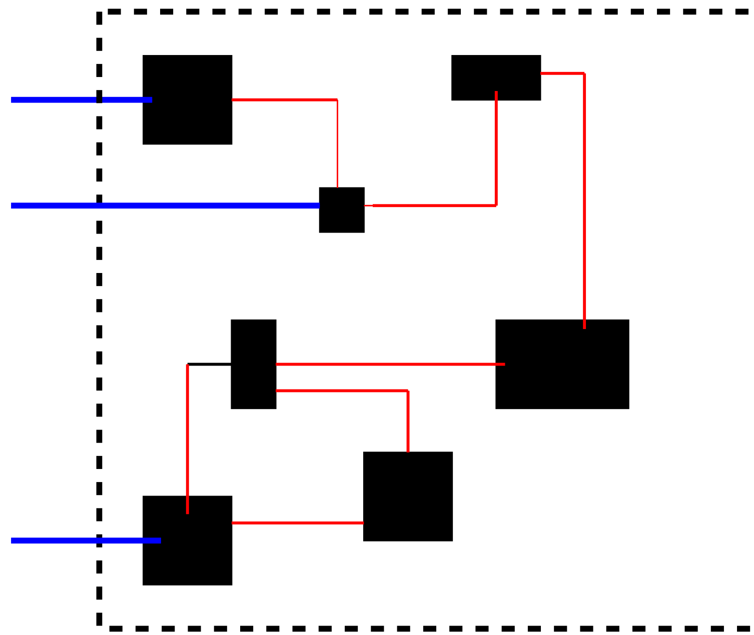
$$\boldsymbol{V}_{\mathbf{available}} \leq \boldsymbol{V} - \boldsymbol{V}(\mathbf{x}^*) \leq \boldsymbol{V}_{\mathbf{required}}$$

Proof:

No changes required from the differential equation case. Verify!

SYSTEM INTERCONNECTION

Interconnected system

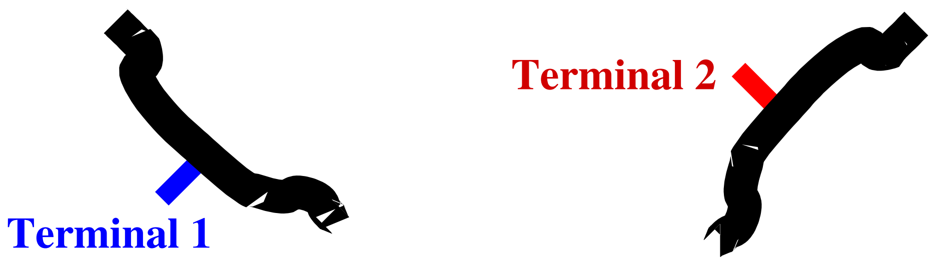


Formalize & prove: interconnection of dissipative systems is dissipative!

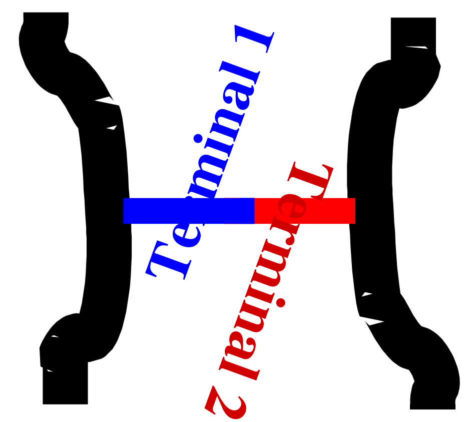
SYSTEM INTERCONNECTION

Think of interconnection in terms of physical terminals.

Before interconnection:



after interconnection:



SYSTEM INTERCONNECTION

Think of interconnection in terms of physical terminals.
Variables on such terminals:

Type of terminal	Variables	Signal space
electrical	(voltage, current)	$\mathbb{R}^2$
mechanical (1-D)	(force, position)	$\mathbb{R}^2$
mechanical (2-D)	((position, attitude), (force, torque))	$(\mathbb{R}^2 \times S^1) \times (\mathbb{R}^2 \times T^*S^1)$
mechanical (3-D)	((position, attitude), (force, torque))	$(\mathbb{R}^2 \times S^2) \times (\mathbb{R}^2 \times T^*S^2)$
thermal	(temp., heat flow)	$\mathbb{R}^2$
fluidic	(pressure, flow)	$\mathbb{R}^2$
thermal - fluidic	(pressure, temp., mass flow, heat flow)	$\mathbb{R}^4$

SYSTEM INTERCONNECTION

Think of interconnection in terms of physical terminals.
Imposes laws on the variables that 'live' on the terminals.

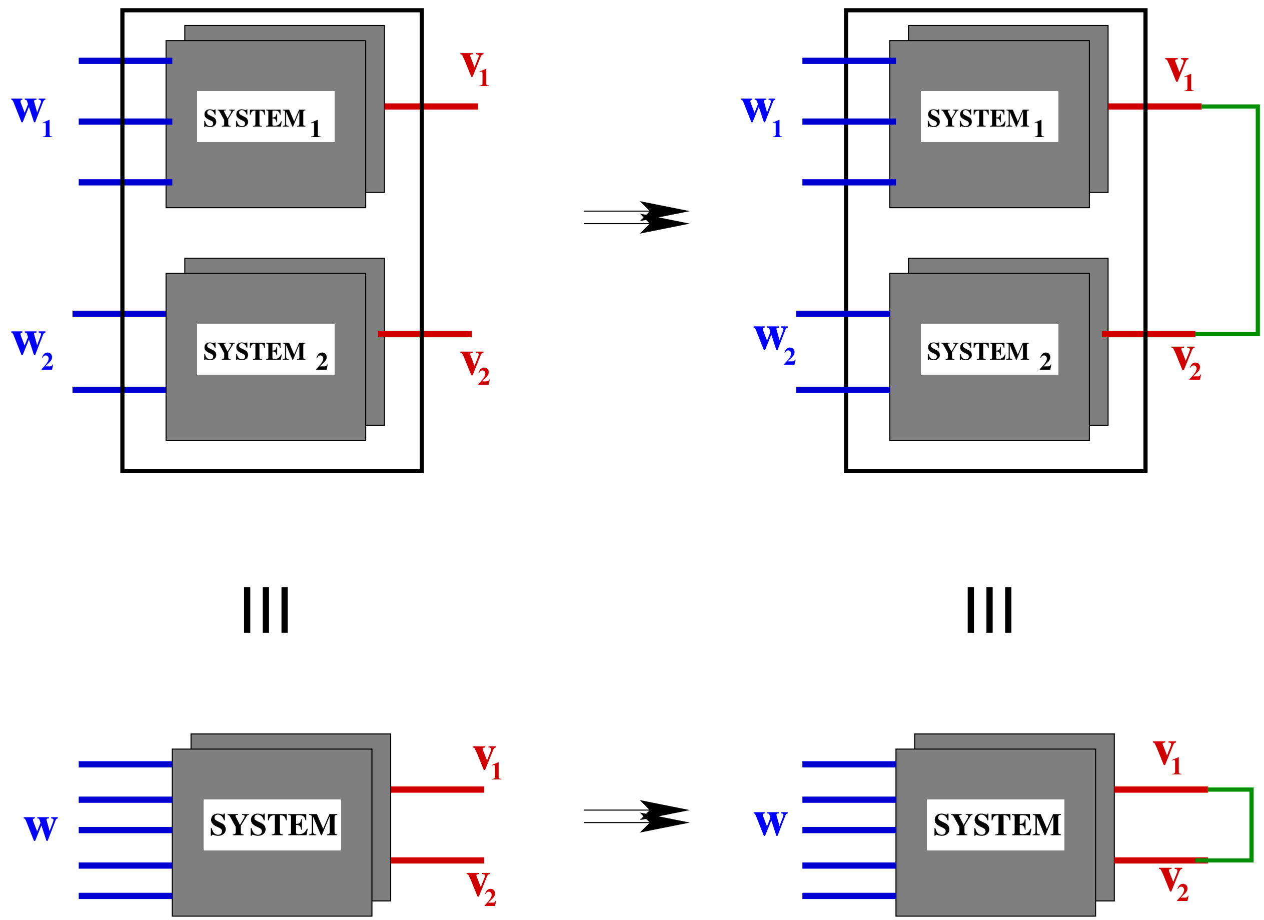
Pair of terminals	Terminal 1	Terminal 2	Law
electrical	(V_1, I_1)	(V_2, I_2)	$V_1 = V_2, I_1 + I_2 = 0$
1-D mech.	(F_1, q_1)	(F_2, q_2)	$F_1 + F_2 = 0, q_1 = q_2$
2-D mech.			
thermal	(Q_1, T_1)	(Q_2, T_2)	$Q_1 + Q_2 = 0, T_1 = T_2$
fluidic	(p_1, f_1)	(p_2, f_2)	$p_1 = p_2, f_1 + f_2 = 0$
info processing	$\mathbb{m}$-input u	$\mathbb{m}$-output y	$u = y$
etc.	etc.	etc.	etc.

