DISSIPATIVE SYSTEMS

Lectures
by

Jan C. Willems

K.U. Leuven

LYAPUNOV FUNCTIONS

Consider the classical dynamical system, the flow

$$\Sigma : \quad \overset{\bullet}{\mathrm{x}} = f(\mathrm{x})$$

with $\mathrm{x} \in \mathbb{X} = \mathbb{R}^{\mathrm{n}}$, the state, and $f : \mathbb{X} \to \mathbb{X}$, the vector-field.
Denote the set of solutions $x : \mathbb{R} \to \mathbb{X}$ by $\mathfrak{B}$, the 'behavior'.

LYAPUNOV FUNCTIONS

Consider the classical dynamical system, the **flow**

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Denote the set of solutions $x : \mathbb{R} \to \mathbb{X}$ by $\mathfrak{B}$, the **'behavior'.**

$$V : \mathbb{X} \to \mathbb{R}$$

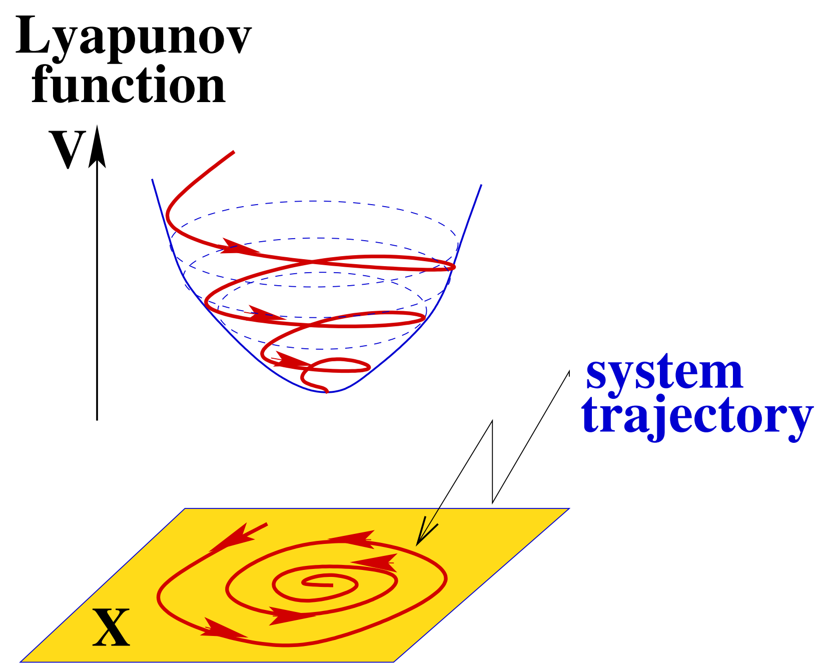
is said to be a **Lyapunov function** for Σ if along $x \in \mathfrak{B}$

$$\frac{d}{dt} V(x) \leq 0$$

Equivalently, if

$$\dot{V}^{\Sigma} := \nabla V \cdot f \leq 0.$$

LYAPUNOV FUNCTIONS



Let $\mathrm{x}^* =$ arg min $\{V(\mathrm{x}) \mid \mathrm{x} \in \mathbb{X}\}$

Typical Lyapunov 'theorem' $\cong$ **'global stability':**

$$V(\mathrm{x}) > 0 \text{ and } \overset{\bullet}{V}^{\Sigma}(\mathrm{x}) < 0 \text{ for } \mathrm{x}^* \neq \mathrm{x} \in \mathbb{X}$$

$$\Rightarrow \quad \forall \ x \in \mathfrak{B}, \text{ there holds } x(t) \to \mathrm{x}^* \text{ for } t \to \infty$$

LYAPUNOV FUNCTIONS

Refinements: LaSalle's invariance principle.

Converse: Kurzweil's thm.

LQ theory $\rightsquigarrow$ $\boldsymbol{A^\top X + XA = Y}$

'(matrix) Lyapunov equation'.

A linear system $\dot{\mathrm{x}} = A\mathrm{x}$ **is stable if and only if it has a quadratic positive definite Lyapunov function.**

Basis for most stability results in diff. eq'ns, physics, (adaptive) control, system identification, even numerical analysis.

LYAPUNOV FUNCTIONS

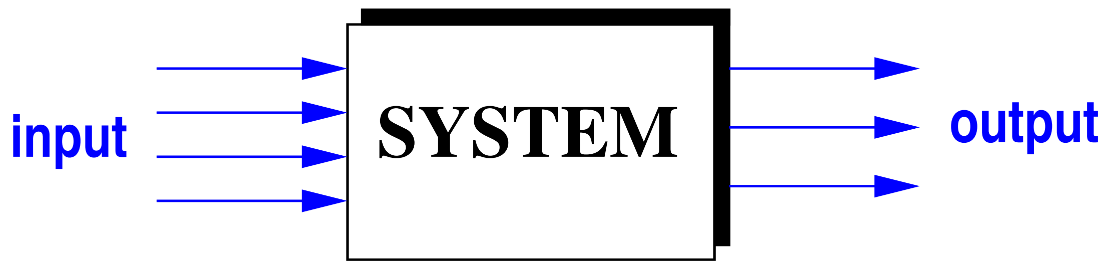
Lyapunov functions play a remarkably central role in the field.

Aleksandr Mikhailovich Lyapunov (1857-1918)

Introduced Lyapunov's 'second method' in his Ph.D. thesis (1899).

OPEN SYSTEMS

'Open' systems are a much more appropriate starting point for the study of dynamics. For example,



OPEN SYSTEMS

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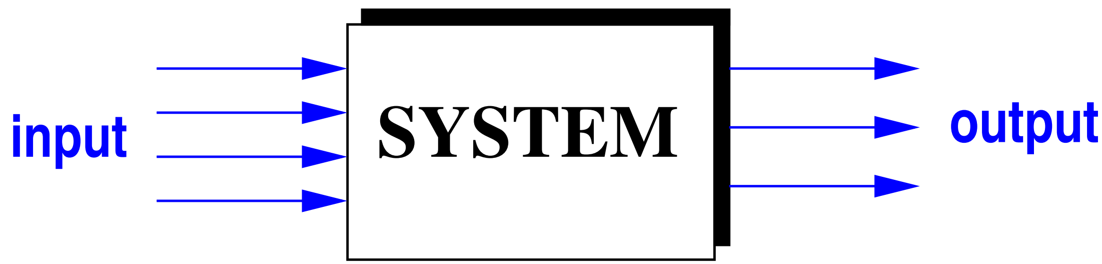
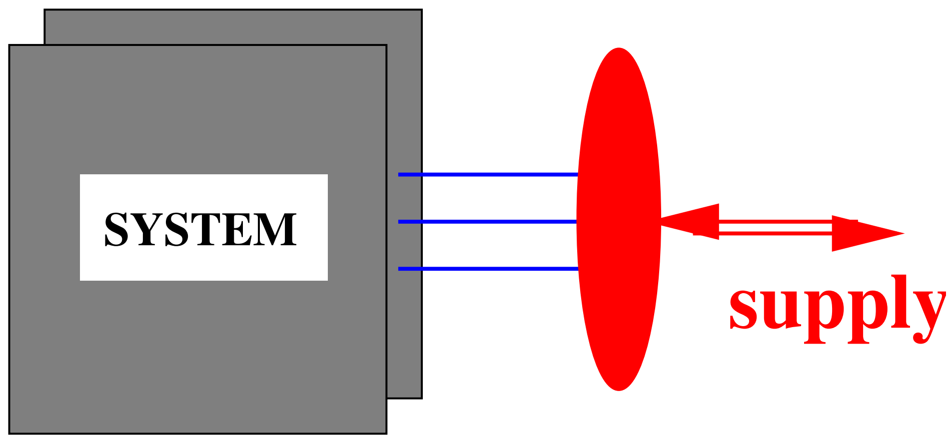


What is the analogue of a Lyapunov function for ‘open’ systems?

$\rightsquigarrow$ **Dissipative systems.**

THEME

A dissipative system absorbs supply.



THEME

A dissipative system absorbs supply.

How do we formalize this?

Physical examples.

Conditions for dissipativeness in terms of

(state space, transfer function) system representations.

Linear-quadratic theory.

Leads to important classical notion of positive realness.

How did this arise?

Direct applications of positive realness:

electrical circuit synthesis, covariance generation.

Applications to stability, stabilization, and robustness.

Part I: General Theory

DISSIPATIVE SYSTEMS: Def'n

Dynamics:

$$\Sigma : \quad \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}), \quad \mathbf{y} = h(\mathbf{x}, \mathbf{u}).$$

$\mathbf{u} \in \mathbb{U} = \mathbb{R}^{\mathtt{m}}, \mathbf{y} \in \mathbb{Y} = \mathbb{R}^{\mathtt{p}}, \mathbf{x} \in \mathbb{X} = \mathbb{R}^{\mathtt{n}}$:

the input, output, and state.

Behavior $\mathfrak{B} :=$ **all** $(u, y, x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$ **satisfying**

$$\frac{d}{dt}x = f(x, u), \;\; y = h(x, u).$$

$$\mathfrak{B}_{\text{external}} := \{(u, y) \mid \exists\, x : (u, y, x) \in \mathfrak{B}\}$$

external behavior.

DISSIPATIVE SYSTEMS: Def'n

Dynamics:

$$\Sigma : \quad \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}), \quad \mathbf{y} = h(\mathbf{x}, \mathbf{u}).$$

Let

$$s : \mathbb{U} \times \mathbb{Y} \rightarrow \mathbb{R}$$

be a function, called the *supply rate*.

DISSIPATIVE SYSTEMS: Def'n

Σ **is said to be** ***dissipative w.r.t.*** s **if** $\exists$

$$V : \mathbb{X} \to \mathbb{R},$$

called the ***storage function,*** **such that**

$$V(x(t_2)) \leq V(x(t_1)) + \int_{t_1}^{t_2} s(u(t), y(t))\, dt$$

$\forall\, t_2 \geq t_1$ **along trajectories:** $\forall\, (u, y, x) \in \mathfrak{B}$.

