MATHEMATICAL MODELS of SYSTEMS

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MATHEMATICAL BACKGROUND

NOTATION

Symbol	Meaning	LaTeX
$\exists$	there exist(s)	\exists
$\exists(!)$	there exist(s a) unique	
$\nexists$	there does not exist(s)	\nexists
$\forall$	for all	\forall
:	such that	
$\wedge$	and	\wedge
$\vee$	or	\vee

Symbol	Meaning	LaTeX
$=$	left is equal to right	
$:=$	left is by definition (or defined to be) equal to right	
$=:$	right is by definition equal to left	
$\Leftrightarrow$	left is equivalent to right	\Leftrightarrow
$:\Leftrightarrow$	left is by definition equivalent to right	
$\Leftrightarrow:$	right is by definition equivalent to to left	
$\Rightarrow$	left implies right	\Rightarrow
$\Leftarrow$	right implies left	\Leftarrow
$\leftrightarrow$	left and right are one-to-one related	\leftrightarrow
$\mapsto$	maps to	\mapsto

Symbol	Meaning	LaTeX
$\in$	belongs to	\in
$\notin$	does not belong to	\notin
$\subset$	is a subset of	\subset
$\supset$	is a superset of: $[A \supset B] :\Leftrightarrow [B \subset A]$	\supset
$\cap$	intersection	\cap
$\cup$	union	\cup
$/$	set difference $A/B := \{a \in A \mid a \notin B\}$	

Symbol	Meaning	LaTeX
2^A	the set of all subsets of A	
A^B	the set of maps from A to B	
	$S/B := \{a \in S \mid a \notin B\}$	
$\emptyset$	the empty set	\emptyset
∞	infinity	\infty
iff	if and only if	

Notation	Meaning	LaTeX
$\mathbb{N}$	the natural numbers $\mathbb{N} := \{1, 2, \ldots, n, \ldots\}$	\mathbb{N}
$\mathbb{Z}$	the integers $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots, n, \ldots\}$	\mathbb{Z}
$\mathbb{Z}_+$	the nonnegative integers $\mathbb{Z}_+ := \{0, 1, 2, \ldots, n, \ldots\}$	
$\mathbb{R}$	the real numbers	\mathbb{R}
$\mathbb{R}_+$	$:= [0, \infty)$, the nonnegative real numbers	
$\mathbb{R}_-$	$:= (-\infty, 0]$, the nonpositive real numbers	
$\mathbb{C}$	the complex numbers	\mathbb{C}

Symbol	**Meaning**	**LaTeX**
$\times$	**Cartesian product**	**\times**
$S^{\mathtt{n}}$	**`n`-fold Cartesian product of S**	**\times**

Sets

- $\{a \mid a$ has property $A\}$

  denotes the set of all elements that have property A

- $\{a \in S \mid a$ has property $A\}$

  denotes the subset of S consisting of

  the elements of S that have property A

- Occasionally, we use *'collection'* and *'family'* as synonyms of 'set'.
- The set of all subsets of the set S is denoted by 2^S. It is called the *power set* of S.
- $\{s_1, s_2, \ldots, s_n\}$

  denotes the set with elements $s_1, s_2, \ldots, s_n$;

  if some of the s_k's are equal, then they count 'only once',

  i.e., we do not consider sets with multiplicity,

  whence, for example, $\{0, 0, 1, 1, 1, \} = \{0, 1\}$.

- $(s_1, s_2, \ldots, s_n)$ denotes an ordered n-tuple
- $S_1 \times S_2 \times \cdots \times S_n := \{(s_1, s_2, \ldots, s_n) \mid s_k \in S_k, k = 1, 2, \ldots, n\}$ (called the *Cartesian product* of $S_1, S_2, \cdots, S_n$)
- $S^n := \underbrace{S \times S \times \cdots \times S}_{n-\text{times}}$
- A *relation* on $S_1, S_2, \cdots, S_n$ (or between the variables of $S_1, S_2, \cdots S_n$) is a subset of $S_1 \times S_2 \times \cdots \times S_n$.
- A relation $R \subset S_1 \times S_2$ is said to be *one-to-one* if for all $s_1 \in S_1$ there is exactly one $s_2 \in S_2$ such that $(s_1, s_2) \in R$ and, vice-versa, for all $s_2 \in S_2$ there is exactly one $s_1 \in S_1$ such that $(s_1, s_2) \in R$, i.e., there is a bijection $f : S_1 \to S_2$ whose graph is R. A one to-one-relation is denoted by $S_1 \overset{R}{\longleftrightarrow} S_2$ or, when $(s_1, s_2) \in \mathbb{R}$, $s_1 \in S_1 \overset{R}{\longleftrightarrow} s_2 \in S_2$ (S_1 and S_2 are often deleted when they are obvious from the context).

- A relation on $S^2 (S^3, S^{\mathrm{n}})$ is called a *binary (ternary,* $\mathtt{n}$*-ary) relation* on S.
- A binary relation R on S is called an *equivalence relation* on S if
  1. $(s, s) \in R$
  2. $[(s_1, s_2) \in R] \Rightarrow [(s_2, s_1) \in R]$
  3. $[(s_1, s_2), (s_2, s_3) \in R] \Rightarrow [(s_1, s_3) \in R]$
- the set $\{s' \in \S \mid (s', s) \in R\}$ is called the *equivalence class* associated by $s \in S$. Note that two equivalence classes either coincide, or are disjoint.
- Let S be a set. A family $S_\alpha, \alpha \in S$ of subsets of S (S might be an infinite set) is called a *partition* of S if
  1. the sets S_α are non-empty,
  2. disjoint, i.e., $[\alpha_1 \neq \alpha_2] \Rightarrow [S_{\alpha_1} \neq S_{\alpha_2}]$,
  3. and their union covers S, i.e., $\bigcup_{\alpha \in S} S_\alpha = S$.