SYSTEM INTERCONNECTION

Formalization of interconnection. (Also) this is (by far) easiest in the behavioral setting.

We proceed as if we want to interconnect two terminals of one and the same system. It is easy to see that this covers the general situation, even when interconnecting many terminals of many different systems.

SYSTEM INTERCONNECTION



SYSTEM INTERCONNECTION

Recall the definition of a behavioral system: $\Sigma = (\mathbb{R}, \mathbb{W}, \mathfrak{B})$, with $\mathbb{R}$ = the time-axis, $\mathbb{W}$ = the space of manifest variables, and $\mathfrak{B}$ = the behavior, $\mathfrak{B} \subseteq (\mathbb{W})^{\mathbb{R}}$.

Let

$$\Sigma = (\mathbb{R}, \mathbb{W} \times \mathbb{V}_1 \times \mathbb{V}_2, \mathfrak{B})$$

be a dynamical system.

The variables v_1 and v_2 are the variables that 'live' on the terminals which will be interconnected. As the idea of what interconnection does, we take: it imposes a static relation among the variables on the interconnected terminals. Interconnections should be 'trivialities' that obey all conceivable conservation laws.

SYSTEM INTERCONNECTION

Let

$$\Sigma = (\mathbb{R}, \mathbb{W} \times \mathbb{V}_1 \times \mathbb{V}_2, \mathfrak{B})$$

be a dynamical system.

$\rightsquigarrow$ the **interconnection constraint**

$$I(\mathrm{v}_1, \mathrm{v}_2) = 0.$$

and the **interconnected system** $\Sigma_I = (\mathbb{R}, \mathbb{W}, \mathfrak{B}_I)$ with

$$\mathfrak{B}_I := \{w \mid \exists (v_1, v_2) \text{ such that}$$

$$(w, v_1, v_2) \in \mathfrak{B} \text{ and } I(v_1(t), v_2(t)) = 0 \; \forall \, t\}.$$

SYSTEM INTERCONNECTION

Extends in a straightforward way to state systems

The state space of the interconnected system is the direct product of the state spaces of the components. Verify!

Note the controllability, observability, etc. may be destroyed by interconnection. Also the input/output structure may be hard to follow through the interconnection. The behavioral approach avoids 'well-posedness' questions.

INTERCONNECTION and DISSIPATIVITY

We will assume that the supply rate is additive among the terminals, i.e., if there are n terminals, with terminal variables

$$\mathrm{w}_1, \mathrm{w}_2, \ldots, \mathrm{w}_\mathrm{n},$$

leading to the space of manifest variables

$$\mathbb{W} = \mathbb{W}_1 \times \mathbb{W}_2 \times \cdots \mathbb{W}_\mathrm{n},$$

then

$$s((\mathrm{w}_1, \mathrm{w}_2, \ldots, \mathrm{w}_\mathrm{n})) = s_1(\mathrm{w}_1) + s_2(\mathrm{w}_2) + \cdots + s_\mathrm{n}(\mathrm{w}_\mathrm{n}).$$

INTERCONNECTION and DISSIPATIVITY

Consider two terminals with variables $\mathrm{v}_1, \mathrm{v}_2$ and supply rates $s_1(\mathrm{v}_1), s_2(\mathrm{v}_2)$. The interconnection constraint

$$I(\mathrm{v}_1, \mathrm{v}_2) = 0.$$

is said to be (supply) neutral :$\Leftrightarrow$

$$I(v_1(t), v_2(t)) = 0 \; \forall \, t \in \mathbb{R}$$

$$\Rightarrow s_1(v_1(t)) + s_2(v_2(t)) \; \forall \, t \in \mathbb{R}$$

INTERCONNECTION and DISSIPATIVITY

Consider two terminals with variables $\mathrm{v}_1, \mathrm{v}_2$ and supply rates $s_1(\mathrm{v}_1), s_2(\mathrm{v}_2)$. The interconnection constraint

$$I(\mathrm{v}_1, \mathrm{v}_2) = 0.$$

is said to be (supply) neutral :$\Leftrightarrow$

$$I(v_1(t), v_2(t)) = 0 \,\forall\, t \in \mathbb{R}$$

$$\Rightarrow s_1(v_1(t)) + s_2(v_2(t)) \,\forall\, t \in \mathbb{R}$$

Examples: Electrical terminals: Terminal var's: voltage, current.

$$s_1(V_1, I_1) = V_1 * I_1, \;\; s_2(V_2, I_2) = V_2 * I_2,$$

$$I(V_1, I_1, V_2, I_2) : V_1 = V_2, I_1 + I_2 = 0.$$

INTERCONNECTION and DISSIPATIVITY

Consider two terminals with variables $\mathrm{v}_1, \mathrm{v}_2$ and supply rates $s_1(\mathrm{v}_1), s_2(\mathrm{v}_2)$. The interconnection constraint

$$I(\mathrm{v}_1, \mathrm{v}_2) = 0.$$

is said to be (supply) neutral $:\Leftrightarrow$

$$I(v_1(t), v_2(t)) = 0 \; \forall\, t \in \mathbb{R}$$

$$\Rightarrow s_1(v_1(t)) + s_2(v_2(t)) \; \forall\, t \in \mathbb{R}$$

Examples: <u>Mechanical terminals</u>:

Terminal variables: force (F), position (q), velocity (v).

$v = \frac{d}{dt} q$ will be among the behavioral eq'ns.

$$s_1(F_1, q_1, v_1) = F_1 * v_1, \;\; s_2(F_2, q_2, v_2) = F_2 * v_2,$$

$$I(F_1, q_1, v_1, F_2, q_2, v_2) : q_1 = q_2, F_1 + F_2 = 0.$$

INTERCONNECTION and DISSIPATIVITY

Consider two terminals with variables $\mathrm{v}_1, \mathrm{v}_2$ and supply rates $s_1(\mathrm{v}_1), s_2(\mathrm{v}_2)$. The interconnection constraint

$$I(\mathrm{v}_1, \mathrm{v}_2) = 0.$$

is said to be (supply) neutral :$\Leftrightarrow$

$$I(v_1(t), v_2(t)) = 0 \; \forall \, t \in \mathbb{R}$$

$$\Rightarrow s_1(v_1(t)) + s_2(v_2(t)) \; \forall \, t \in \mathbb{R}$$

Examples: <u>**Heat flow terminals**</u>

INTERCONNECTION and DISSIPATIVITY

Consider two terminals with variables $\mathrm{v}_1, \mathrm{v}_2$ and supply rates $s_1(\mathrm{v}_1), s_2(\mathrm{v}_2)$. The interconnection constraint

$$I(\mathrm{v}_1, \mathrm{v}_2) = 0.$$

is said to be (supply) neutral :$\Leftrightarrow$

$$I(v_1(t), v_2(t)) = 0 \, \forall \, t \in \mathbb{R}$$

$$\Rightarrow s_1(v_1(t)) + s_2(v_2(t)) \, \forall \, t \in \mathbb{R}$$

Examples: **input/output connection:**

Terminal variables: terminal 1: y_1, terminal 2: u_2

$$s_1(y_1) = -||y_1||^2, \;\; s_2(u_2) = ||u_2||^2, I(y_1, u_2) : u_2 = y_1.$$

So with these supply rates, SIMULINK©'s connections are neutral.