DISSIPATIVE SYSTEMS: Def'n

or, incrementally,

$$\frac{d}{dt}V(x) \leq s(u,y)$$

DISSIPATIVE SYSTEMS: Def'n

$$\frac{d}{dt} V(x) \leq s(u, y)$$

This $\leq$ is called the ***dissipation inequality***. $\Leftrightarrow$

$$\overset{\bullet}{V}^{\Sigma}(\mathrm{x}, \mathrm{u}) := \nabla V(\mathrm{x}) \cdot f(\mathrm{x}, \mathrm{u}) \leq s(\mathrm{u}, h(\mathrm{x}, \mathrm{u})) \quad \forall\, (\mathrm{x}, \mathrm{u}).$$

The function $d : \mathbb{U} \times \mathbb{X} \rightarrow \mathbb{R}$ defined by

$$d(\mathrm{u}, \mathrm{x}) := s(\mathrm{u}, h(\mathrm{x}, \mathrm{u})) - \overset{\bullet}{V}^{\Sigma}(\mathrm{x}, \mathrm{u})$$

is called the ***dissipation rate*** (≥ 0).

If equality holds: ***conservative system.***

DISSIPATIVE SYSTEMS: Def'n

$$\frac{d}{dt} V(x) \leq s(u, y)$$

For power and energy

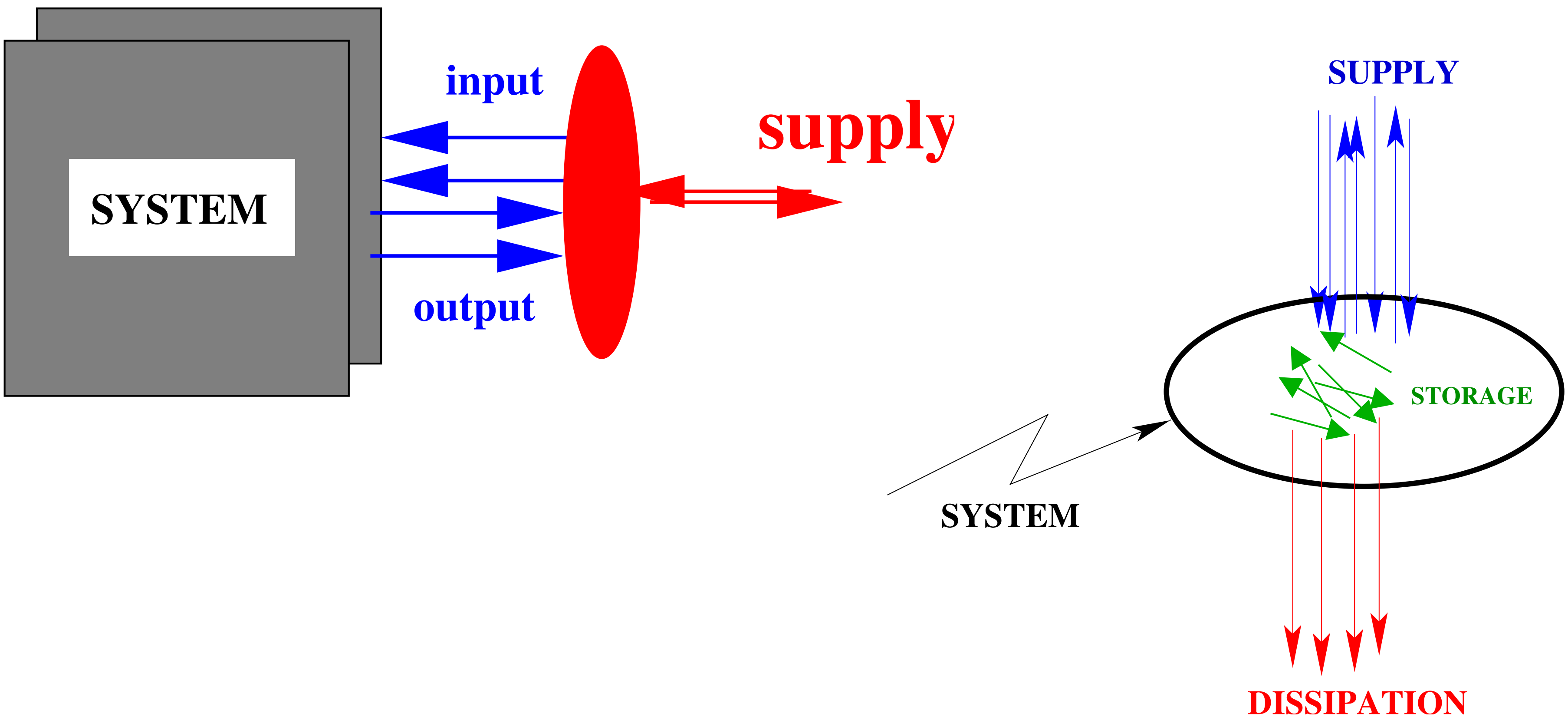
$s(\mathrm{u}, \mathrm{y}) \cong$ **power delivered.**

$V(\mathrm{x}) \cong$ **internal stored energy.**

Dissipativity $:\Leftrightarrow$

rate of increase of stored energy $\leq$ power delivered.

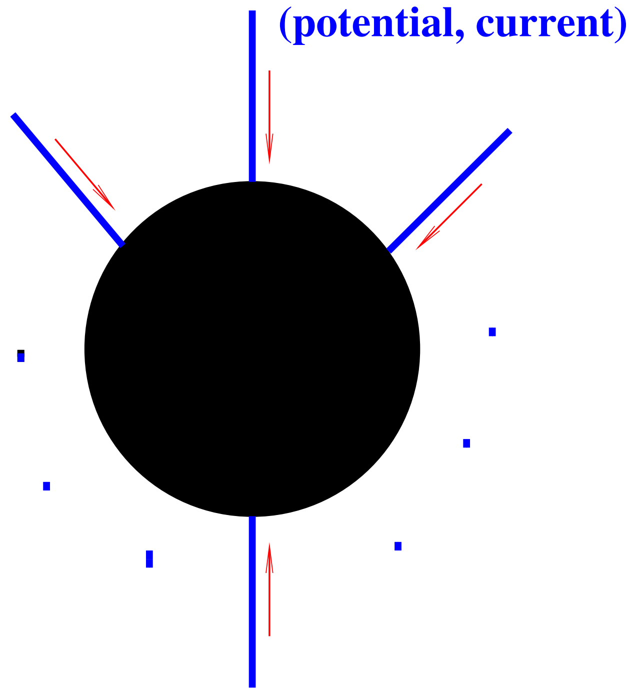
DISSIPATIVE SYSTEMS: Def'n



$$s(\mathrm{u}, h(\mathrm{x}, \mathrm{u})) = \dot{V}^{\Sigma}(\mathrm{x}, \mathrm{u}) + d(\mathrm{u}, \mathrm{x}) \qquad d \geq 0$$

PHYSICAL EXAMPLES

Electrical circuit:



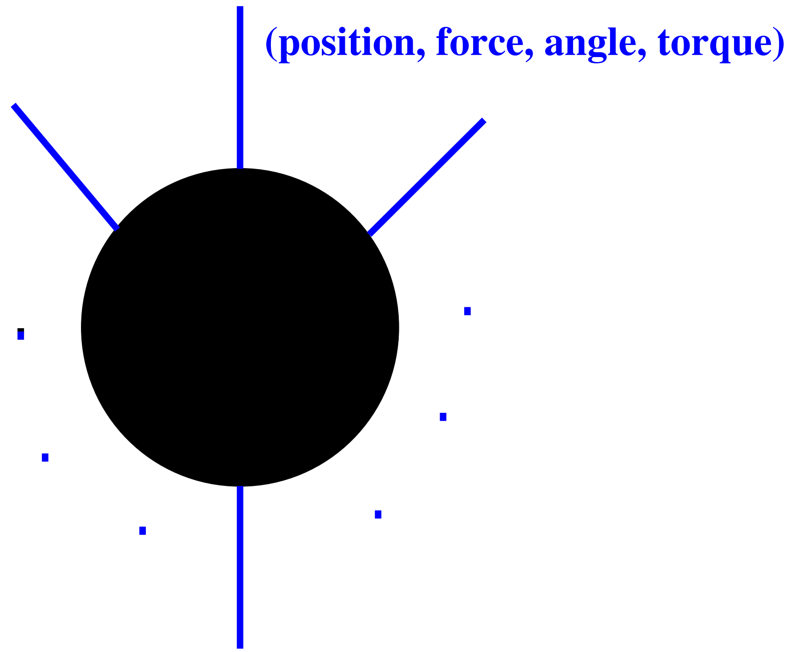
Dissipative w.r.t. $\Sigma_{\ell=1}^{N} V_\ell I_\ell$ **(electrical power).**

PHYSICAL EXAMPLES

System	Supply	Storage
Electrical circuit	$V^\top I$ V : voltage I : current	energy in capacitors and inductors

PHYSICAL EXAMPLES

Mechanical device:



Dissipative w.r.t. $\Sigma_{\ell=1}^{\mathtt{N}}((\frac{d}{dt}q_\ell)^\top F_\ell + (\frac{d}{dt}\theta_\ell)^\top T_\ell)$

