MATHEMATICAL MODELS of SYSTEMS

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Lecture 9

DISTRIBUTED SYSTEMS

THEME

Most physical systems are 'distributed', with independent variables

time *and space*.

This explains the central role in physics of PDE's.

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time *and space*.

This explains the central role in physics of PDE's.

How do we incorporate this structure in our framework?

What does, for example, controllability mean?

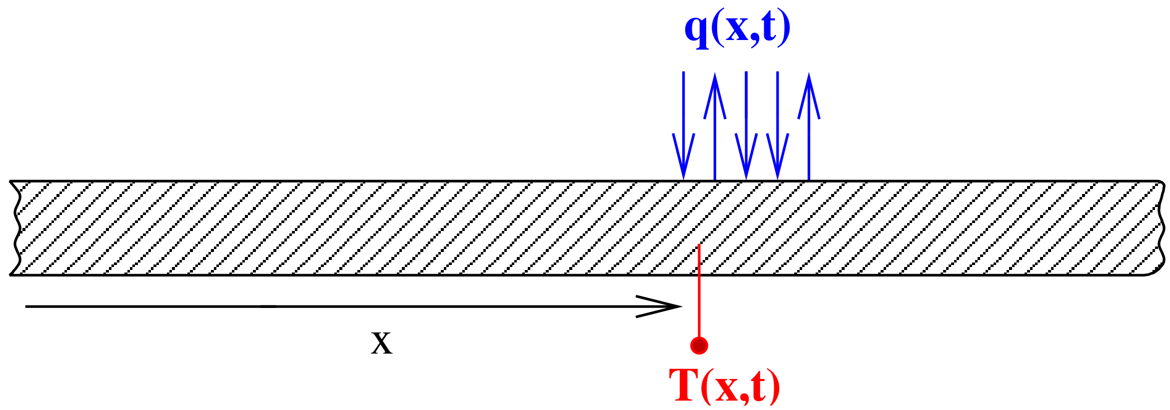
When are such systems dissipative? What is the storage function?

OUTLINE

- **Examples**
- **Behavioral n-D systems**
- **Systems described by linear PDE's**
- **Controllability & Observability**
- **3 central theorems**
- **Dissipative distributed systems**
- **Factorization of polynomial matrices**

EXAMPLES

1. *Heat diffusion*



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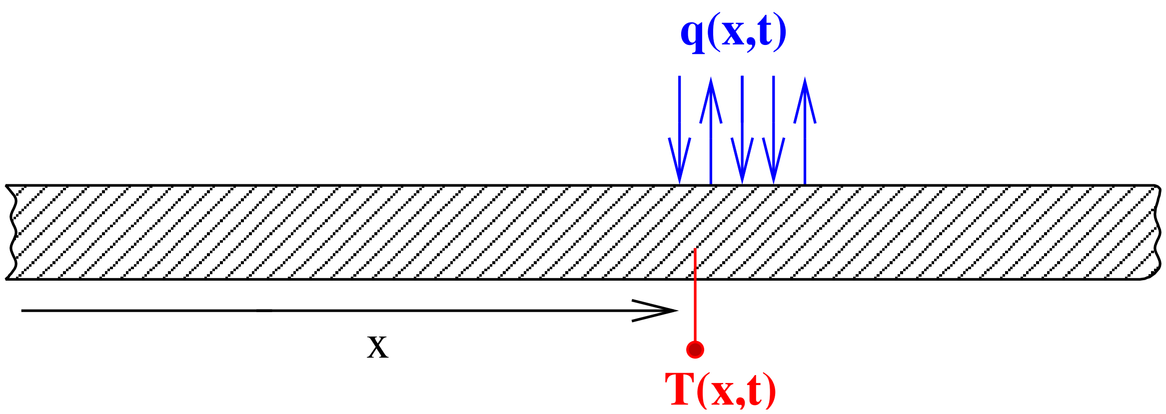

The PDE

$$\frac{\partial}{\partial t}T = \frac{\partial^2}{\partial x^2}T + q$$

describes the evolution of the temperature $T(x,t)$

($x \in \mathbb{R}$ position, $t \in \mathbb{R}$ time) in a medium

and the heat $q(x,T)$ supplied to / radiated away from it.

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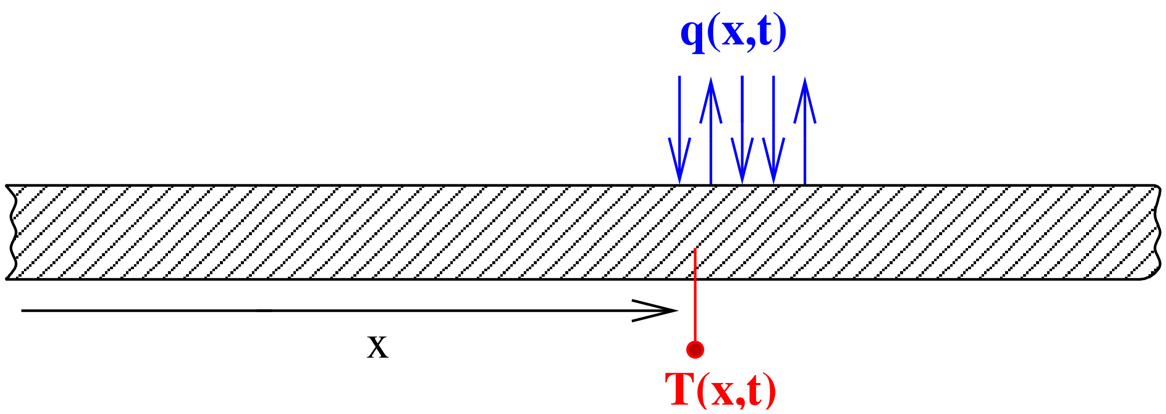

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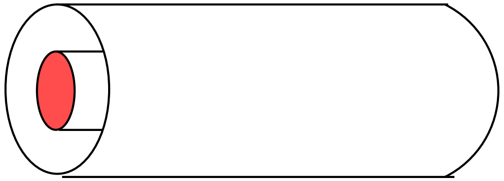
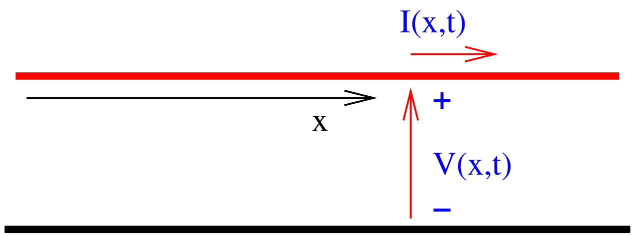
and the heat $q(x,T)$ supplied to / radiated away from it.