INTERCONNECTION and DISSIPATIVITY

Theorem: Assume that

$$\Sigma = (\mathbb{R}, \mathbb{W} \times \mathbb{V}_1 \times \mathbb{V}_2, \mathbb{X}, \mathfrak{B})$$

is dissipative w.r.t.

$$s((\mathrm{w}, \mathrm{v}_1, \mathrm{v}_2)) = s'(\mathrm{w}) + s_1(\mathrm{v}_1) + s_2(\mathrm{v}_2)$$

with storage function V. Assume furthermore that the interconnection constraint $I(\mathrm{v}_1, \mathrm{v}_2) = 0$ is neutral w.r.t. $s_1 + s_2$.

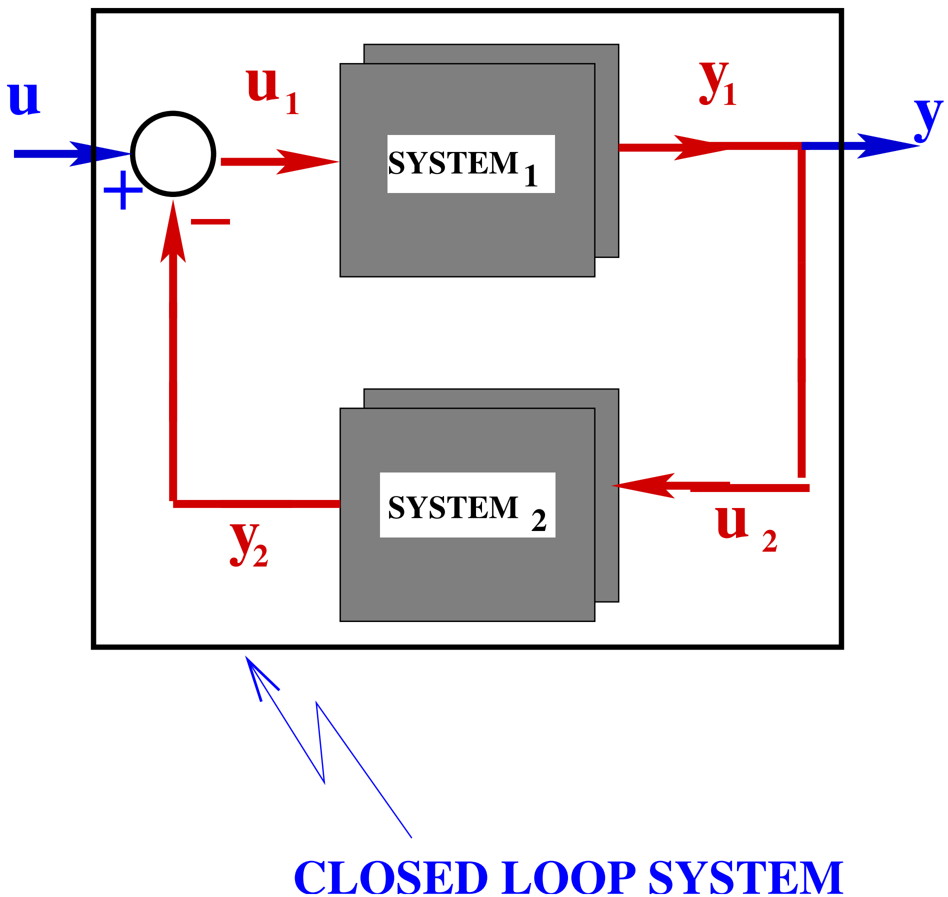
Then the interconnected system $\Sigma_I = (\mathbb{R}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_I)$ is dissipative w.r.t. s' with storage function V.

Proof: trivial

INTERCONNECTION and DISSIPATIVITY

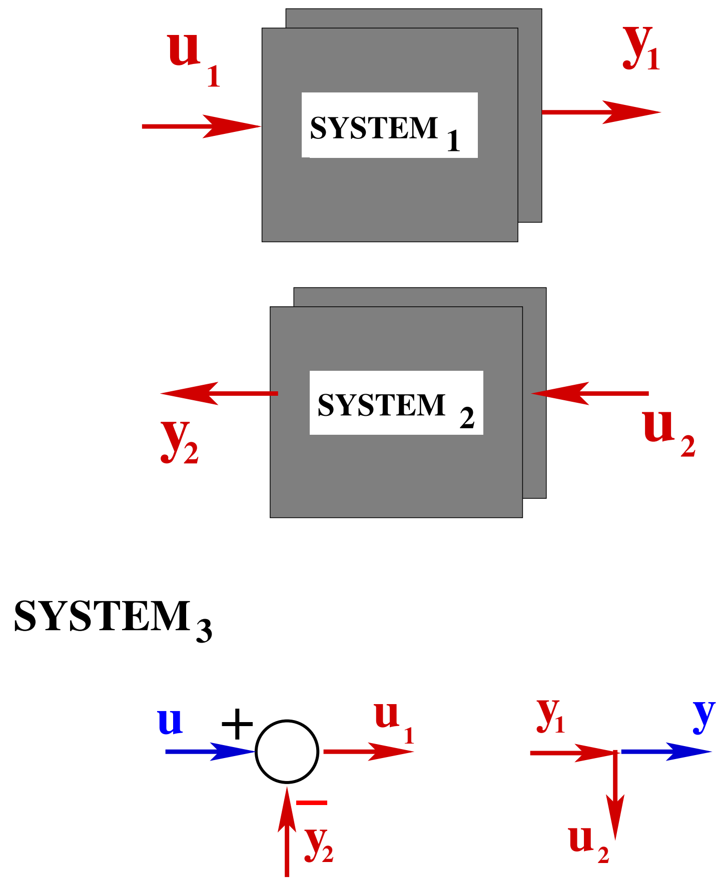
This theorem has a number of interesting applications.

1. Feedback and passivity. Consider the feedback system



INTERCONNECTION and DISSIPATIVITY

Decompose this as (the notation reflects the interconnection constraints):



INTERCONNECTION and DISSIPATIVITY

Now verify:

- **System 3 is dissipative w.r.t.** $\mathbf{u}^\top \mathbf{y} - \mathbf{u}_1^\top \mathbf{y}_1 - \mathbf{u}_2^\top \mathbf{y}_2$.
- **The interconnections are neutral.**

Conclude that if

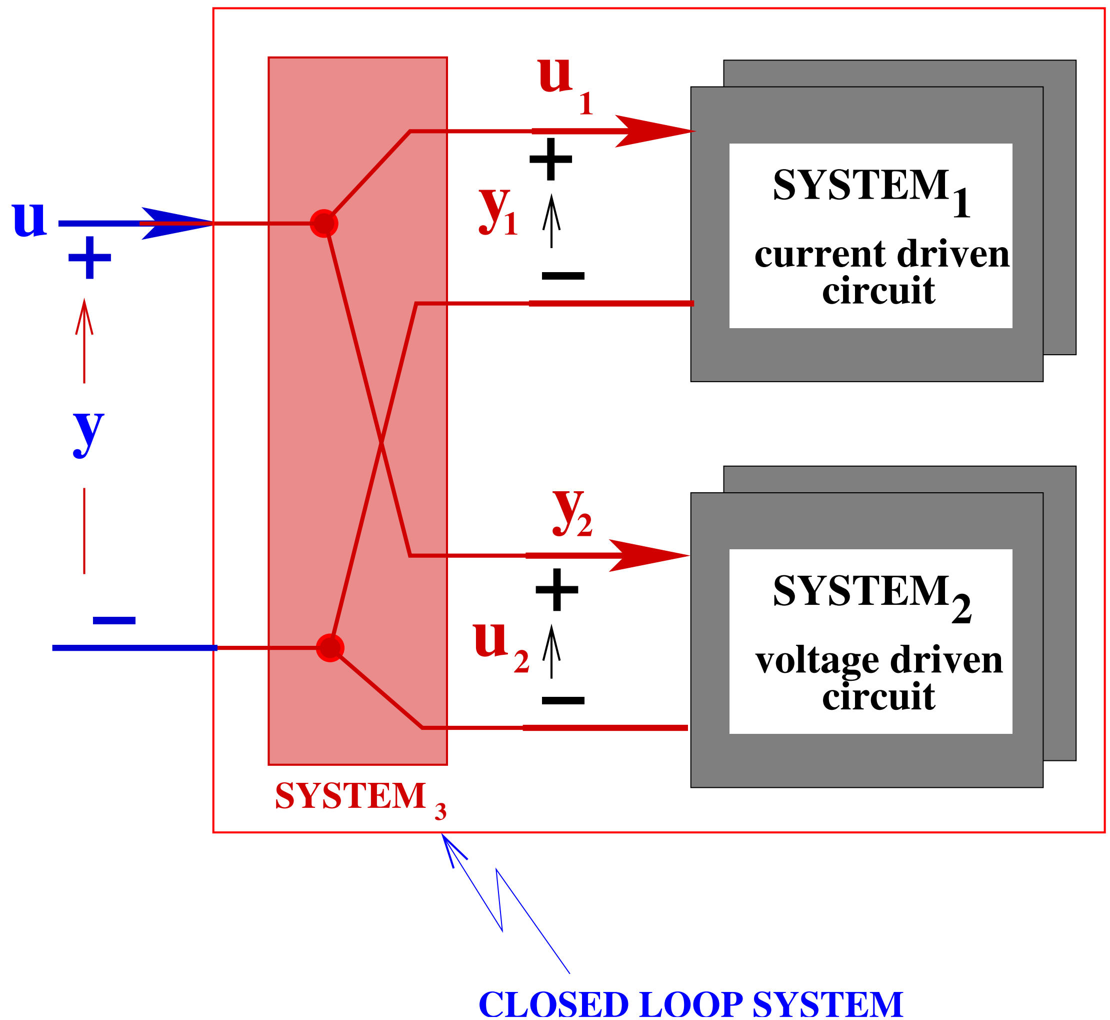
1. **System 1 is diss. (passive) w.r.t.** $\mathbf{u}_1^\top \mathbf{y}_1$ **with st. f'n** $V_1(\mathbf{x}_1)$
2. **System 2 is diss. (passive) w.r.t.** $\mathbf{u}_2^\top \mathbf{y}_2$ **with st. f'n** $V_2(\mathbf{x}_2)$