- **There is an obvious one-to-one relation between the equivalence relations on S and the partitions of S. If the equivalence relation is given, define the partition as the family of disjoint equivalence classes. If the partition is given, define the equivalence relation R by $[(s_1, s_2) \in R] \Leftrightarrow [s_1$ and s_2 belong the same element of the partition].**
- **Let R be an equivalence relation on S. A subset C of S is said to be a *canonical form* (w.r.t. R) if C intersects each equivalence class at least once, and a *trim canonical form* if it intersects each equivalence class exactly once.**

Maps

- **A *map* f from the set A to the set B associates with every element of A an element of B.**
- **A is called the *domain* and B the *co-domain* of f.**
- **We use the notation $f : A \to B$, or $A \xrightarrow{f} B$.**
- **The set of all maps from A to B is denoted by B^A.**
- **If f takes $a \in A$ to $b \in B$, then we write $b = f(a)$, or $f : a \mapsto b$, or $a \overset{f}{\mapsto} b$.**
- **Sometimes the latter notation is used to define f by means of a recipe, for example the map sqrt : $\mathbb{R}_+ \to \mathbb{R}_+$ can be defined by sqrt : $x \in \mathbb{R}_+ \mapsto \sqrt{x} \in \mathbb{R}_+$, or $x \in \mathbb{R}_+ \overset{\textbf{sqrt}}{\mapsto} \sqrt{x} \in \mathbb{R}_+$.**
- **We use the term 'function', 'operator', and 'transformation' as synonyms for 'map'. Sometimes the term function (and certainly 'functional') is**

reserved for maps with co-doamin $\mathbb{R}$ or $\mathbb{C}$. We use instead real (complex) *functional* for maps with co-domain $\mathbb{R}$ ($\mathbb{C}$).

- $\{b \in B \mid \exists a \in A : f(a) = b\}$ is called the *image* of f, denoted $\mathrm{im}(f)$. Sometimes the term *range* of f is used.
- If $f : A \to A$, then f is called a map on A.
- The *identity map*on A (denoted id_A) is defined by $a \in A \stackrel{\mathrm{id}_A}{\mapsto} a \in A$.
- Let $f : A \to B$, and $g : B \to C$. The map $h : a \in A \mapsto g(f(a))$ is called the *composition* of g with f. Notation: $h = g \circ f$, or $h = gf$.
- If f is a map on A, then $f^{\mathrm{n}} := \underbrace{f \circ f \circ \cdots \circ f}_{\mathrm{n-times}}$.
- $f : A \to B$ is said to be *injective* if $[a_1 \neq a_2] \Rightarrow [f(a_1) \neq f(a_2)]$. This property is sometimes called 'one-to-one'.

- $f : A \to B$ is said to be *surjective* if $\forall\, b \in B\ \exists\, a \in A : f(a) = b$, i.e., if $\mathrm{im(f) = B}$. This property is sometimes called ‘onto’.
- $f : A \to B$ is said to be *bijective* if it is injective and surjective.
- The map $g : B \to A$ is said to be a *left inverse* of $f : A \to B$ if $gf = \mathrm{id_A}$. It is easy to see that a left inverse exists iff f is injective.
- The map $g : B \to A$ is said to be a *right inverse* of $f : A \to B$ if $fg = \mathrm{id_B}$. It is easy to see that a right inverse exists iff f is surjective.
- The map $g : B \to A$ is said to be the *inverse* of $f : A \to B$ if $gf = \mathrm{id_A}$ and $fg = \mathrm{id_B}$. It is easy to see that the inverse exists iff f is surjective. Moreover, the inverse is uniquely defined and denoted as f^{-1}. Whence $f^{-1} : B \to A$.
- Let $f : A \to B$ and $A' \subset A$. Then $f|_{A'} : A' \to B$ is defined by $f|'_A : a \in A' \mapsto f(a) \in B$. $f|'_A$ is called the *restriction* of f to A'.

- **Let $f : A \to B$ and $A' \subset A$. Then $f(A') := \{b \in B \mid \exists a \in A' : b = f(a)\}$. Whence $f(A') = \mathrm{im}(f|_{A'})$.**
- **Let $f : A \to B$ and $B' \subset B$. Then $f^{-1}(B') := \{a \in A \mid f(a) \in B'\}$. $f^{-1}(B')$ is called the *pre-image* of B' under f.**

ALGEBRAIC STRUCTURES

► **Let A be a set. A map from A(A^2, A^3, $A^{\mathtt{n}}$) to A is called a** ***unary (binary, ternary,*** **`n`*-ary) operation*** **on A. Such operations are very important elements for defining algebraic structures.**

GROUPS

A *group*

RINGS

A *ring*

FIELDS

A *field*

MODULES and VECTOR SPACES

A *module*

LINEAR ALGEBRA

FUNCTION SPACES

End of Mathematical Background