We wish to develop a theory that treats x and t on the same footing.

2. *Coaxial cable*

!! Model the relation between the voltage $V(x,t)$ and the current $I(x,t)$ in a coaxial cable.





⇝ The PDE's

$$\boxed{\frac{\partial}{\partial x}V = -L_0\frac{\partial}{\partial t}I,} \qquad (VI)$$

$$\boxed{\frac{\partial}{\partial x}I = -C_0\frac{\partial}{\partial t}V.} \qquad (IV)$$

with L_0 the inductance, and C_0 the capacitance per unit length.

These imply the 'consequences'

$$\boxed{\frac{\partial^2}{\partial x^2} V = L_0 C_0 \frac{\partial^2}{\partial t^2} V,} \qquad (V)$$

and

$$\boxed{\frac{\partial^2}{\partial x^2} I = L_0 C_0 \frac{\partial^2}{\partial t^2} I.} \qquad (I)$$

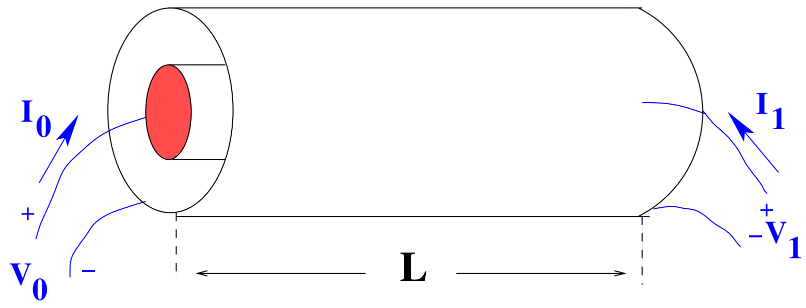
Wave eqn's.

Leads to the questions

- **Are (V), (I) really consequences of $(VI) + (IV)$?**
- **$(V) + (I) \Leftrightarrow (VI) + (IV)$?**
- **$(V) + (I) + (VI) \Leftrightarrow (VI) + (IV)$?**
- **Does (V) express all the constraints on V implied by $(VI) + (IV)$?**
- **Develop a calculus to obtain all consequences, to compute this elimination, to decide equivalence.**

With boundary conditions (cable of length L):

!! Model the relation between the voltages V_0, V_1 and the currents I_0, I_1 at the ends of a uniform cable of length L.



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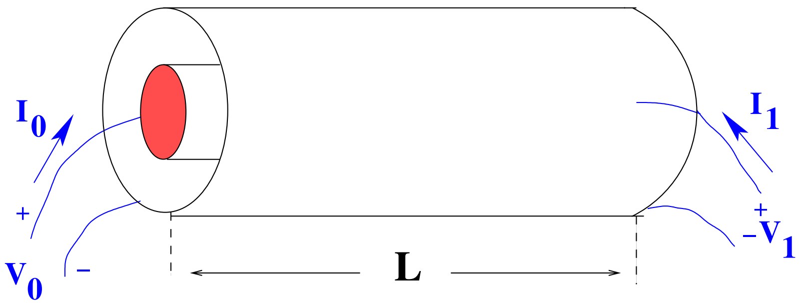
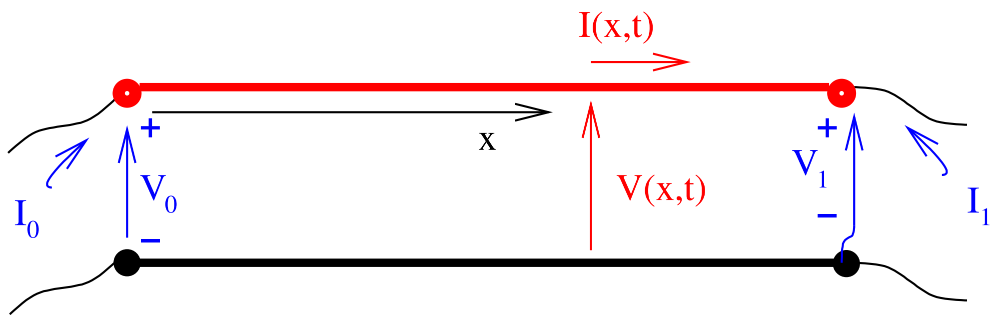


Introduce the voltage $V(x,t)$ and the current flow $I(x,t)$ $0 \leq x \leq L$ in the cable.

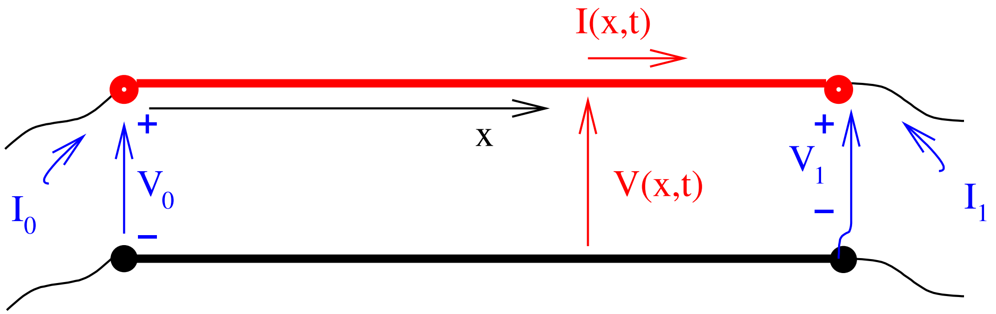


⤳ **The equations:**

$$\begin{aligned} \frac{\partial}{\partial x} V &= -L_0 \frac{\partial}{\partial t} I, \\ \frac{\partial}{\partial x} I &= -C_0 \frac{\partial}{\partial t} V, \end{aligned}$$

$$\begin{aligned} V_0(t) &= V(0,t), \\ V_1(t) &= V(L,t), \\ I_0(t) &= I(0,t), \\ I_1(t) &= -I(L,t). \end{aligned}$$