$\Rightarrow$ **feedback system dissipative (passive) w.r.t.** $\mathbf{u}^\top \mathbf{y}$,
storage function $V_1(\mathbf{x}_1) + V_2(\mathbf{x}_2)$.

Taking $u = 0$, **yields** $V_1(\mathbf{x}_1) + V_2(\mathbf{x}_2)$ **as a Lyapunov f'n.**
This is at the basis of many stability criteria.

INTERCONNECTION and DISSIPATIVITY

Physical interpretation:



INTERCONNECTION and DISSIPATIVITY

<u>Other siutations</u>:

1. **The Popov criterion**

   System 1: SISO LTI diff., diss. w.r.t. $u_1^\top(y_1 + \alpha \dot{y}_1)$ with st. f'n $V(x)$ (i.e., $G(\xi)(1+\alpha\xi)$ p.r.)

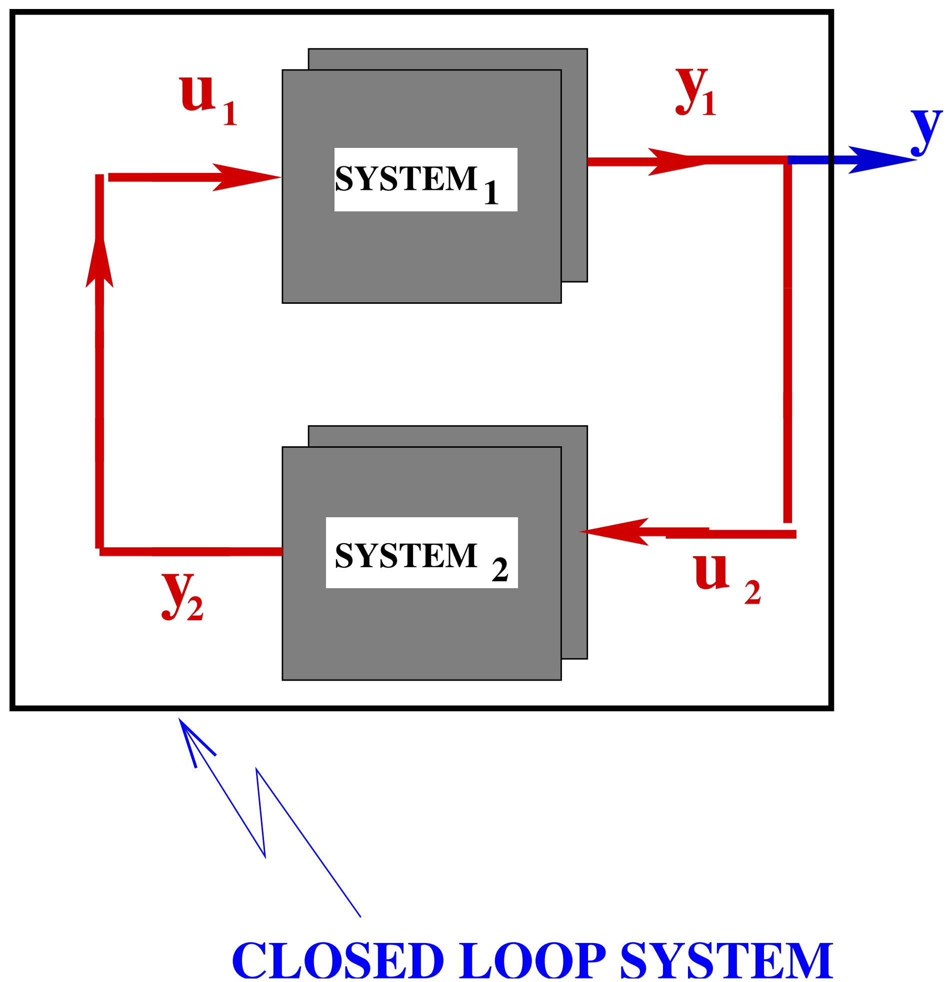
   System 2: a memoryless nonlinearity $u_2 \mapsto y_2 = f(u_2)$, with $\sigma f(\sigma) \geq 0 \; \forall \, \sigma \in \mathbb{R}$. This system is diss. w.r.t. $y_2^\top(u_2 + \alpha \dot{u}_2)$ with st. f'n $\alpha F(u_2)$, $F(\sigma) := \int_0^\sigma (\nu)\, d\nu$.

   $\Rightarrow$ feedback system dissipative w.r.t. $u^\top(y + \alpha \dot{y})$, with storage function $V(x) + \alpha F(y)$.

   Taking $u = 0$, yields $V(x) + \alpha F(y)$ as a Lyapunov f'n.

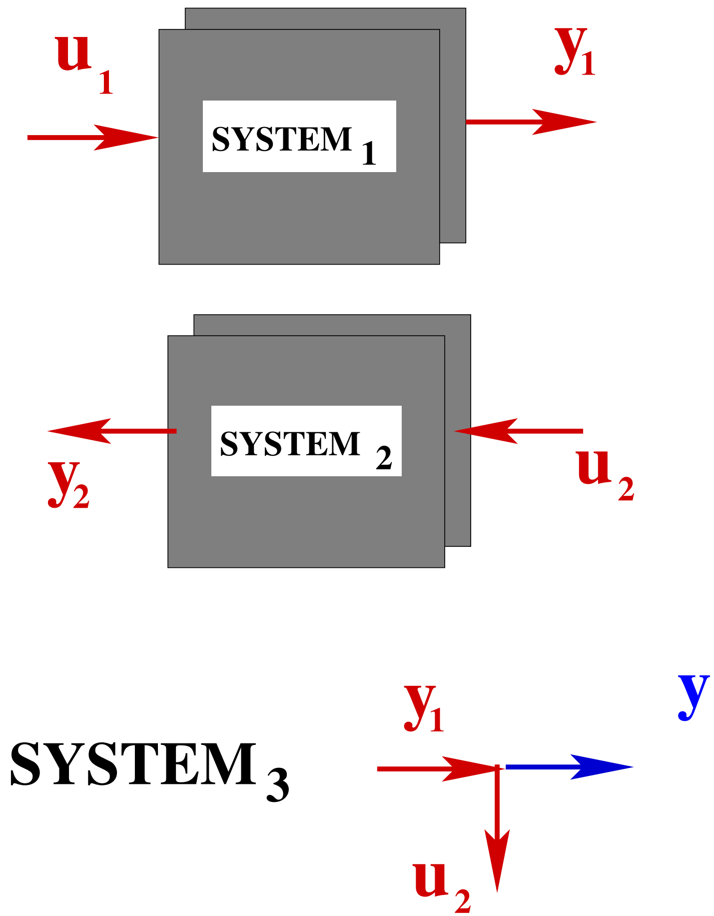
2. **The circle criterion** exercise

2. Feedback and contractivity. Consider the feedback system



INTERCONNECTION and DISSIPATIVITY

Decompose this as (the notation reflects the interconnection constraints):