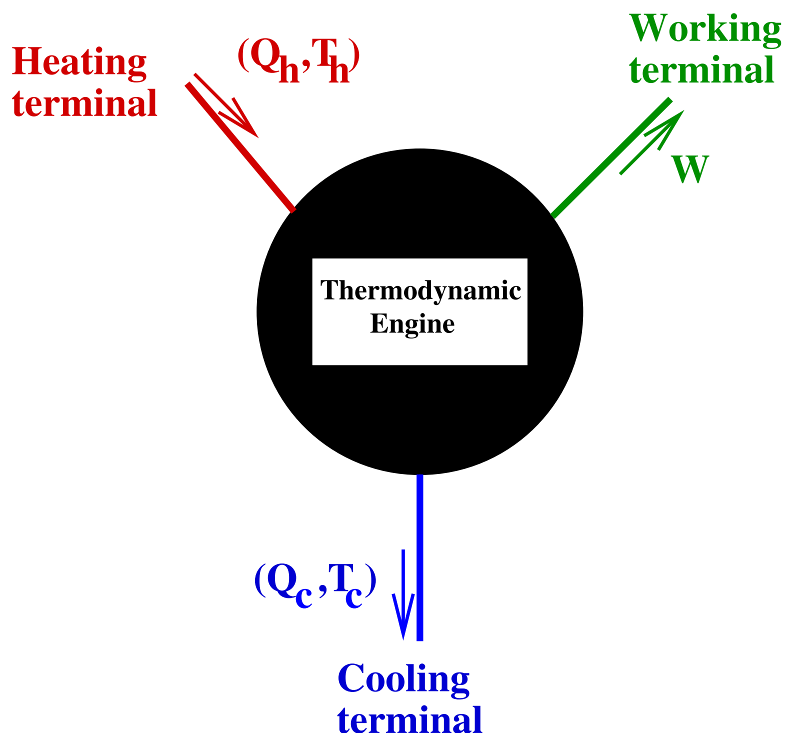
(mechanical power)

PHYSICAL EXAMPLES

System	Supply	Storage
Electrical circuit	$V^\top I$ V : voltage I : current	energy in capacitors and inductors
Mechanical system	$F^\top v + (\frac{d}{dt}\theta)^\top T$ F : force, v : velocity θ: angle, T : torque	potential + kinetic energy

PHYSICAL EXAMPLES

Thermodynamic system:



Conservative w.r.t. $\Sigma_{\ell=1}^{N} Q_\ell + \Sigma_{\ell=1}^{N'} W_\ell,$

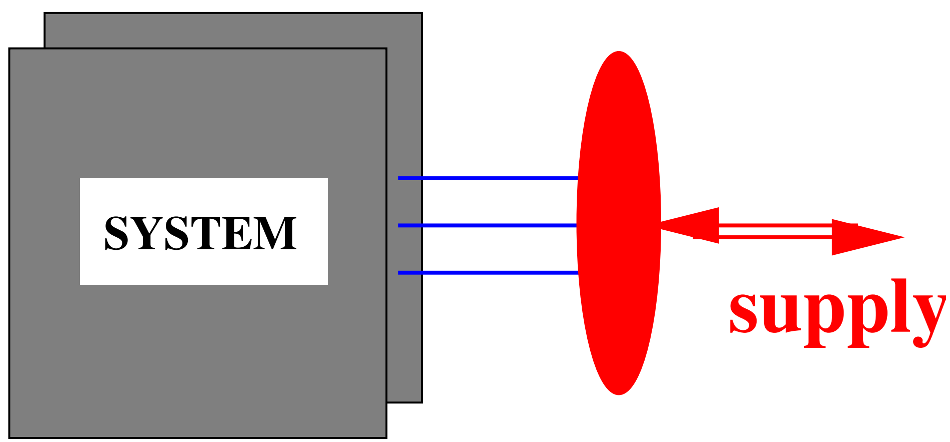
Dissipative w.r.t. $-\Sigma_{\ell=1}^{N} \frac{Q_\ell}{T_\ell}.$

PHYSICAL EXAMPLES

System	Supply	Storage
Electrical circuit	$V^\top I$ V : voltage I : current	energy in capacitors and inductors
Mechanical system	$F^\top v + (\frac{d}{dt}\theta)^\top T$ F : force, v : velocity θ: angle, T : torque	potential + kinetic energy
Thermodynamic system	$Q + W$ Q : heat, W : work	internal energy
Thermodynamic system	$-Q/T$ Q : heat, T : temp.	entropy
etc.	etc.	etc.

CONSTRUCTION of STORAGE F'NS

Given (a representation of) Σ, the dynamics, and given s, the supply rate, is the system dissipative w.r.t. s, i.e., does there exist a storage function V such that the dissipation inequality holds?



Monitor dynamics, power flow. How much 'energy' is stored?

CONSTRUCTION of STORAGE F'NS

Assume:

1. **State space $\mathbb{X}$ of Σ connected:**
   every state reachable from every other state;

2. **Observability: given u, y,**
   $\exists$ at most one x such that $(u, y, x) \in \mathfrak{B}$.

Let $\mathrm{x}^* \in \mathbb{X}$ be an element of $\mathbb{X}$, a 'normalization' point for the storage functions, since these are only defined by an additive constant.

CONSTRUCTION of STORAGE F'NS

Notation: $(\mathrm{x}_1, t_1) \overset{u}{\mapsto} (\mathrm{x}_2, t_2)$

:= u takes the state x_1 at time t_1 to state x_2 at time t_2.

Consider the following two state f'ns, universal storage f'ns:

The available storage: $V_{\text{available}}$, **defined by**

$$V_{\text{available}}(\mathrm{x}) := \underset{T \geq 0, (u,y,x) \in \mathfrak{B} : (\mathrm{x},0) \overset{u}{\mapsto} (\mathrm{x}^*, T)}{\text{supremum}} \{-\int_0^T s(u,y)\, dt\}$$

The required supply: V_{required}, **defined by**

$$V_{\text{required}}(\mathrm{x}) := \underset{T \geq 0, (u,y,x) \in \mathfrak{B} : (\mathrm{x}^*, -T) \overset{u}{\mapsto} (\mathrm{x},0)}{\text{infimum}} \{\int_{-T}^0 s(u,y)\, dt\}$$

CONSTRUCTION of STORAGE F'NS

Note:

if $\mathbf{x}^* \in \mathbb{X}$ is an equilibrium, associated with $\mathbf{u}^* \in \mathbb{U}$, $\mathbf{y}^* \in \mathbb{Y}$:

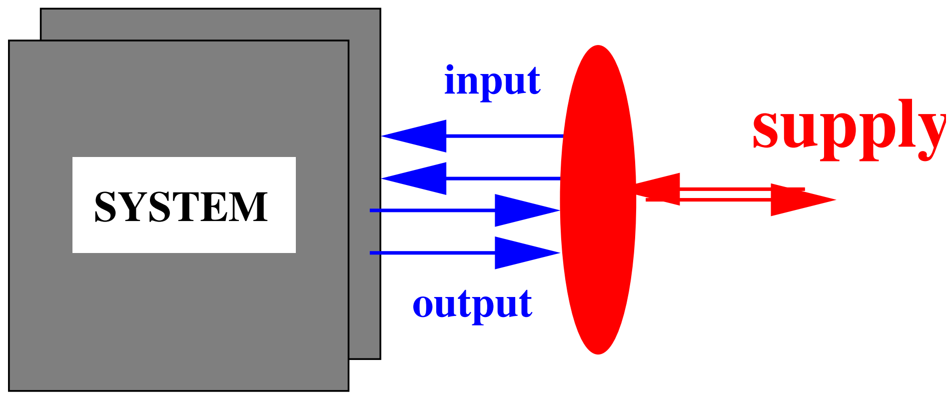
$$f(\mathbf{x}^*, \mathbf{u}^*) = 0, \mathbf{y}^* = h(\mathbf{x}^*, \mathbf{u}^*),$$

and

$$s(\mathbf{x}^*, \mathbf{u}^*) = 0,$$

then in def. of $V_{\text{available}}$ and V_{required}, we can take lim $T \to \infty$.

CONSTRUCTION of STORAGE F'NS



!! Maximize the supply extracted, starting in fixed initial state

$\rightsquigarrow$ **available storage.**

!! Minimize the supply needed to set up a fixed initial state

$\rightsquigarrow$ **required supply.**

CONSTRUCTION of STORAGE F'NS

Basic theorem: Let Σ and s be given.

The following are equivalent:

1. **Σ is dissipative w.r.t. s** (i.e. $\exists$ a storage f'n V)
2. $$\oint s(u,y) \ dt \geq 0$$
   for all **periodic** $(u,y) \in \mathfrak{B}_{\mathrm{external}}$, equivalently, by observability, for all periodic $(u,y,x) \in \mathfrak{B}$.
3. $V_{\mathrm{available}} < \infty$
4. $V_{\mathrm{required}} > -\infty$

CONSTRUCTION of STORAGE F'NS

Basic theorem: Let Σ and s be given.

Moreover, assuming that any of these conditions are satisfied, then

$$V_{\text{available}} \quad \text{and} \quad V_{\text{required}}$$

are both storage functions, the set of storage f'ns is convex, and

$$V_{\text{available}} - V_{\text{available}}(\mathbf{x}^*) \leq V - V(\mathbf{x}^*) \leq V_{\text{required}} - V_{\text{required}}(\mathbf{x}^*)$$

In fact, $V_{\text{available}}(\mathbf{x}^*) = V_{\text{required}}(\mathbf{x}^*) = 0$.

PROOF of the BASIC TH'M

$\mathbf{1. \Rightarrow 2.:}$

Σ is dissipative w.r.t. s $\Rightarrow$

$$\oint s(u,y)\ dt \geq 0$$

for all periodic $(u,y) \in \mathfrak{B}_{\text{external}}$:

Use the dissipation inequality (and observability).

PROOF of the BASIC TH'M

$\mathbf{2. \Rightarrow 3.:}$ $V_{\text{available}} : \mathbb{X} \to \mathbb{R}$

(i) $V_{\text{available}}(\mathrm{x}) > -\infty$: sup over non-empty set by reachability.

(ii) $V_{\text{available}}(\mathrm{x}) < \infty$:

Note that by 2., $(u, y, x) \in \mathfrak{B}$ and $(\mathrm{x}^*, T_1) \overset{u}{\mapsto} (\mathrm{x}^*, T_2)$ implies $\int_{T_1}^{T_2} s(u, y)\, dt \geq 0$.