Viewed as a black box



Relation between V_0, V_1:

$$\frac{\partial^2}{\partial x^2} V = L_0 C_0 \frac{\partial^2}{\partial t^2} V, \quad V_0(\cdot) = V(0,\cdot), \quad V_1(\cdot) = V(L,\cdot),$$

and between I_0, I_1:

$$\frac{\partial^2}{\partial x^2} I = L_0 C_0 \frac{\partial^2}{\partial t^2} I, \quad I_0(\cdot) = I(0,\cdot), \quad I_1(\cdot) = I(L,\cdot).$$

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and between I_0, I_1:

$$\frac{\partial^2}{\partial x^2} I = L_0 C_0 \frac{\partial^2}{\partial t^2} I, \quad I_0(\cdot) = I(0, \cdot), \; I_1(\cdot) = I(L, \cdot).$$

- **Two terminal variables are 'free', the other two are 'bound', (free = one voltage, one current, bound = one voltage, one current), but**

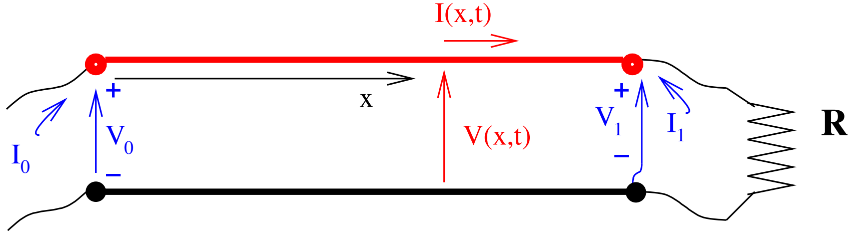
  there is no reasonable choice of inputs and outputs!

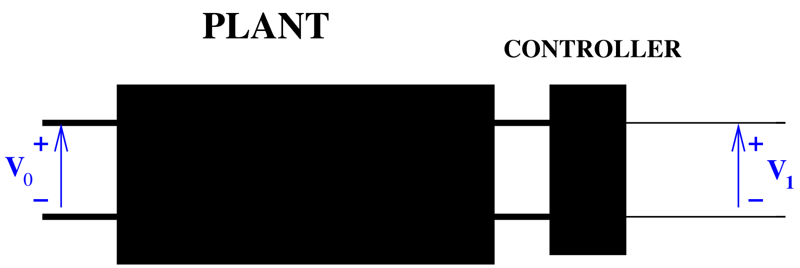
  for 'off-the-shelf' modeling.

- **What is the role of $V(x,t)$ and $I(x,t)$, $\;0 \leq x \leq L$, in modeling the relation between V_0, I_0, V_1, I_1?**

If terminated by an impedance $\rightsquigarrow$ undesirable reflections.

characteristic impedance $$R = \sqrt{\frac{L_0}{C_0}}$$ $\Rightarrow$ **no reflections!**





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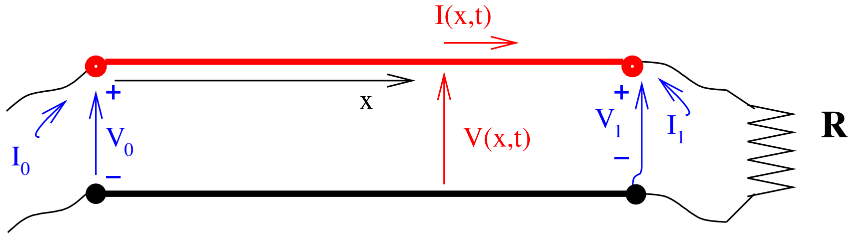
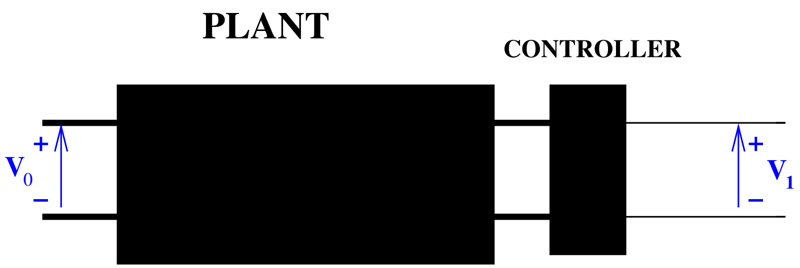
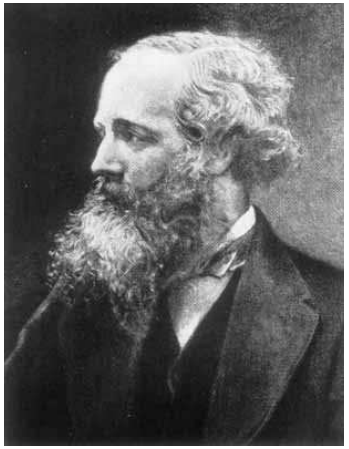




We view this termination as a behavioral controller. In this ex., the classical sensor-to-actuator feedback interpretation is an illusion.

$\exists$ very many such examples of controllers.

3. Maxwell's eqn's



$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

Set of independent variables $= \mathbb{R} \times \mathbb{R}^3$ (time and space),

dependent variables $= (\vec{E}, \vec{B}, \vec{j}, \rho)$

(electric field, magnetic field, current density, charge density),

$\in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$,

the behavior = set of solutions to these PDE's.

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Which PDE's describe $(\rho, \vec{E}, \vec{j})$ in Maxwell's equations ?

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Which PDE's describe $(\rho, \vec{E}, \vec{j})$ in Maxwell's equations ?