INTERCONNECTION and DISSIPATIVITY

Now verify:

- **System 3 is dissipative w.r.t.** $||\mathrm{y}_1||^2 - ||\mathrm{u}_2||^2$.
- **The interconnections are neutral.**

Conclude that if

1. **System 1 is dissipative w.r.t.** $||\mathrm{u}_1||^2 - ||\mathrm{y}_1||^2$ **with storage f'n** $V_1(\mathrm{x}_1)$ **and**
2. **System 2 is dissipative w.r.t.** $||\mathrm{u}_2||^2 - ||\mathrm{y}_2||^2$ **with storage f'n** $V_2(\mathrm{x}_2)$,

$\Rightarrow$ **feedback system diss. w.r.t.** $s = 0$, **storage f'n** $V_1(\mathrm{x}_1) + V_2(\mathrm{x}_2)$.

This yields $V_1(\mathrm{x}_1) + V_2(\mathrm{x}_2)$ **as a Lyapunov f'n.**
This is at the basis of many stability criteria.

INTERCONNECTION and DISSIPATIVITY

Refinement:

Let $|\rho| \leq 1$. System 3 is dissipative w.r.t.

$$||\mathbf{y_1}||^2 - |\rho|^2||\mathbf{u_2}||^2 - (1 - |\rho|^2)||\mathbf{y}||^2.$$

Conclude that if

1. **System 1 diss. w.r.t. $||\mathbf{u_1}||^2 - ||\mathbf{y_1}||^2$ st. f'n $V_1(\mathbf{x_1})$**
2. **System 2 diss. w.r.t. $|\rho|^2||\mathbf{u_2}||^2 - ||\mathbf{y_2}||^2$ st. f'n $V_2(\mathbf{x_2})$,**

$\Rightarrow$ feedback system dissipative w.r.t. $-(1 - |\rho|^2)||\mathbf{y}||^2$ with storage f'n $V_1(\mathbf{x_1}) + V_2(\mathbf{x_2})$.

$\rightsquigarrow$ $V_1(\mathbf{x_1}) + V_2(\mathbf{x_2})$ as a Lyapunov f'n, with strictness on $\dot{V}^{\Sigma}$.
This is at the basis of many asymptotic stability criteria.

RECAP

- The basic th'm on dissipative systems holds for general state systems.
- System interconnection is readily formalized in the setting of behavioral systems.
- Under reasonable assumptions:

  interconnection of dissipative systems is dissipative.
- Essential for preservation of dissipativity by interconnection:

  interconnection constraints that are 'supply neutral'.
- Important application: the construction of Lyapunov functions for feedback systems with passivity or contractivity conditions on the open loop systems.

RLCT CIRCUITS

THE REALIZATION PROBLEM

Given a set of building blocks,
and a way to interconnect these building blocks,
what behaviors can be obtained?

Example 1: State representation algorithms. Building blocks:
adders, amplifiers, forks, integrators
(as in analog computers)

$$\rightsquigarrow \quad \textbf{LTIDS} \quad \dot{x} = Ax + Bu, \quad y = Cx + Du.$$

Example 2: Electrical circuit synthesis. Building blocks:
resistors, capacitors, inductors, connectors,
transformers, gyrators.

BUILDING BLOCKS

Module Types:

Resistors, Capacitors, Inductors, Transformers, Connectors.

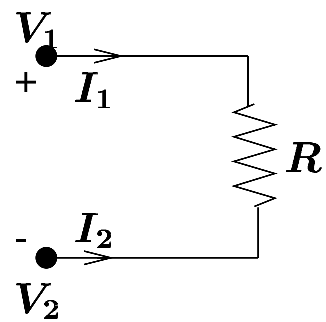
All terminals are of the same type: electrical,
and there are 2 variables associated with each terminal,

$$(V, I)$$

V **the *potential*,**
I **the *current* (counted > 0 when it flows into the module).**

$\rightsquigarrow$ **signal space of each terminal: $\mathbb{R}^2$.**

BUILDING BLOCKS



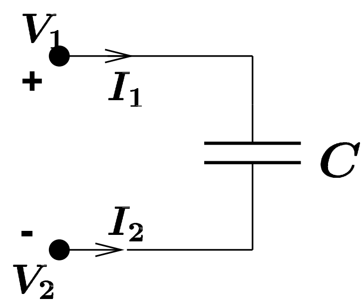
Resistor: 2-terminal module.

Parameter: $R > 0$ (resistance in ohms, say).

Device laws:

$$V_1 - V_2 = R\, I_1\ ; \quad I_1 + I_2 = 0.$$

BUILDING BLOCKS



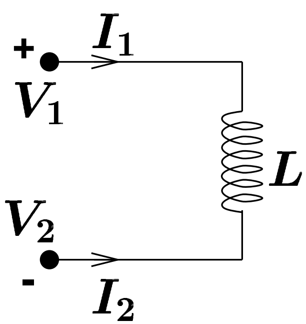
Capacitor: 2-terminal module.

Parameter: $C > 0$ (capacitance in farads, say).

Device laws:

$$C\,\frac{d}{dt}(V_1 - V_2) = I_1\,; \quad I_1 + I_2 = 0.$$

BUILDING BLOCKS



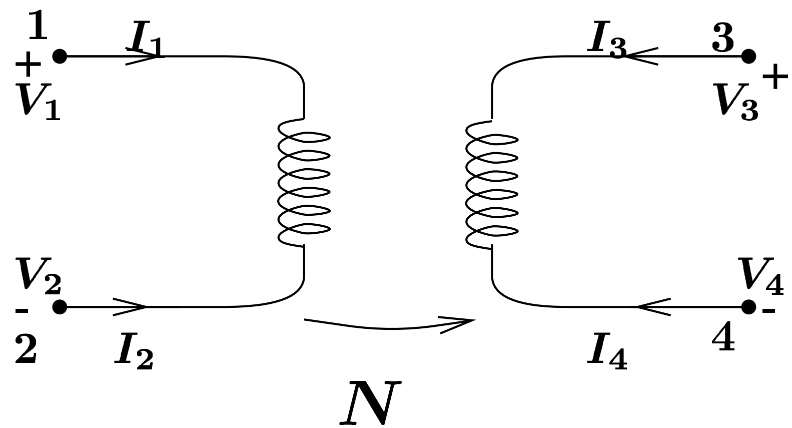
Inductor: 2-terminal module.

Parameter: $L > 0$ (inductance in henrys, say).

Device laws:

$$L\,\frac{d}{dt}I_1 = V_1 - V_2\,; \quad I_1 + I_2 = 0.$$

BUILDING BLOCKS

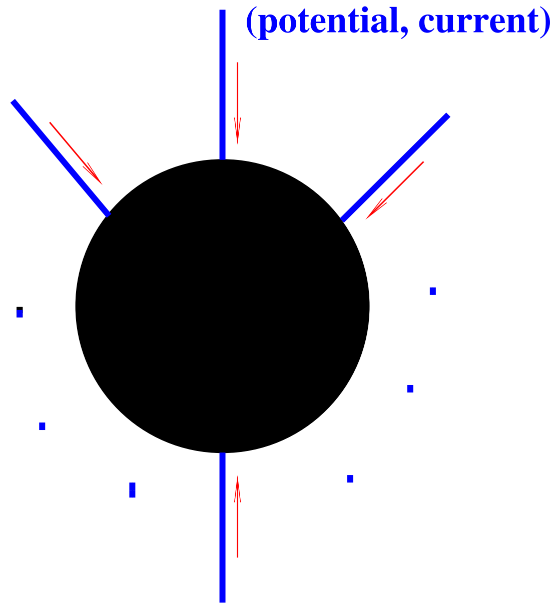


Transformer: **4-terminal module; terminals (1,2): primary;**
terminals (3,4): secondary.
Parameter: $N \in \mathbb{R}$ **(the turns ratio,** $\in (0, \infty)$ **).**
Device laws:

$$V_3 - V_4 = N(V_1 - V_2); I_1 = -NI_3;$$

$$I_1 + I_2 = 0; I_3 + I_4 = 0.$$

BUILDING BLOCKS



Connector: **many-terminal module.**

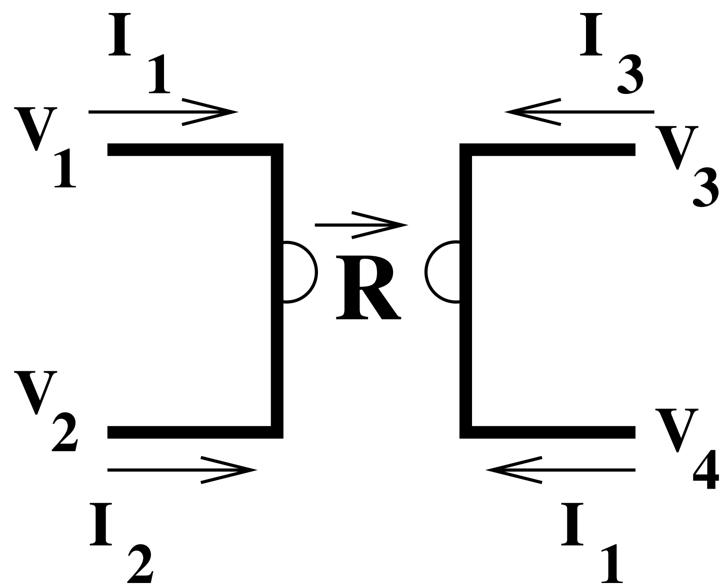
Parameter: $\mathtt{n}$ **(number of terminals, an integer).**

Device laws:

$$V_1 = V_2 = \cdots = V_{\mathrm{n}}\,; \quad I_1 + I_2 + \cdots + I_{\mathrm{n}} = 0.$$

BUILDING BLOCKS

In more advanced applications, we also meet the



Gyrator: 4-terminal module; (1,2): primary; (3,4): secondary.

Parameter: $R \in \mathbb{R}$ (gyrator resistance, in Ohms, say).

Device laws:

$$V_1 - V_2 = RI_3; V_3 - V_4 = -RI_1;$$

$$I_1 + I_2 = 0; I_3 + I_4 = 0.$$

INTERCONNECTION

Assume that terminal 1, with terminal variables V_1, I_1, is connected to terminal 2, with terminal variables V_2, I_2.

$\rightsquigarrow$ Interconnection constraint:

$$\mathcal{I}(V_1, I_1, V_2, I_2): \qquad V_1 = V_2, I_1 + I_2 = 0.$$

INTERCONNECTION

Assume that terminal 1, with terminal variables V_1, I_1, is connected to terminal 2, with terminal variables V_2, I_2.

$\rightsquigarrow$ Interconnection constraint:

$$I(V_1, I_1, V_2, I_2): \qquad V_1 = V_2, I_1 + I_2 = 0.$$

Now interconnect terminals of a (finite) number of building blocks.

The result is called a(n electrical) circuit.

INTERCONNECTION

Assume that terminal 1, with terminal variables V_1, I_1, is connected to terminal 2, with terminal variables V_2, I_2.

$\rightsquigarrow$ Interconnection constraint:

$$I(V_1, I_1, V_2, I_2): \qquad V_1 = V_2, I_1 + I_2 = 0.$$

Call the ‘unconnected’ terminals, the external terminals.

Number them: $(1, 2, \ldots, |E|)$.

Take as manifest variables of the circuit, the external terminal voltages and currents : $\Pi_{\mathrm{k} \in |E|} \ (V_{\mathrm{k}}, I_{\mathrm{k}})$.

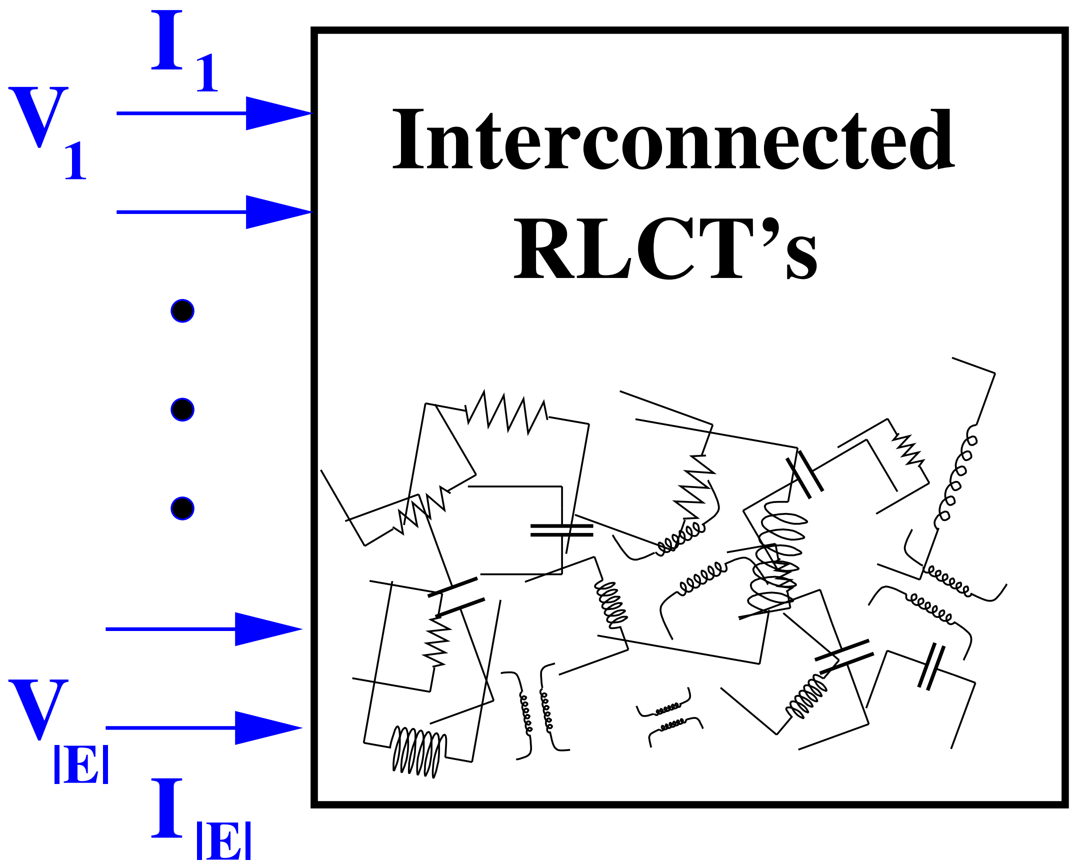
Denote $\Pi_{\mathrm{k} \in |E|} \ (V_{\mathrm{k}}, I_{\mathrm{k}})$ as $(V, I) \in \mathbb{R}^{2|E|}$.

By carrying out the interconnections, we end up with a system

$$(\mathbb{R}, \mathbb{R}^{2|E|}, \mathfrak{B}),$$

with external behavior: $\mathfrak{B} \subseteq (\mathbb{R}^{2|E|})^{\mathbb{R}}$.

INTERCONNECTION



CIRCUIT SYNTHESIS

The electrical circuit synthesis problem can be stated as follows:

***Realizability:* Which external behaviors can be obtained by interconnecting a finite number of R's, C's, L's, and T's? (or without T's, or with also G's?)**

***Synthesis:* If a behavior is realizable, give a wiring diagram (an architecture) that leads to the desired external behavior.**

CIRCUIT SYNTHESIS

The electrical circuit synthesis problem can be stated as follows:

Realizability: ***Which external behaviors can be obtained by interconnecting a finite number of R's, C's, L's, and T's? (or without T's, or with also G's?)***

Synthesis: ***If a behavior is realizable, give a wiring diagram (an architecture) that leads to the desired external behavior.***

This problem is of great importance (historical and otherwise) in electrical engineering. Important names:

Otto Brune	**R.M. Foster**	**W. Cauer**
E.A. Guillemin	**Sidney Darlington**	**A.D. Fialkow**
B.D.H. Tellegen	**Dante Youla**	**Vitold Belevitch**
etc., etc.		