Concatenate $(\mathrm{x}^*, -T') \overset{u'}{\mapsto} (\mathrm{x}, 0)$ with $(\mathrm{x}, 0) \overset{u}{\mapsto} (\mathrm{x}^*, T)$. Then

$$-\int_0^T s(u, y)\, dt \leq \int_{-T'}^0 s(u', y')\, dt.$$

Take the supremum over the left hand side.

Note $V_{\text{available}}(\mathrm{x}^*) = 0$: the sup then occurs for $T = 0$.

PROOF of the BASIC TH'M

3. $\Rightarrow$ 1. : $V_{\text{available}}$ **satisfies the dissipation inequality:**

$$V_{\text{available}}(x(t_1))$$

$$= \operatorname*{supremum}_{T \geq 0, (u,y,x) \in \mathfrak{B} : (x(t_1), t_1) \overset{u}{\mapsto} (\text{x}^*, t_1 + T)} \left\{ -\int_{t_1}^{t_1+T} s(u,y)\, dt \right\}$$

$$\geq \operatorname*{supremum}_{T \geq 0, (u,y,x) \in \mathfrak{B} : (x(t_1), t_1) \overset{u}{\mapsto} (\text{x}^*, t_2 + T)} \left\{ -\int_{t_1}^{t_2+T} s(u,y)\, dt \right\}$$

$$\geq -\int_{t_1}^{t_2} s(u,y)\, dt + \operatorname*{supremum}_{T \geq 0, (u,y,x) \in \mathfrak{B} : (x(t_2), t_2) \overset{u}{\mapsto} (\text{x}^*, t_2 + T)} \left\{ -\int_{t_2}^{t_2+T} s(u,y)\, dt \right\}$$

$$= -\int_{t_1}^{t_2} s(u,y)\, dt + V_{\text{available}}(x(t_2)).$$

PROOF of the BASIC TH'M

$\mathbf{2.} \Rightarrow \mathbf{4.} \Rightarrow \mathbf{1.}:$

The proof with V_{required} as a storage function is analogous.

PROOF of the BASIC TH'M

Convexity of the set of storage functions: obvious.

Bound $V_{\text{available}} \leq V - V(\mathrm{x}^*)$

Consider a trajectory $(\mathrm{x}, 0) \overset{u}{\mapsto} (\mathrm{x}^*, T)$. The dissipation inequality implies

$$V(\mathrm{x}) - V(\mathrm{x}^*) \geq -\int_0^T s(u(t), y(t))\, dt$$

Take the supremum of the right hand side.

PROOF of the BASIC TH'M

Bound $V_{\text{available}} \le V - V(\mathrm{x}^*)$

Consider a trajectory $(\mathrm{x}, 0) \overset{u}{\mapsto} (\mathrm{x}^*, T)$. The dissipation inequality implies

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Take the supremum of the right hand side.

Bound $V - V(\mathrm{x}^*) \le V_{\text{required}}$

is proven analogously.

SMOOTHNESS

To make all arguments rigorous requires certain assumptions.

The behavior $\mathfrak{B}$ of

$$\Sigma : \quad \overset{\bullet}{\mathrm{x}} = f(\mathrm{x}, \mathrm{u}), \quad \mathrm{y} = h(\mathrm{x}, \mathrm{u}).$$

must have the ‘state’ property, i.e.

$$(u_1, y_1, x_1), (u_2, y_2, x_2) \in \mathfrak{B}, t \in \mathbb{R}, \text{ and } x(t_1) = x(t_2)$$
$$\Rightarrow (u_1, y_1, x_1) \wedge_t (u_2, y_2, x_2) \in \mathfrak{B}$$

($\wedge_t$ denotes *concatenation* at t).

This can be achieved by assuming that the set of admissible input functions $\mathfrak{U} \subseteq \mathbb{U}^{\mathbb{R}}$ is closed under concatenation, and the sol’n set of x’s consists of abs. cont. f’ns.

SMOOTHNESS

To make all arguments rigorous requires certain assumptions.

The behavior $\mathfrak{B}$ of

$$\Sigma : \quad \overset{\bullet}{\mathrm{x}} = f(\mathrm{x}, \mathrm{u}), \quad \mathrm{y} = h(\mathrm{x}, \mathrm{u}).$$

must have the property that $s(u, y)$ is locally integrable , i.e.

$$\int_{t_1}^{t_2} s(u(t), y(t))\, dt < \infty \quad \forall\, (u, y, x) \in \mathfrak{B}, t_1, t_2 \in \mathbb{R}$$

SMOOTHNESS

To make all arguments rigorous requires certain assumptions.

The equivalence of the global and local versions of the dissipation inequality

1. $V(x(t_2)) \leq V(x(t_1)) + \int_{t_1}^{t_2} s(u(t), y(t))\, dt$
   $\forall (u, y, x) \in \mathfrak{B}$
2. $\frac{d}{dt} V(x) \leq s(u, y) \quad \forall (u, y, x) \in \mathfrak{B}$
3. $\nabla V(\mathrm{x}) \cdot f(\mathrm{x}, \mathrm{u}) \leq s(\mathrm{u}, h(\mathrm{x}, \mathrm{u})) \quad \forall \mathrm{u} \in \mathbb{U}, \mathrm{x} \in \mathbb{X}$

also requires certain smoothness on $\mathfrak{B}$, f and on V.

Obviously, V must be differentiable.
While for a given V one may simply wish to assume this, for $V_{\text{available}}$ and V_{required}, if needed,
this has to be proven.

SMOOTHNESS

To make all arguments rigorous requires certain assumptions.

1. $V(x(t_2)) \leq V(x(t_1)) + \int_{t_1}^{t_2} s(u(t), y(t))\, dt$
   $\forall (u, y, x) \in \mathfrak{B}$
2. $\frac{d}{dt} V(x) \leq s(u, y) \quad \forall (u, y, x) \in \mathfrak{B}$
3. $\nabla V(\mathrm{x}) \cdot f(\mathrm{x}, \mathrm{u}) \leq s(\mathrm{u}, h(\mathrm{x}, \mathrm{u})) \quad \forall \mathrm{u} \in \mathbb{U}, \mathrm{x} \in \mathbb{X}$

Note that assuming 1. for a ‘small’ behavior (e.g., having $\mathfrak{C}^\infty$, and/or compact support conditions), deducing from there 3., will yield, by integrating, 1. for a ‘large’ behavior (e.g. with locally integrable u’s, absolutely continuous x’s).

RECAP

- A system is dissipative :$\Leftrightarrow$

$$V(x(t_2)) \leq V(x(t_1)) + \int_{t_1}^{t_2} s(u,y)\, dt.$$

- $\exists$ many physical examples of dissipative open systems.
- $\exists$ storage function $\Leftrightarrow$

$$\oint s(u,y)\ dt \geq 0$$

for all periodic trajectories.

- Universal storage functions:

the available storage, the required supply.

Part II: Linear Quadratic Theory

LINEAR DIFFERENTIAL SYSTEMS

These admit many representations:

1. State space representation:

$$\Sigma : \quad \dot{\mathrm{x}} = A\mathrm{x} + B\mathrm{u}, \quad \mathrm{y} = C\mathrm{x} + D\mathrm{u}$$

$\mathbf{u} \in \mathbb{R}^{\mathtt{m}}, \mathbf{y} \in \mathbb{R}^{\mathtt{p}}, \mathbf{x} \in \mathbb{R}^{\mathtt{n}}; A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}, C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}, D \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$.

Notation: $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$**. Assume (in Part II & III) controllability and observability.** ***Behavior***

$$\mathfrak{B} := (u, y, x) : u \in \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{m}}), y \in \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{m}}), x : \mathbb{R} \to \mathbb{R}^{\mathtt{n}} \text{ abs. cont.}$$

satisfying $$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \;\; y(t) = Cx(t) + Du(t) \quad \text{a.e.}$$

Occasionally (when y is unimportant, we will denote $(u, x) \in \mathfrak{B}$).

LINEAR DIFFERENTIAL SYSTEMS

These admit many representations:

2. Transfer function $G \in \mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{m}}.$

Usual interpretation via exponential or frequency response, or Laplace transform, or differential equation (kernel representation)

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u$$

with $P \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi], Q \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}[\xi], G = P^{-1}Q$**, a left co-prime factorization, or (image representation)**

$$u = D(\frac{d}{dt})\ell, \quad y = N(\frac{d}{dt})\ell,$$

with $D \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi], Q \in \mathbb{R}^{\mathtt{m}\times\mathtt{p}}[\xi], G = ND^{-1}$**, a right co-prime fact.**

LINEAR DIFFERENTIAL SYSTEMS

These admit many representations:

3. Impulse response

$$y(t) = H_0 u(t) + \int_0^t H(t-t')u(t')\,dt' ,$$

possibly ‘completed’.

LINEAR DIFFERENTIAL SYSTEMS

These admit many representations:

4. Relations among these representations

$$G(\xi) = D + C(I\xi - A)^{-1}B$$

$G \in \mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{m}} =$ **the Laplace transform of**

$$t \in \mathbb{R}_+ \mapsto H_0\delta + H(t) \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}.$$

$$H_0 = D, H(t) = Ce^{At}B.$$

QUADRATIC SUPPLY RATES

$s(\mathrm{u},\mathrm{y}) =$ **a quadratic form in** (u,y)**.**

$\rightsquigarrow$ **a q.f. in** (u,x) **(**y **often not relevant, and by observability the properties - as periodicity, or** $\mathcal{L}_2$**, of** (u,y,x) **and** (u,x) **coincide):**

$$s(\mathrm{u},\mathrm{x}) = \begin{bmatrix}\mathrm{u}\\ \mathrm{x}\end{bmatrix}^\top S \begin{bmatrix}\mathrm{u}\\ \mathrm{x}\end{bmatrix}, \quad S = \begin{bmatrix} R & L \\ L^\top & Q\end{bmatrix}, \quad R = R^\top, Q = Q^\top$$

with as important special cases

$$s(\mathrm{u},\mathrm{y}) = ||\mathrm{u}||^2 - ||\mathrm{y}||^2,$$

$$\mathtt{m} = \mathtt{p} \quad \text{and} \quad s(\mathrm{u},\mathrm{y}) = \mathrm{u}^\top \mathrm{y}.$$

Relevant in electrical circuits (supply rate: $<$voltage,current$>$), mechanics: (supply rate $<$force,velocity$>$), scattering repr., etc.