Eliminate $\vec{B}$ from Maxwell's equations $\rightsquigarrow$

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\ \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} &= 0. \end{aligned}$$

Potential functions

The following equations in the

scalar potential $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$

and the

vector potential $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$,

generate exactly the solutions to Maxwell's equations:

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}
$$

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▶ **When and why is a representation in terms of a potential possible?** **'Image representation' issue.**

Leads to the following questions:

- ▶ **Is there a fundamental reason why the behavior of $(\rho, \vec{E}, \vec{j})$ is also described by a PDE?** **'Elimination' issue.**

- ▶ **When and why is a representation in terms of a potential possible?**

  'Image representation' issue.

- ▶ **Derive algorithms for elimination, image representation.**

BEHAVIORAL $\mathtt{n}$-D SYSTEMS

A system = $\boxed{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$

$\mathbb{T}$, **the set of *independent* variables,**

$\mathbb{W}$, **the set of *dependent* variables,**

$\boxed{\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}}$ **: *the behavior*** **(= the admissible trajectories).**

$$\boxed{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$$

For a trajectory $w : \mathbb{T} \to \mathbb{W}$**, we thus have:**

$w \in \mathfrak{B}$ **: the model allows the trajectory** w**,**

$w \notin \mathfrak{B}$ **: the model forbids the trajectory** w**.**

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$\mathbb{T} = \mathbb{R}$ **(in continuous-time systems),**

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$ **(in** $\mathtt{n}$**-D systems),**

$\mathbb{W} \subseteq \mathbb{R}^{\mathtt{w}}$ **(in lumped systems),**

or a finite set (in DES).

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$\boxed{\mathbb{T} = \mathbb{R}^{\mathtt{n}} \text{ (in n-D systems),}}$

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Emphasis today: $\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,

$\mathfrak{B}$ **= solutions of system of linear constant coefficient PDE's.**

First principles models invariably contain auxiliary variables,

in addition to the variables the model aims at.

$\leadsto$ Manifest and latent variables.

Manifest = the variables the model aims at,

Latent = auxiliary variables.

We want to capture this in a mathematical definition.

A system with latent variables = $\boxed{\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})}$

$\mathbb{T}$, **the set of** ***independent*** **variables.**

$\mathbb{W}$, **the set of** ***manifest*** **dependent variables**

(= the variables that the model aims at).

$\mathbb{L}$, **the set of** ***latent*** **dependent variables**

(= the auxiliary modeling variables).

$\boxed{\mathfrak{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}}$ **:** ***the full behavior***

(= the pairs $(w, \ell) : \mathbb{T} \to \mathbb{W} \times \mathbb{L}$ **that the model declares possible).**

The manifest behavior

The latent variable system $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ induces the *manifest system* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, with *manifest behavior*

$$\mathfrak{B} = \{ w : \mathbb{T} \to \mathbb{W} \mid \exists\ \ell : \mathbb{T} \to \mathbb{L} \text{ such that } (w, \ell) \in \mathfrak{B}_{\text{full}} \}$$

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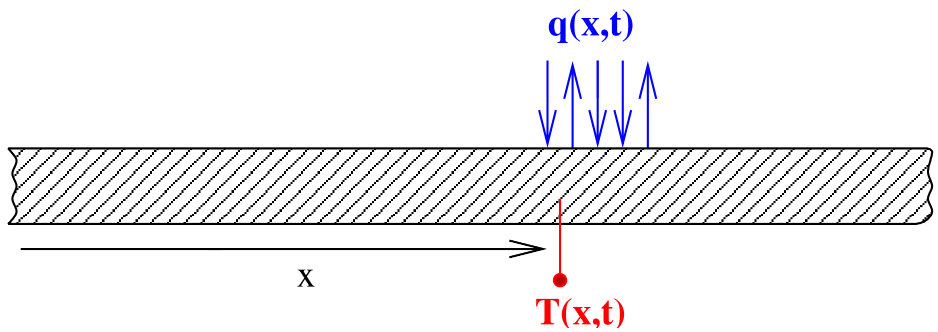
In convenient equations for $\mathfrak{B}$, the latent variables are *'eliminated'*.

Examples

1. Heat diffusion



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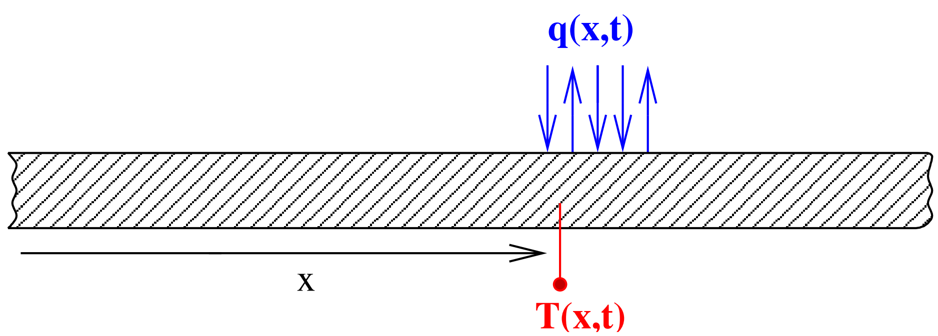
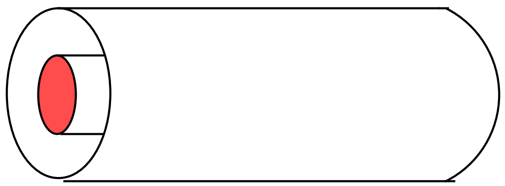


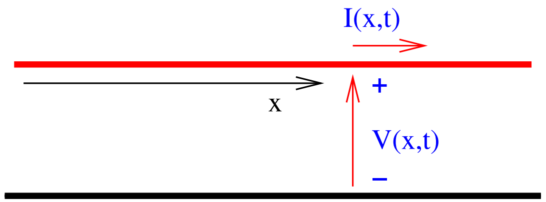
$\mathbb{T} = \mathbb{R}^2$ **(time and space);**

$\mathbb{W} = \mathbb{R}_+ \times \mathbb{R}$ **(temperature and heat);**

$\mathfrak{B}$ **= sol'ns to the PDE, the diffusion eq'n.**

2. Coaxial cable





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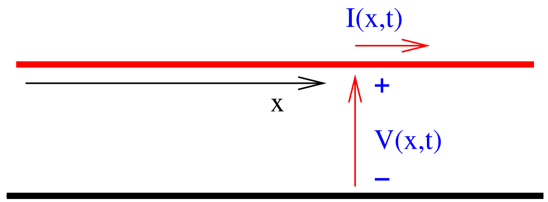


Consider the voltage as the variable the model aims at.