CIRCUIT SYNTHESIS

We list seven necessary conditions!

We now discuss these conditions, aiming at demonstrating

- the relevance of passivity and positive realness
- the ease of analysis provided by the behavioral approach

CIRCUIT SYNTHESIS

We list seven necessary conditions!

1. $\mathfrak{B} \in \mathfrak{L}^{2|E|}$

i.e., $\Sigma = (\mathbb{R}, \mathbb{R}^{2|E|}, \mathfrak{B})$ **is a LTIDS. There are** ∞ **ways of stating what this means.**

For example, there exists a polynomial matrix $R^{\bullet \times 2|E|} \in \mathbb{R}[\xi]$ **such that** $\mathfrak{B}$ **consists of the solutions of**

$$R(\frac{d}{dt}) \begin{bmatrix} V \\ I \end{bmatrix} = 0.$$

Proof: Elimination th'm.

CIRCUIT SYNTHESIS

We list seven necessary conditions!

1. $\mathfrak{B} \in \mathfrak{L}^{2|E|}$
2. **KVL**

$$(V, I) \in \mathfrak{B} \text{ and } \alpha \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}) \Rightarrow (V + \alpha e, I) \in \mathfrak{B}$$

with

$$e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Proof: **Verify for each of the modules, and for the int. constraint.**

CIRCUIT SYNTHESIS

We list seven necessary conditions!

1. $\mathfrak{B} \in \mathfrak{L}^{2|E|}$
2. **KVL**
3. **KCL**

$$(V, I) \in \mathfrak{B} \Rightarrow e^\top I = 0$$

with

$$e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Proof: Verify for each of the modules, and for the int. constraint.

CIRCUIT SYNTHESIS

We list seven necessary conditions!

1. $\mathfrak{B} \in \mathfrak{L}^{2|E|}$
2. **KVL**
3. **KCL**
4. **The input cardinality,** $\mathtt{m}(\mathfrak{B}) = |E|$

In other words, there exist a partition of (V, I) in $|E|$ inputs and $|E|$ outputs, with, if you insist, a proper transfer function.

Consider this together with the next property.

CIRCUIT SYNTHESIS

We list seven necessary conditions!

1. $\mathfrak{B} \in \mathfrak{L}^{2|E|}$
2. KVL
3. KCL
4. The input cardinality, $\mathtt{m}(\mathfrak{B}) = |E|$
5. Hybridicity

There exists an I/O repr. for which the input and output var.

$$(u_1, u_2, \ldots, u_{|E|}), \ (y_1, y_2, \cdots, y_{|E|})$$

pair as follows: $\{u_{\mathrm{k}}, y_{\mathrm{k}}\} = \{V_{\mathrm{k}}, I_{\mathrm{k}}\}$

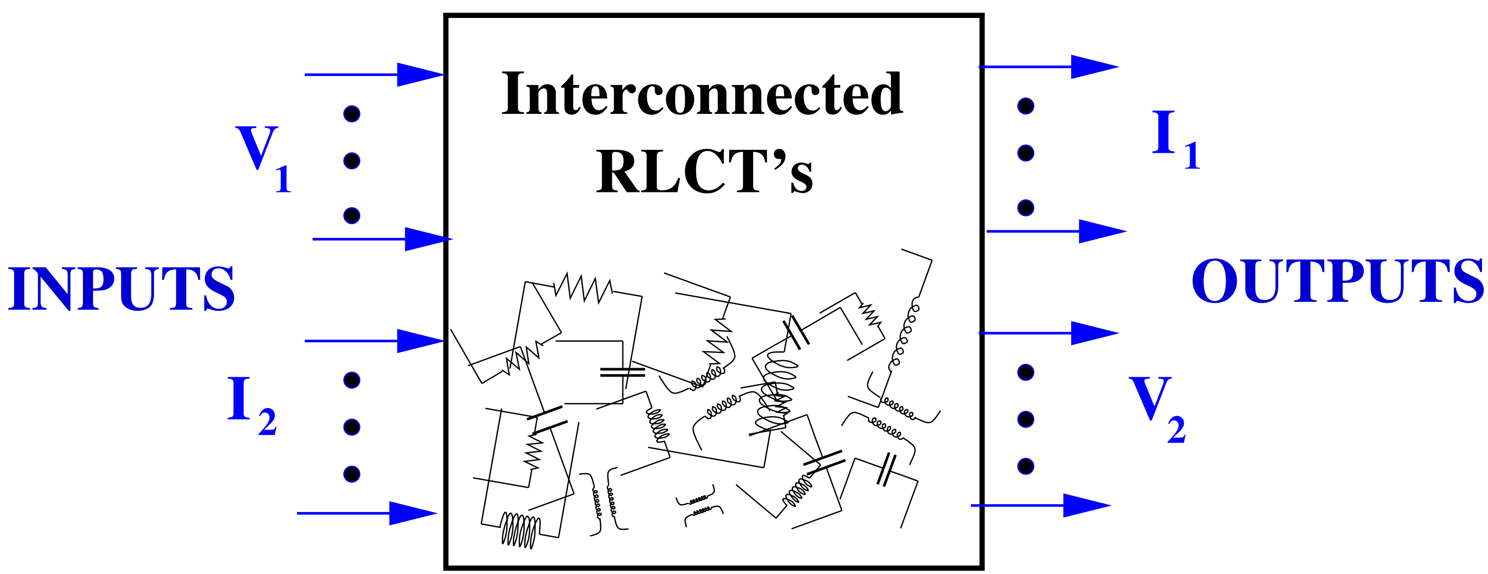
In other words, each terminal is either

current controlled or voltage controlled.

CIRCUIT SYNTHESIS

We list seven necessary conditions!

1. $\mathfrak{B} \in \mathfrak{L}^{2|E|}$
2. **KVL**
3. **KCL**
4. **The input cardinality,** $\mathtt{m}(\mathfrak{B}) = |E|$
5. **Hybridicity**



CIRCUIT SYNTHESIS

We list seven necessary conditions!

1. $\mathfrak{B} \in \mathfrak{L}^{2|E|}$
2. KVL
3. KCL
4. The input cardinality, $\mathtt{m}(\mathfrak{B}) = |E|$
5. Hybridicity
6. Passivity. From hybridicity, $\mathfrak{B}$ admits a representation as

$$\dot{\mathrm{x}} = A\mathrm{x} + B\mathrm{u}, \ \ \mathrm{y} = C\mathrm{x} + D\mathrm{u}$$

This system is dissipative w.r.t. the supply rate $u^\top y = V^\top I$, and with a quadratic positive definite storage f'n

$$V(\mathrm{x}) = \mathrm{x}^\top K \mathrm{x}, K = K^\top > 0.$$

This states that the net electrical energy goes into the circuit.

CIRCUIT SYNTHESIS

We list seven necessary conditions!

1. $\mathfrak{B} \in \mathfrak{L}^{2|E|}$
2. **KVL**
3. **KCL**
4. **The input cardinality,** $\mathtt{m}(\mathfrak{B}) = |E|$
5. **Hybridicity**
6. **Passivity.** **It is easiest to prove properties 4, 5, and 6 together.**

6: a circuit is an interconnection of passive elements, with neutral interconnection laws.

4 and 5: holds for passive circuits, prove it by considering one interconnection at the time.

CIRCUIT SYNTHESIS

We list seven necessary conditions!

1. $\mathfrak{B} \in \mathfrak{L}^{2|E|}$
2. **KVL**
3. **KCL**
4. **The input cardinality,** $\mathtt{m}(\mathfrak{B}) = |E|$
5. **Hybridicity**
6. **Passivity.**
7. **Reciprocity.** **The transfer f'n G is signature symmetric, i.e.**

$$\Sigma G = G^\top \Sigma.$$

Σ **is the signature matrix** $\Sigma = \mathrm{diag}(s_1, s_2, \ldots, s_{|E|})$,
with $s_\mathrm{k} = +1$ **if terminal** k **is voltage controlled,**
and $s_\mathrm{k} = -1$ **if terminal** k **is current controlled.**

CIRCUIT SYNTHESIS

We list seven necessary conditions!

1. $\mathfrak{B} \in \mathfrak{L}^{2|E|}$
2. KVL
3. KCL
4. The input cardinality, $\mathtt{m}(\mathfrak{B}) = |E|$
5. Hybridicity
6. Passivity.
7. Reciprocity.