LQ THEOREM

Theorem: Let $\Sigma = \left[\begin{array}{c|c} A & B \\ \hline \bullet & \bullet \end{array}\right]$ and $s(\mathrm{u}, \mathrm{x}) = \begin{bmatrix} \mathbf{u} \\ \mathbf{x} \end{bmatrix}^\top S \begin{bmatrix} \mathbf{u} \\ \mathbf{x} \end{bmatrix}$ be given.

The following are equivalent:

1. Σ is dissipative w.r.t. s.

LQ THEOREM

Theorem: Let $\Sigma = \left[\begin{array}{c|c} A & B \\ \hline \bullet & \bullet \end{array}\right]$ and $s(\mathrm{u},\mathrm{x}) = \begin{bmatrix} \mathrm{u} \\ \mathrm{x} \end{bmatrix}^\top S \begin{bmatrix} \mathrm{u} \\ \mathrm{x} \end{bmatrix}$ be given.

The following are equivalent:

2. **Behavioral characterizations:**

2.1

$$\oint s(u(t), x(t))\, dt \geq 0$$

for all periodic $(u, x) \in \mathfrak{B}$

2.2

$$\int_{-\infty}^{\infty} s(u(t), x(t))\, dt \geq 0$$

for all $(u, x) \in \mathfrak{B} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathrm{m}} \times \mathbb{R}^{\mathrm{n}})$

2.3 $\cdots\ \forall\ (u, x) \in \mathfrak{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^{\mathrm{m}} \times \mathbb{R}^{\mathrm{n}}); \mathcal{D} = \mathfrak{C}^{\infty}$ with comp. supp.

LQ THEOREM

Theorem: Let $\Sigma = \left[\begin{array}{c|c} A & B \\ \hline \bullet & \bullet \end{array}\right]$ and $s(\mathrm{u}, \mathrm{x}) = \begin{bmatrix} \mathrm{u} \\ \mathrm{x} \end{bmatrix}^\top S \begin{bmatrix} \mathrm{u} \\ \mathrm{x} \end{bmatrix}$ be given.

The following are equivalent:

3.1 Σ is dissipative w.r.t. s with a quadratic storage function.

3.2 Linear matrix inequality (LMI):

there exists $K = K^\top \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$ such that

$$\begin{bmatrix} A^\top K + KA - Q & KB - L^\top \\ B^\top K - L & -R \end{bmatrix} \leq 0$$

LQ THEOREM

Theorem: Let $\Sigma = \left[\begin{array}{c|c} A & B \\ \hline \bullet & \bullet \end{array}\right]$ and $s(\mathrm{u},\mathrm{x}) = \begin{bmatrix} \mathrm{u} \\ \mathrm{x} \end{bmatrix}^\top S \begin{bmatrix} \mathrm{u} \\ \mathrm{x} \end{bmatrix}$ be given.

The following are equivalent:

4. **Frequency-domain characterization**

$$R + L(i\omega I - A)^{-1}B + B^\top(-i\omega I - A^\top)^{-1}L$$
$$+B^\top(-i\omega I - A^\top)^{-1}Q(i\omega I - A)^{-1}B \geq 0$$

for all $\omega \in \mathbb{R}$, $i\omega \notin \sigma(A)$

$\sigma(\bullet)$ denotes the spectrum, the set of eigenvalues of $\bullet$

5. Characterization in terms of impulse response: ??

(LMI)

The matrix eq'n: $K = K^\top$

$$\begin{bmatrix} A^\top K + KA - Q & KB - L^\top \\ B^\top K - L & -R \end{bmatrix} \leq 0$$

has become a (the?) key equation in systems and control theory.
Note that this (LMI) states exactly that

$$\frac{d}{dt}x = Ax + Bu$$

$$\Rightarrow \quad \frac{d}{dt}x^\top K x \leq \begin{bmatrix} u \\ x \end{bmatrix}^\top \begin{bmatrix} R & L \\ L^\top & Q \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix},$$

i.e. that $\mathrm{x}^\top K \mathrm{x}$ is a (quadratic) storage f'n.

(LMI)

The matrix eq'n: $K = K^\top$

$$\begin{bmatrix} A^\top K + KA - Q & KB - L^\top \\ B^\top K - L & -R \end{bmatrix} \leq 0$$

has become a (the?) key equation in systems and control theory.

Solution set is convex, compact, and attains its infimum K_- and its supremum K_+:

$$K_- \leq K \leq K_+$$

$\mathrm{x}^\top K_- \mathrm{x}$ **= available storage,** $\mathrm{x}^\top K_+ \mathrm{x}$ **= required supply.**

(LMI)

The matrix eq'n: $K = K^\top$

$$\begin{bmatrix} A^\top K + KA - Q & KB - L^\top \\ B^\top K - L & -R \end{bmatrix} \leq 0$$

has become a (the?) key equation in systems and control theory.

If $R > 0$, then equivalent to Algebraic Riccati inequality (ARIneq)

$$K = K^\top$$

$$A^\top K + KA - Q + (KB - L^\top)R^{-1}(B^\top K - L) \leq 0$$

(LMI)

If $R > 0$, then equivalent to **Algebraic Riccati inequality (ARIneq)**

$$K = K^\top$$

$$A^\top K + KA - Q + (KB - L^\top)R^{-1}(B^\top K - L) \leq 0$$

In fact, there exist sol'ns to (ARIneq)
$\Leftrightarrow$ there exist sol'ns to the **Algebraic Riccati equation (ARE)**

$$K = K^\top$$

$$A^\top K + KA - Q + (KB - L^\top)R^{-1}(B^\top K - L) = 0$$

In particular, the extreme sol'n K_- and K_+ of (LMI) satisfy (ARE).
There exist various other characterizations of K_-, K_+.

PROOF of LQ TH'M and (LMI)

We will prove the equivalence of the following 10 statements:

I. $\exists \; V \cdots$ (1, page 15)

II. $\exists$ quadratic $V \cdots$ (3.1, page 15)

III. $\oint \geq 0$ for all periodic $\cdots$ (2.1, page 15)

IV. $\int \geq 0$ for all $\mathcal{L}_2 \cdots$ (2.2, page 15)

V. $\int \geq 0$ for all $\mathcal{L}_2$ of compact support $\cdots$

VI. $\int \geq 0$ for all $\mathfrak{C}^\infty$ of compact support $\cdots$ (2.3, page 15)

VII. Frequency domain condition (4, page 15)

VIII. (LMI) (3.2, page 15)

IX. For $R > 0$, solvability of the (ARIneq) (page 16)

X. For $R > 0$, solvability of the (ARE) (page 16); K_-, K_+ sol'ns.

PROOF of LQ TH'M and (LMI)

I $\Rightarrow$ VIII **I: $\exists$ $V \cdots$** $\Rightarrow$ **VIII: $\exists$ sol'n to the (LMI)**

The difficult part is the following proposition, which we take for granted

Proposition: Assume that

$$\underset{T\geq 0, (\mathrm{x},0)\overset{u}{\mapsto}(0,T)}{\text{supremum}} \quad \{-\int_0^T s(u,x)\,dt\} < \infty \qquad \forall\, \mathrm{x} \in \mathbb{R}^{\mathrm{n}}$$

Then this supremum is a quadratic form in x,

$$\mathrm{x}^\top K \mathrm{x}, \qquad \text{and} \qquad K = K^\top.$$

It follows from the basic th'm that $\mathrm{x}^\top K \mathrm{x}$ satisfies the dissipation inequality, equivalently, the (LMI).

PROOF of LQ TH'M and (LMI)

I $\Rightarrow$ VIII **VIII $\Leftrightarrow$ IX**

Include VIII $\Leftrightarrow$ IX only in the case $R > 0$.

VIII: $\exists$ sol'n to the (LMI) $\Rightarrow$ IX: $\exists$ sol'n to the (ARIneq)

Schur complement: **Let $M_{11} = M_{11}^\top, M_{22} = M_{22}^\top > 0$. Then**

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix} \geq 0 \quad \Leftrightarrow \quad M_{11} - M_{12} M_{22}^{-1} M_{12}^\top \geq 0.$$

PROOF of LQ TH'M and (LMI)

I $\Rightarrow$ VIII **VIII $\Leftrightarrow$ IX** **IX $\Leftrightarrow$ X**

Include IX $\Leftrightarrow$ X only in the case $R > 0$.

IX: $\exists$ sol'n to (ARIneq) $\Leftrightarrow$ X: $\exists$ sol'n to (ARE), K_-, K_+ sol'ns

$\Rightarrow$ is trivial. To show $\Rightarrow$, use the following propostion, a clever idea due to C. Scherer.

Proposition: Assume $F = F^\top \geq 0, H = H^\top$, and (A, F) controllable. Then if the ARineq

$$X = X^\top, \quad A^\top X + XA + XFX + H \leq 0$$

has a sol'n, so does the (ARE)

$$Y = Y^\top, A^\top Y + YA + YFY + H = 0$$

Proof: Define $P := -(A^\top X + XA + XFX + H)$ and consider the (ARE)

$$D = D^\top, \quad (A + FX)^\top D + D(A + FX) + DFD - P = 0.$$

This is a 'standard' (in the sense that $F \geq 0, P \geq 0, (A, F)$ contr.) (ARE) of the theory of LQ optimal control. We assume that it is known that a sol'n D exists. Now prove by a straightforward calculation that $Y = X + D$ solves the (ARE). Now, there even exist a sol'ns $D \geq 0$ and ≤ 0. Hence the infimal and supremal sol's of (LMI) and (ARIeq) solve (ARE).