$\mathbb{T} = \mathbb{R}^2$ **(time and space);**

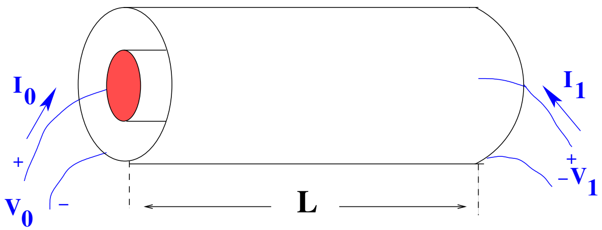
$\mathbb{W} = \mathbb{R}$ **(voltage);**

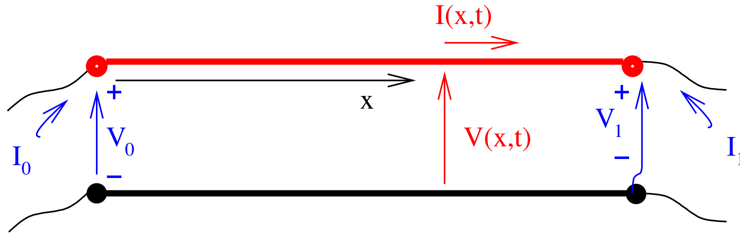
$\mathbb{L} = \mathbb{R}$ **(current);**

$\mathfrak{B}_{\text{full}}$ **= sol'ns to the PDE's;**

$\mathfrak{B}$**= sol'ns to** $\frac{\partial^2}{\partial x^2} V = L_0 C_0 \frac{\partial^2}{\partial t^2} V$?

3. Coaxial cable of length L.





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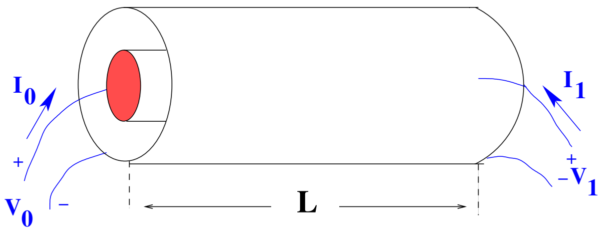

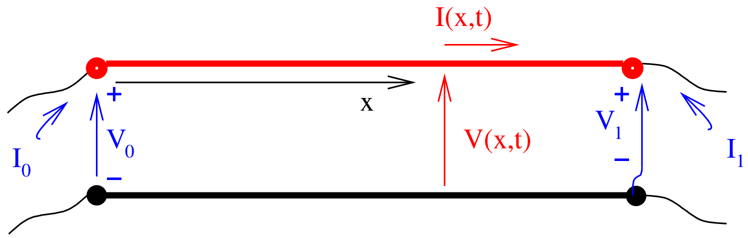

Consider the terminal variables as the variables the model aims at.

$\mathbb{T} = \mathbb{R}$ **(time);**

$\mathbb{W} = \mathbb{R}^4$ **(2 voltages, 2 currents),**

latent variables = $V(x,\cdot), I(x,\cdot); 0 \leq x \leq L$

(voltage and current in the coax)

$\mathfrak{B}_{\text{full}}$ **= sol'ns to the PDE's + boundary conditions.**

$\mathfrak{B}$**= sol'ns to ... ?**

4. Maxwell's eqn'ns

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If we view the electrical variables as manifest, and $\vec{B}$ as latent

$\mathbb{T} = \mathbb{R}^4, \mathbb{W} = \mathbb{R}^7, \mathbb{L} = \mathbb{R}^3,$

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If we consider the representation in terms of the potentials $\phi, \vec{A}$

$\mathbb{T} = \mathbb{R}^4, \mathbb{W} = \mathbb{R}^{10}, \mathbb{L} = \mathbb{R}^4,$

$\mathfrak{B}_{\text{full}} =$ **solutions to potential eqn's,** $\mathfrak{B} =$ **solutions to ME?**

LINEAR DIFFERENTIAL SYSTEMS

We now discuss the fundamentals of the theory of $\mathtt{n}$-D systems

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- *linear,* **meaning** $[(w_1, w_2 \in \mathfrak{B}) \wedge (\alpha, \beta \in \mathbb{R})] \Rightarrow [\alpha w_1 + \beta w_2 \in \mathfrak{B}]$;

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- ***linear,*** **meaning** $[(w_1, w_2 \in \mathfrak{B}) \wedge (\alpha, \beta \in \mathbb{R})] \Rightarrow [\alpha w_1 + \beta w_2 \in \mathfrak{B}]$;

- ***shift-invariant,*** **meaning** $[(w \in \mathfrak{B}) \wedge (x \in \mathbb{R}^{\mathtt{n}})] \Rightarrow [\sigma^x w \in \mathfrak{B}]$,

  where σ^x **denotes the** $x-$**shift: for** $x = (x_1, x_2, \ldots, x_{\mathtt{n}}) \in \mathbb{R}^{\mathtt{n}}$,

  $(\sigma^x f)(x'_1, x'_2, \ldots, x'_{\mathtt{n}}) := f(x'_1 + x_1, x'_2 + x_2, \ldots, x'_{\mathtt{n}} + x_{\mathtt{n}})$

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- ***differential,*** **meaning** $\mathfrak{B}$ **consists of the solutions of a system of PDE's.**

$\mathtt{n}$-D systems described by PDE's

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, $\mathtt{n}$ **independent variables,**

$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, $\mathtt{w}$ **dependent variables,**

$\mathfrak{B} =$ **the solutions of a linear constant coefficient system of PDE's.**

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Let $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, **and consider**

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0 \quad (*)$$

Define its behavior

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds } \} = \ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}))$$

n-D systems described by PDE's

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, n independent variables,

$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, w dependent variables,

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$\mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$ mainly for convenience, but important for some results.