This curious properties may be translated as:

The influence of terminal $\mathtt{k}'$ on terminal $\mathtt{k}''$ is equal to the influence of terminal $\mathtt{k}''$ on terminal $\mathtt{k}'$.

CIRCUIT SYNTHESIS

We list seven necessary conditions!

1. $\mathfrak{B} \in \mathfrak{L}^{2|E|}$
2. **KVL**
3. **KCL**
4. **The input cardinality,** $\mathtt{m}(\mathfrak{B}) = |E|$
5. **Hybridicity**
6. **Passivity.**
7. **Reciprocity.**

Proof: **Show that each of the modules satisfy property (7). Show that this property remain valid after interconnection, i.e. proceed again one interconnection at the time.**

CIRCUIT SYNTHESIS

We list seven necessary conditions!

1. $\mathfrak{B} \in \mathfrak{L}^{2|E|}$
2. **KVL**
3. **KCL**
4. **The input cardinality,** $\mathtt{m}(\mathfrak{B}) = |E|$
5. **Hybridicity**
6. **Passivity.**
7. **Reciprocity.**

If $\mathfrak{B}$ is controllable then these conditions are also sufficient for realizability. However, in order to obtain a 'clean' statement, it is convenient to eliminate $I_{|E|} = -I_1 - I_2 - \cdots - I_{|E|-1}$, and look at the behavior of $(V_1 - V_{|E|}, V_2 - V_{|E|}, \ldots, V_{|E|-1} - V_{|E|}, I_1, I_2, \ldots, I_{|E|-1})$.

CIRCUIT SYNTHESIS

We list seven necessary conditions!

1. $\mathfrak{B} \in \mathfrak{L}^{2|E|}$
2. **KVL**
3. **KCL**
4. **The input cardinality,** $\mathtt{m}(\mathfrak{B}) = |E|$
5. **Hybridicity**
6. **Passivity.**
7. **Reciprocity.**

The transfer function $G \in \mathbb{R}^{(|E|-1)\times(|E|-1)}$ ***is realizable using RLCT's if and only if it is signature symmetric and positive real.***

CIRCUIT SYNTHESIS

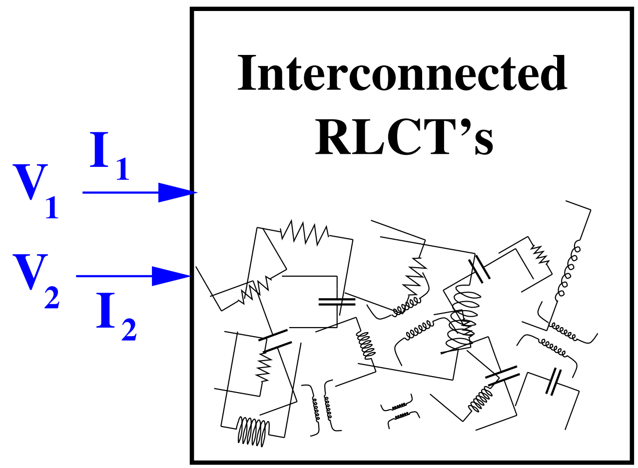
We list seven necessary conditions!

1. $\mathfrak{B} \in \mathfrak{L}^{2|E|}$
2. **KVL**
3. **KCL**
4. **The input cardinality,** $\mathtt{m}(\mathfrak{B}) = |E|$
5. **Hybridicity**
6. **Passivity.**
7. **Reciprocity.**

The transfer function $G \in \mathbb{R}^{(|E|-1)\times(|E|-1)}$ ***is realizable using RLCTG's if and only if it is positive real.***

SYNTHESIS of DRIVING POINT IMPEDANCES

Consider a 2-terminal circuit



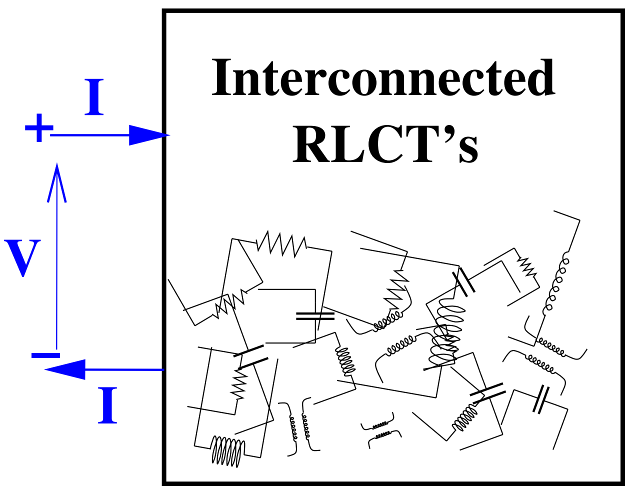
KCL $\Rightarrow$ $I_1 + I_2 = 0.$ **Set** $\boldsymbol{I := I_1 = -I_2}$.

KVL $\Rightarrow$ **the beh. eq'ns involve only** $V_1 - V_2$**. Set** $\boldsymbol{V := V_1 - V_2}$.

The behavior of (V, I) **is called the port description.**

SYNTHESIS of DRIVING POINT IMPEDANCES

Port description:



Z, the transfer f'n $I \mapsto V$ is called the **driving point impedance**.
Note that Z need not be proper.

Which driving point impedances are realizable?

SYNTHESIS of DRIVING POINT IMPEDANCES

Which driving point impedances are realizable?

$Z \in \mathbb{R}(\xi)$ is the driving point impedance
of an electrical circuit that consists of an interconnection
of a finite number of
positive R's, positive L's, positive C's, and transformers
if and only if Z is positive real.

SYNTHESIS of DRIVING POINT IMPEDANCES

Which driving point impedances are realizable?

$Z \in \mathbb{R}(\xi)$ **is the driving point impedance of an electrical circuit that consists of an interconnection of a finite number of positive R's, positive L's, positive C's, and transformers if and only if Z is positive real.**

This result led to the introduction of positive real functions.
First proven by Otto Brune in his M.I.T. Ph.D. dissertation (see O. Brune, *Synthesis of a finite two-terminal network whose driving point impedance is a prescribed function of frequency,* Journal of Mathematics and Physics, volume 10, pages 191-236, 1931).

SYNTHESIS of DRIVING POINT IMPEDANCES

Which driving point impedances are realizable?

$Z \in \mathbb{R}(\xi)$ is the driving point impedance
of an electrical circuit that consists of an interconnection
of a finite number of
positive R's, positive L's, positive C's, and transformers
if and only if Z is positive real.

Are transformers needed?

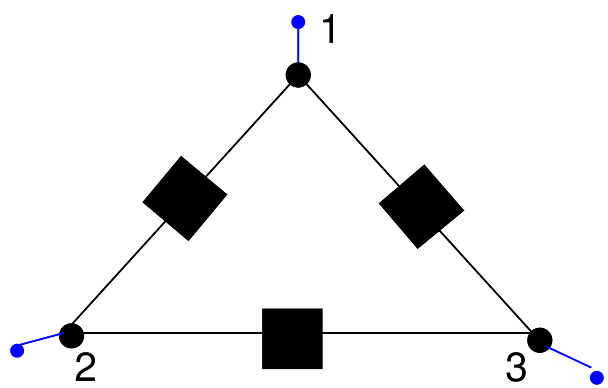
In 1949, Bott and Duffin proved 'no' in a one-page (!) paper (see R. Bott and R.J. Duffin, *Impedance synthesis without transformers,* Journal of Applied Physics, vol. 20, page 816, 1949).
However, their synthesis has common factors, non-controllability!

REMARK

TERMINALS versus PORTS

Note that we have used throughout the terminal description of circuits. It is simply more appropriate and more general (even when using only 'port' devices).

Example:



However, port descriptions are more parsimonious in the choice of variables (it halves their number).

RECAP

- **Realizability theory: an important engineering oriented problem area.**
- **The analysis and synthesis of RLCT circuits is an important application of passive systems.**
- **7 necessary conditions for realizability by passive R,L,C,T's:**
  differential system, KVL, KCL, input cardinality, hybridicity, passivity, and reciprocity.
- **In the controllable case these conditions are also sufficient.**
- **It is the circuit synthesis problem that led to positive realness.**