PROOF of LQ TH'M and (LMI)

I $\Rightarrow$ VIII **VIII $\Leftrightarrow$ IX** **IX $\Leftrightarrow$ X** **VIII $\Leftrightarrow$ II $\Rightarrow$ I**

VIII: $\exists$ sol'n to (ARE) $\Leftrightarrow$ **II: $\exists$ quadratic $V \cdots$** $\Rightarrow$ **$\exists$ $V \cdots$**

Trivial.

PROOF of LQ TH'M and (LMI)

I $\Rightarrow$ VIII **VIII $\Leftrightarrow$ IX** **IX $\Leftrightarrow$ X** **VIII $\Leftrightarrow$ II $\Rightarrow$ I** **I $\Leftrightarrow$ III**

I: $\exists\ V \cdots$ $\Rightarrow$ **III: $\oint \geq 0$ for all periodic $\cdots$**

Basic theorem of dissipative systems.

PROOF of LQ TH'M and (LMI)

I $\Rightarrow$ VIII **VIII $\Leftrightarrow$ IX** **IX $\Leftrightarrow$ X** **VIII $\Leftrightarrow$ II $\Rightarrow$ I** **I $\Leftrightarrow$ III** **III $\Rightarrow$ VII**

III: $\oint \geq 0$ for all periodic $\cdots$ $\Rightarrow$ **VII: Frequency condition**

Use your frequency domain intelligence.

Consider the (complex) periodic inputs $u(t) = ae^{i\omega t}$.
For all $\omega \in \mathbb{R} : i\omega \notin \sigma(A)$, there is an associated periodic $x(t) = be^{i\omega t}$ with $b = (i\omega I - A)^{-1}Ba$.
Calculate $\oint$ and obtain the frequency condition.

PROOF of LQ TH'M and (LMI)

I $\Rightarrow$ VIII **VIII $\Leftrightarrow$ IX** **IX $\Leftrightarrow$ X** **VIII $\Leftrightarrow$ II $\Rightarrow$ I** **I $\Leftrightarrow$ III**
III $\Rightarrow$ VII **VII $\Rightarrow$ IV**

VII: Frequency condition $\Rightarrow$ **IV: $\int \geq 0$ for all $\mathcal{L}_2 \cdots$**

Assume that $(u, x) \in \mathfrak{B} \cap \mathcal{L}_2$. Use Parseval's theorem to compute $\int_{-\infty}^{\infty} s(u(t), x(t))\, dt$.

PROOF of LQ TH'M and (LMI)

I $\Rightarrow$ VIII **VIII $\Leftrightarrow$ IX** **IX $\Leftrightarrow$ X** **VIII $\Leftrightarrow$ II $\Rightarrow$ I** **I $\Leftrightarrow$ III**

III $\Rightarrow$ VII **VII $\Rightarrow$ IV** **IV $\Rightarrow$ V $\Rightarrow$ VI**

IV: $\int \geq 0 \ \forall \ \mathcal{L}_2$ $\Rightarrow$ **V: $\forall$ c. supp.** $\Rightarrow$ **VI: $\forall$ $\mathfrak{C}^\infty$ c. supp.**

Trivial.

PROOF of LQ TH'M and (LMI)

I $\Rightarrow$ VIII   **VIII $\Leftrightarrow$ IX**   **IX $\Leftrightarrow$ X**   **VIII $\Leftrightarrow$ II $\Rightarrow$ I**   **I $\Leftrightarrow$ III**

III $\Rightarrow$ VII   **VII $\Rightarrow$ IV**   **IV $\Rightarrow$ V $\Rightarrow$ VI**   **VI $\Rightarrow$ III**

VI: $\int \geq 0 \; \forall \; \mathfrak{C}^\infty$ c. supp. $\Rightarrow$ **III: $\oint \geq 0$ for all periodic $\cdots$**

Assume the contrary, truncate this periodic sol'n after a large number of periods, make the truncation into a compact support sol'n, and smooth (e.g. by convoluting with a $\mathfrak{C}^\infty$ compact support kernel) in order to obtain a compact support $\mathfrak{C}^\infty$ solution that violates VI.

PROOF of LQ TH'M and (LMI)

I $\Rightarrow$ VIII **VIII $\Leftrightarrow$ IX** **IX $\Leftrightarrow$ X** **VIII $\Leftrightarrow$ II $\Rightarrow$ I** **I $\Leftrightarrow$ III**

III $\Rightarrow$ VII **VII $\Rightarrow$ IV** **IV $\Rightarrow$ V $\Rightarrow$ VI** **VI $\Rightarrow$ III**

That the set of sol's of the (LMI) (and hence of the (ARIneq) for $R > 0$) is convex and compact is trivial. The inequality

$$K_- \leq K \leq K_+$$

follows immediately from the interpretation of K_- and K_+ in terms of the available storage and the required supply.

RECAP

- **A linear differential system with a quadratic supply rate is dissipative $\Leftrightarrow$ there exists a quadratic storage function.**
- **Leads *linea recta* to the (LMI).**
- **The set of sol'ns of this (LMI) is convex, compact, and attains its infimum K_- and its supremum K_+.**
- **These correspond to the available storage and required supply.**
- **The (LMI) is very closely related to algebraic Riccati inequality and the algebraic Riccati equation. The extreme sol'ns K_-, K_+ of the (LMI) are sol'ns of the (ARE) (when $R > 0$).**
- **There is also an explicit condition for dissipativity in terms of the frequency response.**

Part III: Contractivity, Passivity

NON-NEGATIVE STORAGE F'NS

Do storage functions need be ≥ 0?

Since one can always add a constant, one should really ask:

Are storage functions bounded from below?

We did NOT demand this. The reason is physics:

> **in mechanics (e.g. a mass in an inverse square gravitational field), the energy need not be bounded from below,**
>
> **in thermodynamics, the entropy (often the log of the temp.) need not be bounded from above or below.**

Nevertheless, in applications (stability, circuit synthesis) $\geq$ of the storage f'n is essential. We will cover the LQ cases

$$s(\mathrm{u}, \mathrm{y}) = ||\mathrm{u}||^2 - ||\mathrm{y}||^2$$ **(contractivity)**

$$s(\mathrm{u}, \mathrm{y}) = \mathrm{u}^\top \mathrm{y}$$ **(positive realness)**

CONTRACTIVITY

We start with a bit of notation and recalling a couple of definitions:

$$
\begin{aligned}
\mathbb{C}_{-} &:= \{s \in \mathbb{C} \mid \mathrm{Real}(s) < 0\} \\
\mathbb{C}_{+} &:= \{s \in \mathbb{C} \mid \mathrm{Real}(s) > 0\} \\
\mathbb{C}_{\mathbf{0}-} &:= \{s \in \mathbb{C} \mid \mathrm{Real}(s) \leq 0\} \\
\mathbb{C}_{\mathbf{0}+} &:= \{s \in \mathbb{C} \mid \mathrm{Real}(s) \geq 0\}
\end{aligned}
$$

$\bar{\ }$ = complex conjugate.

CONTRACTIVITY

We start with a bit of notation and recalling a couple of definitions:

$A \in \mathbb{R}^{n \times n}$ **is Hurwitz** $:\Leftrightarrow \sigma(A) \subset \mathbb{C}_{-}$.

Equivalently, of course, all trajectories of $\dot{x} = Ax$ **go to zero as** $t \to \infty$.

$A \in \mathbb{R}^{n \times n}$ **is almost Hurwitz** $:\Leftrightarrow$

1. $\sigma(A) \subset \mathbb{C}_{0-}$,
2. **the eigenvalues on the imaginary axis are semi-simple.**

Equivalently, of course, all trajectories of $\dot{x} = Ax$ **are bounded on** $[0, \infty)$.

CONTRACTIVITY

We start with a bit of notation and recalling a couple of definitions:

Let $G \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}(\xi)$. Its $\mathcal{H}_\infty$-norm equals

$$||G||_{\mathcal{H}_\infty} := \text{supremum} \ \ \{||G(s)|| \mid s \in \mathbb{C}_+\}.$$

$||G||_{\mathcal{H}_\infty} < \infty \Leftrightarrow G$ **proper, no poles in** $\mathbb{C}_{0+}$ **(**$\Leftrightarrow A$ **Hurwitz).**
Then

$$||G||_{\mathcal{H}_\infty} = \text{supremum} \ \ \{||G(i\omega)|| \mid \omega \in \mathbb{R}\}.$$

$||G||_{\mathcal{H}_\infty}$ **equals the** $\mathcal{L}_2$ **induced norm of the operator** $u \mapsto y$,

$$y(t) = H_0 u(t) + \int_{0 \text{ or } -\infty}^{t} H(t-t')u(t')\, dt'.$$

Call G **contractive** $:\Leftrightarrow ||G||_{\mathcal{H}_\infty} \leq 1.$

CONTRACTIVITY

Theorem: Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, controllable & observable, and $s(\mathrm{u}, \mathrm{y}) = ||\mathrm{u}||^2 - ||\mathrm{y}||^2$. The following are equivalent:

1. Σ diss. w.r.t. s, with a storage f'n that is **bounded from below**, or, equivalently, **non-negative**.

CONTRACTIVITY

Theorem: Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, controllable & observable, and $s(\mathrm{u},\mathrm{y}) = ||\mathrm{u}||^2 - ||\mathrm{y}||^2$. The following are equivalent:

1. Σ diss. w.r.t. s, with a storage f'n that is **bounded from below**, or, equivalently, **non-negative**.

2. The **(LMI)**

$$K = K^\top > 0$$

$$\begin{bmatrix} A^\top K + KA + C^\top C & KB + C^\top D \\ B^\top K + D^\top C & -I + D^\top D \end{bmatrix} \leq 0$$

has a solution. Equivalently, the supremal sol'n $K_+ > 0$.