Identical theory for $\mathfrak{D}'(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$.

Polynomial matrix notation for PDE's:

PDE:

$$\begin{aligned} w_1(x_1,x_2) + \frac{\partial^2}{\partial x_2^2} w_1(x_1,x_2) + \frac{\partial}{\partial x_1} w_2(x_1,x_2) &= 0 \\ w_2(x_1,x_2) + \frac{\partial^3}{\partial x_2^3} w_1(x_1,x_2) + \frac{\partial^4}{\partial x_1^4} w_2(x_1,x_2) &= 0 \end{aligned}$$

$$\updownarrow$$

Notation:

$$\xi_1 \leftrightarrow \frac{\partial}{\partial x_1} \qquad \xi_2 \leftrightarrow \frac{\partial}{\partial x_2}$$

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad R(\xi_1,\xi_2) = \begin{bmatrix} 1+\xi_2^2 & \xi_1 \\ \xi_2^3 & 1+\xi_1^4 \end{bmatrix}.$$

$$R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}) w = 0$$

Examples:

Diffusion eq'n, Wave eq'n, Co-axial cable

Maxwell's equations

NOMENCLATURE

$\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ **: the set of such systems with** $\mathtt{n}$ **in-,** $\mathtt{w}$ **dependent variables**

$\mathfrak{L}^{\bullet}$ **: with any - finite - number of (in)dependent variables**

Elements of $\mathfrak{L}^{\bullet}$ **:** ***linear differential systems***

$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$ **: a *kernel representation* of the corresponding** $\Sigma \in \mathfrak{L}^{\bullet}$ **or** $\mathfrak{B} \in \mathfrak{L}^{\bullet}$

First principles models $\rightsquigarrow$ latent variables. In the case of systems described by linear constant coefficient PDE's: $\rightsquigarrow$

$$\boxed{R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell}$$

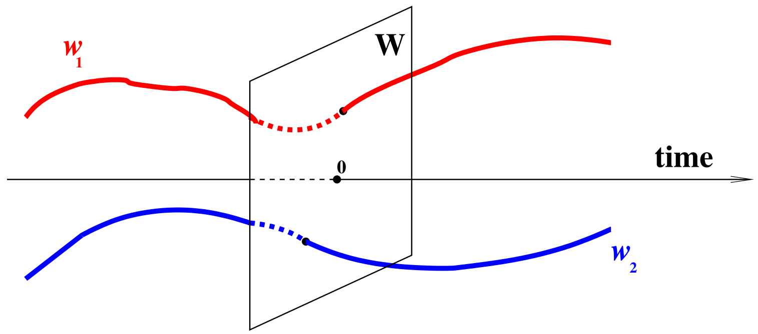
with $R, M \in \mathbb{R}^{\bullet \times \bullet}[\xi]$.

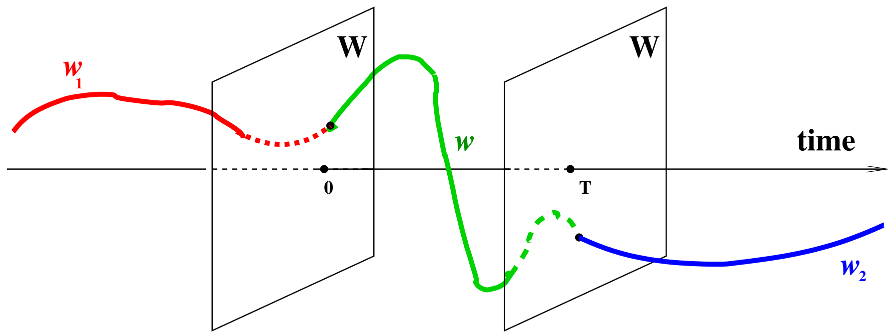
CONTROLLABILITY and OBSERVABILITY

Controllability $:\Leftrightarrow$

system trajectories must be ‘patch-able’, ‘concatenable’.

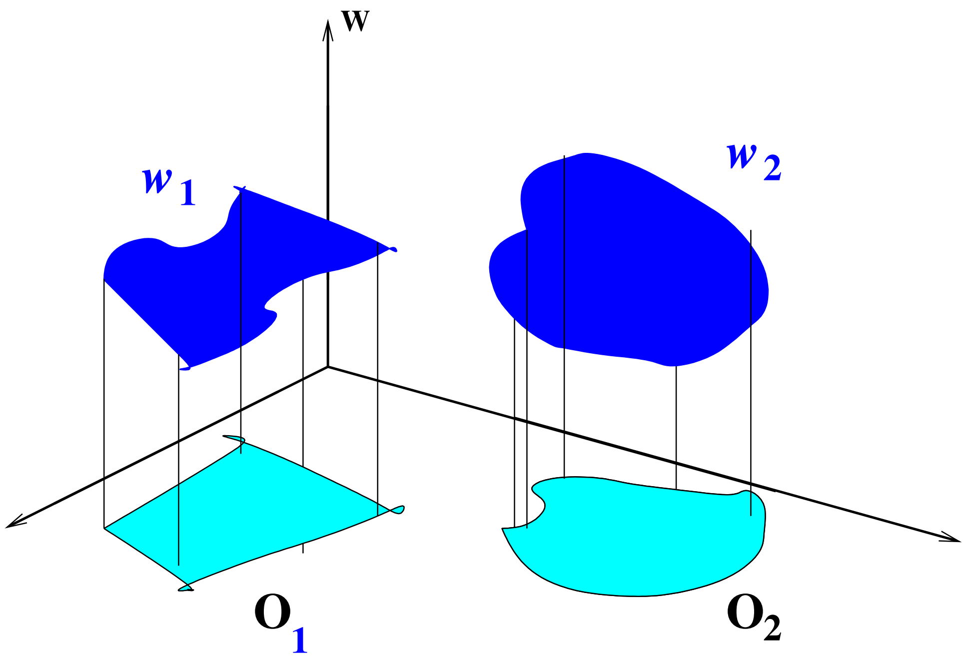
Case $\mathtt{n} = 1, \mathbb{T} = \mathbb{R}$, **any** $w_1, w_2 \in \mathfrak{B}$ **concatenable:**



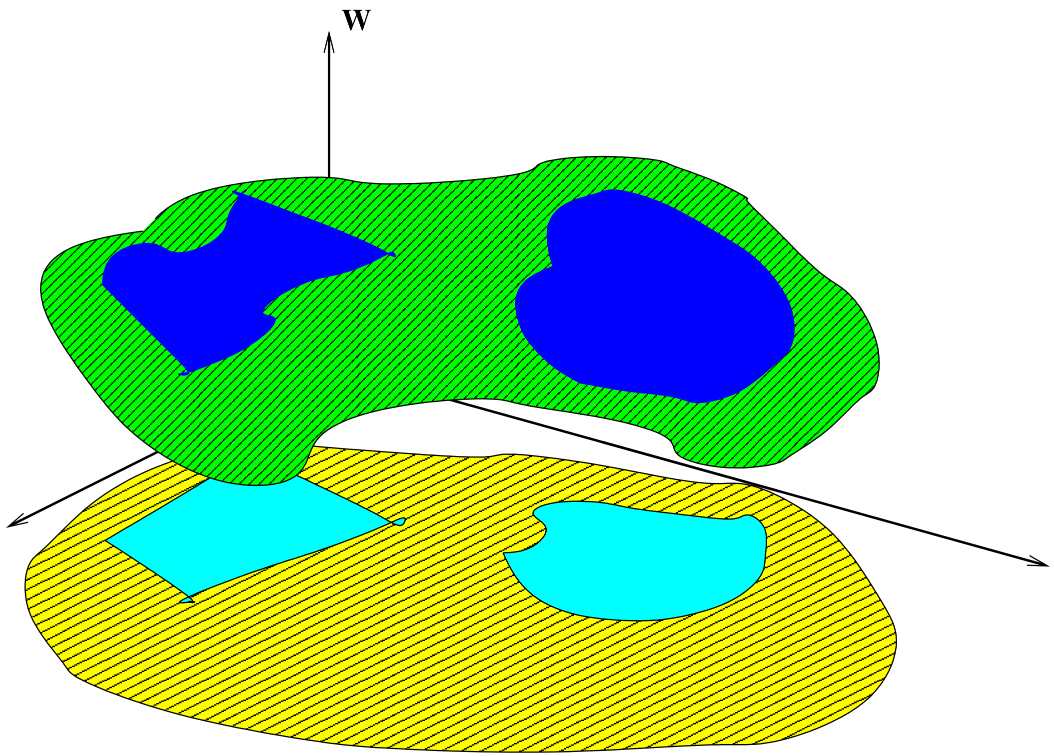


General $\mathtt{n}$, $\mathbb{T} = \mathbb{R}^{\mathtt{n}}$.

Consider any two elements w_1, w_2 of the behavior and any two open sets with non-overlapping closure $O_1, O_2 \subset \mathbb{R}^{\mathtt{n}}$:



Controllability = patchability:



Controllability of $\mathfrak{B}$:=

for any $O_1, O_2 \subset \mathbb{R}^n$, open, non-overlapping closure,

any $w_1, w_2 \in \mathfrak{B}$,

there is a sol'n $w \in \mathfrak{B}$ that 'patches' w_1 on O_1 with w_2 on O_2.

Consider the system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$.

Each element of the behavior $\mathfrak{B}$ hence consists of

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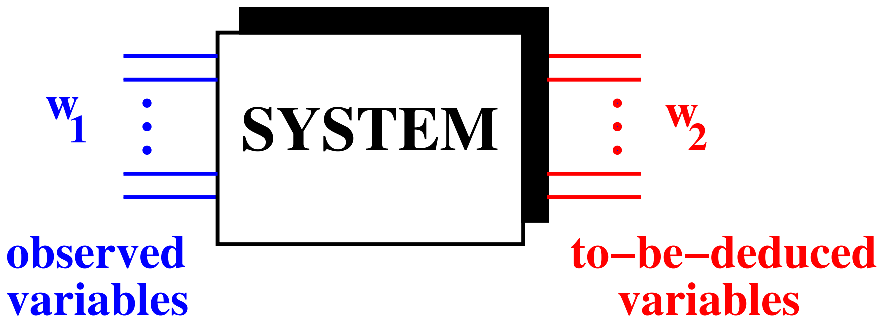

w_1 : observed; w_2 : to-be-deduced.

w_2 **is said to be** ***observable*** **from** w_1

if $((w_1, w_2') \in \mathfrak{B}$**, and** $(w_1, w_2'') \in \mathfrak{B}) \Rightarrow (w_2' = w_2'')$,

i.e., if on $\mathfrak{B}$**, there exists a map** $w_1 \mapsto w_2$.

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observed = manifest

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Def's for ODE's, PDE's, difference eq'ns, exactly the same!

3 THEOREMS

Theorem 1 **Algebraization:**

$$\boxed{\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}} \stackrel{\mathbf{1:1}}{\longleftrightarrow} \textbf{sub-modules of } \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]}$$

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$$\boxed{(\mathfrak{B}_{\mathbf{full}} \in \mathfrak{L}_{\mathtt{n}}^{\bullet}) \Rightarrow (\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\bullet})}$$

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Theorem 3 **Image representation:**

Controllabilility $\Leftrightarrow$ **($\exists$ Image representation)**

Algebraization of $\mathfrak{L}^\bullet$

Note that

$$R(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathrm{n}}})w=0$$

and

$$U(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathrm{n}}})R(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathrm{n}}})w=0$$

have the same behavior if the polynomial matrix U is uni-modular (i.e., when $\det(U)$ is a non-zero constant).

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have the same behavior if the polynomial matrix U is uni-modular (i.e., when $\det(U)$ is a non-zero constant).

$\Rightarrow$ R defines $\mathfrak{B} = \ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}))$, but not vice-versa!