CONTRACTIVITY

Theorem: Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, controllable & observable, and $s(\mathrm{u}, \mathrm{y}) = ||\mathrm{u}||^2 - ||\mathrm{y}||^2$. The following are equivalent:

1. Σ diss. w.r.t. s, with a storage f'n that is bounded from below, or, equivalently, non-negative.

3. Behavioral characterization:

$$\int_{-\infty}^{0} (||u(t)||^2 - ||y(t)||^2) \, dt \geq 0$$

for all $(u, y) \in \mathfrak{B}_{\mathrm{ext}} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathrm{m}} \times \mathbb{R}^{\mathrm{p}})$.

This is called half-line dissipativity.

Note: upper bound 0 on $\int$ immaterial, may as well take $\int_{-\infty}^{t}$.

CONTRACTIVITY

Theorem: Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, controllable & observable, and $s(\mathrm{u},\mathrm{y}) = ||\mathrm{u}||^2 - ||\mathrm{y}||^2$. The following are equivalent:

1. Σ diss. w.r.t. s, with a storage f'n that is **bounded from below**, or, equivalently, **non-negative**.
4. **Frequency-domain characterization:** G **is contractive**, i.e. $||G||_{\mathcal{H}_\infty} \leq 1$.
5. Σ is dissipative w.r.t. s, and **A is Hurwitz**.

CONTRACTIVITY

Theorem: **Consider** $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$**, controllable & observable, and** $s(\mathrm{u},\mathrm{y}) = ||\mathrm{u}||^2 - ||\mathrm{y}||^2$**. The following are equivalent:**

1. **Σ diss. w.r.t. s, with a storage f'n that is bounded from below, or, equivalently, non-negative.**

1'. **Σ diss. w.r.t. s, with all storage f'n bounded from below.**

CONTRACTIVITY

Theorem: Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, controllable & observable, and $s(\mathrm{u},\mathrm{y}) = ||\mathrm{u}||^2 - ||\mathrm{y}||^2$. The following are equivalent:

1. Σ diss. w.r.t. s, with a storage f'n that is bounded from below, or, equivalently, non-negative.

2'. All solutions of the (LMI)

$$K = K^\top$$

$$\begin{bmatrix} A^\top K + KA + C^\top C & KB + C^\top D \\ B^\top K + D^\top C & -I + D^\top D \end{bmatrix} \leq 0$$

are > 0. Equivalently, the infimal sol'n $K_- > 0$.

CONTRACTIVITY

Theorem: Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, controllable & observable, and $s(\mathrm{u}, \mathrm{y}) = ||\mathrm{u}||^2 - ||\mathrm{y}||^2$. The following are equivalent:

1. Σ diss. w.r.t. s, with a storage f'n that is bounded from below, or, equivalently, non-negative.

6.,7.,... Various variations, with ≥ 0 instead of > 0, $\mathfrak{C}^\infty$, compact support, ARineq, (ARE), etc.

PROOF of CONTRACTIVITY TH'M

Preliminary: The **inertia** of $M \in \mathbb{R}^{n \times n}$ is defined as the triple

$$\mathrm{In}(M) := (\nu(M), \zeta(M), \pi(M))$$

with $\nu(M), \zeta(M), \pi(M) =$ the number (counting multiplicity) of eigenvalues of M with respectively real part $> 0, = 0, < 0$.
Of course $\nu(M) + \zeta(M) + \pi(M) = n$.
Btw, $\pi(M) - \nu(M)$ is called the **signature** of M.

Recall the following result involving the inertia and the **Lyapunov equation**

$$A^\top P + PA + Q = 0.$$

Theorem: Assume that (A, P, Q) satisfy the Lyapunov eq'n, with $P = P^\top, Q = Q^\top \geq 0$, and (A, Q) is observable.
Then $\zeta(A) = 0$, and $\mathrm{In}(P) = \mathrm{In}(A)$.

PROOF of CONTRACTIVITY TH'M

Note that each of the conditions of the th'm implies that $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is dissipative w.r.t. $s(\mathrm{u}, \mathrm{y}) = ||\mathrm{u}||^2 - ||\mathrm{y}||^2$. Therefore, we can freely use the LQ th'm, and assume the existence of K_-, K_+, etc.

Each sol'n $K = K^\top$ of the (LMI) satisfies

$$A^\top K + KA + C^\top C \leq 0.$$

Since (A, C) is observable, so is $(A, A^\top K + KA)$.

The inertia theorem therefore implies that all these K's are non-singular and have the same number of positive and negative eigenvalues.

PROOF of CONTRACTIVITY TH'M

1. $\Rightarrow$ 2.

By the basic th'm on dissipativity, there holds for any storage function V,

$$\mathrm{x}^\top K_- \mathrm{x} - \leq V(\mathrm{x}) - V(0) \leq \mathrm{x}^\top K_+ \mathrm{x}.$$

Hence V bounded from below implies $\mathrm{x}^\top K_+ \mathrm{x}$ bounded from below. Since K_+ is non-singular, it is bounded from below if and only if $K_+ \geq 0$, but since it also non-singular, $K_+ > 0$.

PROOF of CONTRACTIVITY TH'M

1. $\Rightarrow$ 2.

2. $\Rightarrow$ 3. By 2., $K_+ > 0$. By the inertia th'm, therefore, A is Hurwitz. Assume that 3. does not hold. Then there is $(u', y') \in \mathfrak{B}_{\text{ext}} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{m}} \times \mathbb{R}^{\mathtt{p}})$ such that

$$\int_{-\infty}^{0} (||u'(t)||^2 - ||y'(t)||^2) \, dt < 0.$$

Consider the input u'', with $u''(t) = u'(t)$ for $t \leq 0$, and $u''(t) = 0$ for $t > 0$. Since A is Hurwitz, the resulting y'' with $y''(t) = y'(t)$ for $t \leq 0$ yields $(u'', y'') \in \mathfrak{B}_{\text{ext}} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{m}} \times \mathbb{R}^{\mathtt{p}})$, and

$$\int_{-\infty}^{+\infty} (||u''(t)||^2 - ||y''(t)||^2) \, dt < 0.$$

Contradicts dissipativeness.

PROOF of CONTRACTIVITY TH'M

1. $\Rightarrow$ 2.

2. $\Rightarrow$ 3.

3. $\Rightarrow$ 4. 3. implies dissipativeness. Therefore

$$\text{supremum} \quad \{||G(i\omega)|| \mid \omega \in \mathbb{R}\} \leq 1.$$

We need to prove that 3. implies that G has no poles $\mathbb{C}_{0+}$, i.e., that A is Hurwitz. If this were not the case, choose a (compact support) input that is zero for $t \geq 0$, and such that $x(0)$ yields a y with $\int_0^\infty ||y(t)||^2 = \infty$. Hence for T sufficiently large

$$\int_{-\infty}^{T} (||u(t)||^2 - ||y(t)||^2)\, dt < 0.$$

Contradicts 3.

PROOF of CONTRACTIVITY TH'M

1. $\Rightarrow$ 2.

2. $\Rightarrow$ 3.

3. $\Rightarrow$ 4.

4. $\Rightarrow$ 5. obvious

PROOF of CONTRACTIVITY TH'M

1. $\Rightarrow$ 2.

2. $\Rightarrow$ 3.

3. $\Rightarrow$ 4.

4. $\Rightarrow$ 5.

5. $\Rightarrow$ 1.

By dissipativeness there exist sol'ns $K = K^\top$ to the (LMI). By the inertia theorem they are all > 0. 1. follows.

PROOF of CONTRACTIVITY TH'M

1. $\Rightarrow$ 2.

2. $\Rightarrow$ 3.

3. $\Rightarrow$ 4.

4. $\Rightarrow$ 5.

5. $\Rightarrow$ 1.

2. $\Rightarrow$ $2'$. **use the inertia theorem.**

PROOF of CONTRACTIVITY TH'M

1. $\Rightarrow$ 2.

2. $\Rightarrow$ 3.

3. $\Rightarrow$ 4.

4. $\Rightarrow$ 5.

5. $\Rightarrow$ 1.

2. $\Rightarrow$ $2'$.

2'. $\Rightarrow$ 1'.

follows from

$$\mathrm{x}^\top K_- \mathrm{x} - \leq V(\mathrm{x}) - V(0) \leq \mathrm{x}^\top K_+ \mathrm{x}$$

and $K - > 0$.

PROOF of CONTRACTIVITY TH'M

1. $\Rightarrow$ **2.**

2. $\Rightarrow$ **3.**

3. $\Rightarrow$ **4.**

4. $\Rightarrow$ **5.**

5. $\Rightarrow$ **1.**

2. $\Rightarrow$ $2'$**.**

2'. $\Rightarrow$ **1'.**

1'. $\Rightarrow$ 1**.** **trivial.**

PROOF of CONTRACTIVITY TH'M

1. $\Rightarrow$ **2.**

2. $\Rightarrow$ **3.**

3. $\Rightarrow$ **4.**

4. $\Rightarrow$ **5.**

5. $\Rightarrow$ **1.**

2. $\Rightarrow$ $2'$**.**

2'. $\Rightarrow$ **1'.**

1'. $\Rightarrow$ 1**.**

POSITIVE REALNESS

A VERY important notion in system theory:

$g \in \mathbb{R}(\xi)$ **is positive real (p.r.):** $\Leftrightarrow$

$$\{s \in \mathbb{C}_+\} \;\Rightarrow\; \{g(s) \in \mathbb{C}_+\}.$$

$\exists$ **numerous equivalent conditions for positive realness. E.g.:**

$$\{s \in \mathbb{C}_{0+}, \quad s \;\; \text{not a pole of } g\} \;\Rightarrow\; \{g(s) \in \mathbb{C}_{0+}\}$$

and, more intricate,

1. $\mathrm{Real}(g(i\omega)) \geq 0$ **for all** $\omega \in \mathbb{R}$
2. g **has no poles in** $\mathbb{C}_+$
3. **the im. axis poles of** g **are simple, with residue real and** > 0
4. $\dfrac{g(s)}{s}$ **is proper, and its limit for** $s \to \infty$ **is** ≥ 0

POSITIVE REALNESS

Matrix case:

$G \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}(\xi)$ **is positive real (p.r.):**$\Leftrightarrow$

$$\{s \in \mathbb{C}_+\} \;\; \Rightarrow \;\; \{G(s) + G^\top(\bar{s}) \geq 0\}.$$

$G^\top(\bar{s})$ **is the Hermitian conjugate of** $G(s)$**.**

POSITIVE REALNESS

Theorem: Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, controllable & observable, and $s(\mathrm{u}, \mathrm{y}) = \mathrm{u}^\top \mathrm{y}$. The following are equivalent:

1. Σ diss. w.r.t. s, with a storage f'n that is bounded from below, or, equivalently, non-negative.

POSITIVE REALNESS

Theorem: Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, controllable & observable, and $s(\mathrm{u}, \mathrm{y}) = \mathrm{u}^\top \mathrm{y}$. The following are equivalent:

1. Σ diss. w.r.t. s, with a storage f'n that is bounded from below, or, equivalently, non-negative.
2. The (LMI) (note: the corresponding storage f'n is $\frac{1}{2}\mathrm{x}^\top K \mathrm{x}$)

$$K = K^\top > 0$$

$$\begin{bmatrix} A^\top K + KA & KB - C^\top \\ B^\top K - C & -D - D^\top \end{bmatrix} \leq 0$$

has a solution. Equivalently, the supremal sol'n $K_+ > 0$.

POSITIVE REALNESS

Theorem: Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, controllable & observable, and $s(\mathrm{u}, \mathrm{y}) = \mathrm{u}^\top \mathrm{y}$. The following are equivalent:

1. Σ diss. w.r.t. s, with a storage f'n that is bounded from below, or, equivalently, non-negative.
3. Behavioral characterization:

$$\int_{-\infty}^{0} u(t)^\top y(t)\, dt \geq 0$$

for all $(u, y) \in \mathfrak{B}_{\mathrm{ext}} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{m}} \times \mathbb{R}^{\mathtt{p}})$.

This is called half-line dissipativity.

POSITIVE REALNESS

Theorem: Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, controllable & observable, and $s(\mathrm{u}, \mathrm{y}) = \mathrm{u}^\top \mathrm{y}$. The following are equivalent:

1. Σ diss. w.r.t. s, with a storage f'n that is **bounded from below**, or, equivalently, **non-negative**.

4. **Frequency-domain characterization:** **G is positive real**.

5. Σ is dissipative w.r.t. s, and A is almost Hurwitz.

POSITIVE REALNESS

Theorem: Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, controllable & observable, and $s(\mathrm{u},\mathrm{y}) = \mathrm{u}^\top \mathrm{y}$. The following are equivalent:

1. Σ diss. w.r.t. s, with a storage f'n that is **bounded from below**, or, equivalently, **non-negative**.

1'. Σ diss. w.r.t. s, with **all** storage f'n bounded from below.

POSITIVE REALNESS

Theorem: Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, controllable & observable, and $s(\mathrm{u},\mathrm{y}) = \mathrm{u}^\top \mathrm{y}$. The following are equivalent:

1. Σ diss. w.r.t. s, with a storage f'n that is bounded from below, or, equivalently, non-negative.

2'. All solutions of the (LMI)

$$K = K^\top$$

$$\begin{bmatrix} A^\top K + KA & KB - C^\top \\ B^\top K - C & -D - D^\top \end{bmatrix} \leq 0$$

are > 0. Equivalently, the infimal sol'n $K_- > 0$.

POSITIVE REALNESS

Theorem: Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, controllable & observable, and $s(\mathrm{u}, \mathrm{y}) = \mathrm{u}^\top \mathrm{y}$. The following are equivalent:

1. Σ diss. w.r.t. s, with a storage f'n that is bounded from below, or, equivalently, non-negative.

6.,7.,... Various variations, with ≥ 0 instead of > 0, $\mathfrak{C}^\infty$, compact support, ARineq, (ARE), etc.

POSITIVE REALNESS

Theorem: Consider $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, controllable & observable, and $s(\mathrm{u}, \mathrm{y}) = \mathrm{u}^\top \mathrm{y}$. The following are equivalent:

1. Σ diss. w.r.t. s, with a storage f'n that is bounded from below, or, equivalently, non-negative.

It is customary to refer to this case as PASSIVITY.

POSITIVE REALNESS

In the special case $D = 0$ (G strictly proper), the (LMI) becomes

$$\begin{aligned} K = K^\top &> 0, \\ A^\top K + KA &\leq 0, \\ KB &= C^\top. \end{aligned}$$

The fact that solvability of this (LMI) is equivalent to positive realness of $G(\xi) = C(I\xi - A^{-1})B$ is usually called the **KYP-lemma** (after Kalman, Yakubovich, Popov).

POSITIVE REALNESS

In this case, it is possible to express passivity as a 'sort of' condition on the impulse response:

$$\begin{bmatrix} \frac{H(0)+H(0)^\top}{2} & H(t_2-t_1) & H(t_3-t_1) & \cdots & H(t_k-t_1) \\ H^\top(t_2-t_1) & \frac{H(0)+H(0)^\top}{2} & H(t_3-t_2) & \cdots & H(t_k-t_2) \\ H^\top(t_3-t_1) & H^\top(t_3-t_2) & \frac{H(0)+H(0)^\top}{2} & \cdots & H(t_k-t_3) \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ H^\top(t_k-t_1) & H(t_k-t_2)^\top & H^\top(t_k-t_3) & \cdots & \frac{H(0)+H(0)^\top}{2} \end{bmatrix} \geq 0$$

for all $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k$ and all $k \in \mathbb{N}$.

The proof will not be given.

PROOF of the P.R. TH'M

We give the proof only in the case $D = 0$. It makes some points of independent interest, namely that the choice of inputs and outputs in a system is not something that is 'fixed'. The system eq'ns are

$$\Sigma : \overset{\bullet}{\mathrm{x}} = A\mathrm{x} + B\mathrm{u}, \quad \mathrm{y} = C\mathrm{x} \quad \rightsquigarrow \text{ behavior } \mathfrak{B}.$$

With the new input and output

$$\mathrm{u}' = \frac{1}{2}(\mathrm{u} + \mathrm{y}), \quad \mathrm{y}' = \frac{1}{2}(\mathrm{u} - \mathrm{y}),$$

the system equations become

$$\Sigma' : \overset{\bullet}{\mathrm{x}} = (A - BC)\mathrm{x} + 2B\mathrm{u}', \quad \mathrm{y}' = -C\mathrm{x} + \mathrm{u}' \quad \rightsquigarrow \text{ beh. } \mathfrak{B}'$$

Obviously $(u, y, x) \in \mathfrak{B} \Leftrightarrow (\frac{1}{2}(u + y), \frac{1}{2}(u - y), x) \in \mathfrak{B}'$.

PROOF of the P.R. TH'M

Define $s'(\mathrm{u}', \mathrm{y}') = ||\mathrm{u}'||^2 - ||\mathrm{y}'||^2$. Note that $s'(\mathrm{u}', \mathrm{y}') = \mathrm{u}^\top \mathrm{y} = s(\mathrm{u}, \mathrm{y})$.

Hence Σ is dissipative w.r.t. s with storage function V
$\Leftrightarrow$ Σ' is dissipative w.r.t. s' with storage function V.

Conclude that

1. of the contractivity th'm $\Leftrightarrow$ 1. of the p.r. th'm.
1'. of the contractivity th'm $\Leftrightarrow$ 1'. of the p.r. th'm.
2. of the contractivity th'm $\Leftrightarrow$ 2. of the p.r. th'm.
2'. of the contractivity th'm $\Leftrightarrow$ 2'. of the p.r. th'm.

and from the relation between s and s'

3. of the contractivity th'm $\Leftrightarrow$ 3. of the p.r. th'm.

PROOF of the P.R. TH'M

Note that the transfer functions G' of Σ' and G of Σ are related by the fractional transformation

$$G' = (I - G)(I + G)^{-1}.$$

Now, for $M \in \mathbb{C}^{n \times n}$, there holds

$$M + M^{\star} \geq 0 \Leftrightarrow I + M \text{ invertible and } ||(I - M)(I + M)^{-1}|| \leq 1.$$

Conclude that

$$||G'||_{\mathcal{H}_\infty} \leq 1 \Leftrightarrow G \text{ positive real.}$$

Hence, 4. of the contractivity th'm $\Leftrightarrow$ 4. of the p.r. th'm.

PROOF of the P.R. TH'M

We still need to prove that 2. $\Leftrightarrow$ 5. This uses the following

Lemma: Assume

$$K = K^\top, A^\top K + KA \leq 0, \text{ and } KB = C^\top.$$

Then

$$A \text{ is almost Hurwitz } \Leftrightarrow K > 0.$$

Assume that 5. holds. Then, by dissipativeness, the (LMI) has a sol'n $K = K^\top$. By the lemma, $K = K^\top > 0$, whence 2. holds. Conversely, if 2. holds, then, by the lemma, A is almost Hurwitz.

PROOF of the P.R. TH'M

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I wasn't able to construct a proof of $\Rightarrow$ of the lemma in time.

In cauda venenum

RECAP

- Non-negativity of the storage function is important in the analysis of physical systems, and in stability applications.
- For $s(\mathrm{u}, \mathrm{y}) = ||\mathrm{u}||^2 - ||\mathrm{y}||^2$, positivity of the (all) storage function comes down to the condition $||G||_{\mathcal{H}_\infty} \leq 1$.
- For $s(\mathrm{u}, \mathrm{y}) = \mathrm{u}^\top \mathrm{y}$, positivity of the (all) storage function comes down to positive realness of G.
- Recently, a n.a.s.c. for the existence of a postive storage function in the general LQ case has been published by Trentelman and Rapisarda (SIAM J. Control & Opt., 2003?).