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Define the ***annihilators*** **of** $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ **by**

$$\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}] \mid n^{\top}(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{B} = 0\}.$$

$\mathfrak{N}_{\mathfrak{B}}$ **is clearly an** $\mathbb{R}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ **sub-module of** $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$.

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Let $< R >$ denote the sub-module of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ spanned by the transposes of the rows of R. Obviously $< R > \subseteq \mathfrak{N}_{\mathfrak{B}}$. But, indeed:

$$\boxed{\mathfrak{N}_{\mathfrak{B}} = < R >!}$$

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Note: Depends on $\mathfrak{C}^\infty$; ($\Leftarrow$) false for compact support soln's:

for any $p \neq 0, \quad p(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$ has $w = 0$

as its only compact support sol'n.

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In particular $f(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w = 0$ is a consequence of $R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w = 0$ if and only if $f \in < R >$.

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Theorem 1 **(Algebraic structure of $\mathfrak{L}_n^w$):**

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- $\mathfrak{L}_n^w \overset{1:1}{\longleftrightarrow}$ **sub-modules of** $\mathbb{R}^w[\xi_1, \cdots, \xi_n]$

- $$R_1(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = 0 \text{ and } R_2(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = 0$$

define the same system iff

$$< R_1 > \; = \; < R_2 > .$$

Elimination

The full behavior of $R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell$,

$$\mathfrak{B}_{\text{full}} = \{(w, \ell) \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{w+\ell}) \mid R(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_n})w = M(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_n})\ell\}$$

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Its manifest behavior equals

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \mid \exists\ \ell \text{ such that } R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell\}.$$

Does $\mathfrak{B}$ belong to $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$?

Theorem 2 (Elimination): It does!

Does $\mathfrak{B}$ belong to $\mathfrak{L}_{\mathrm{n}}^{\mathrm{w}}$?

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Proof: The theorem is a straightforward consequence of the 'fundamental principle': the equation

$$\boxed{A(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})f = y}$$

$A \in \mathbb{R}^{\mathrm{n}_1 \times \mathrm{n}_2}[\xi_1, \cdots, \xi_{\mathrm{n}}]$, $y \in \mathfrak{C}^\infty(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathrm{n}_1})$ **given,** $f \in \mathfrak{C}^\infty(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathrm{n}_2})$ **unknown, is solvable if and only if for** $n \in \mathbb{R}^{\mathrm{n}_1}[\xi_1, \cdots, \xi_{\mathrm{n}}]$

$$(n^\top A = 0) \;\Rightarrow\; (n^\top(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})y = 0).$$

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$$(R, M) \mapsto R'$$

- **Not generalizable to smooth nonlinear systems.**

  Why are differential equations models so prevalent?

Examples

1.

$$\boxed{\frac{\partial^2}{\partial x^2} V = L_0 C_0 \frac{\partial^2}{\partial t^2} V,}$$

describes indeed the behavior of V in the coax.

2. Which PDE's describe $(\rho, \vec{E}, \vec{j})$ in Maxwell's equations ?

Eliminate $\vec{B}$ from Maxwell's equations $\rightsquigarrow$

$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\
\varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{j} &= 0.
\end{aligned}
$$

Elimination theorem $\Rightarrow$

this exercise is exact & successful (+ gives algorithm).

It follows from all this that $\mathfrak{L}^{\bullet}_{\mathtt{n}}$ has very nice properties. It is closed under:

- **Intersection:** $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}) \Rightarrow (\mathfrak{B}_1 \cap \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}})$.
- **Addition:** $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}) \Rightarrow (\mathfrak{B}_1 + \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}})$.
- **Projection:** $(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}_{\mathtt{n}}) \Rightarrow (\Pi_{w_1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}_{\mathtt{n}})$.
- **Action of a linear differential operator:**

$$(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}_{\mathtt{n}}, P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_1}[\xi_1, \cdots, \xi_{\mathtt{n}}]) \Rightarrow (P(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2}_{\mathtt{n}}).$$

- **Inverse image of a linear differential operator:**

$$(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2}_{\mathtt{n}}, P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_1}[\xi_1, \cdots, \xi_{\mathtt{n}}]) \Rightarrow (P(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}))^{-1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}_{\mathtt{n}}).$$

Image representations

Representations of $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$:

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$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\ell$$

called a *'latent variable' representation* of the manifest behavior

$$\mathfrak{B} = (R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}))^{-1} M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\ell}).$$

Missing link:

$$\boxed{w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell}$$

called an *'image' representation* of $\mathfrak{B} = \mathrm{im}(M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}))$.

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¿¿ Which kernels are also images ??

Theorem 3 (Controllability and image representation):

The following are equivalent for $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$:

1. $\mathfrak{B}$ is controllable,
2. $\mathfrak{B}$ admits an image representation,
3. for any $a \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$,
$$a^{\top}[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}]\mathfrak{B} \text{ equals 0 or all of } \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}),$$
4. $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]/\mathfrak{N}_{\mathfrak{B}}$ is torsion free,

etc.

Are Maxwell's equations controllable ?

The following equations in the *scalar potential* $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, generate exactly the solutions to Maxwell's equations:

$$\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow$ $\exists$ potential!

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- Kalman controllability is a straightforward special case.

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<u>Observability</u> means: $M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})$ **is injective:**

ℓ **can be deduced from** w **in**

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\ell.$$

$\exists$ **equivalent** $R'(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0, \ell = M'(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w.$

Not all controllable systems admit an observable image representation.

For $\mathtt{n} = 1$, they do. For $\mathtt{n} > 1$, exceptionally so.

Observability means: $M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})$ is injective:

ℓ can be deduced from w in

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})\ell.$$

$\exists$ equivalent $R'(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w = 0, \ell = M'(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w.$

The latent variable in an image representation ℓ may be 'hidden'.

Example: Maxwell's equations do not allow a potential representation that is observable.

DISSIPATIVE DISTRIBUTED SYSTEMS

A dissipative system absorbs supply, 'globally' over time and space.

¿¿ Can this be expressed 'locally', as

$$\boxed{\text{rate of change in storage + spatial flux} \leq \text{supply rate}}$$

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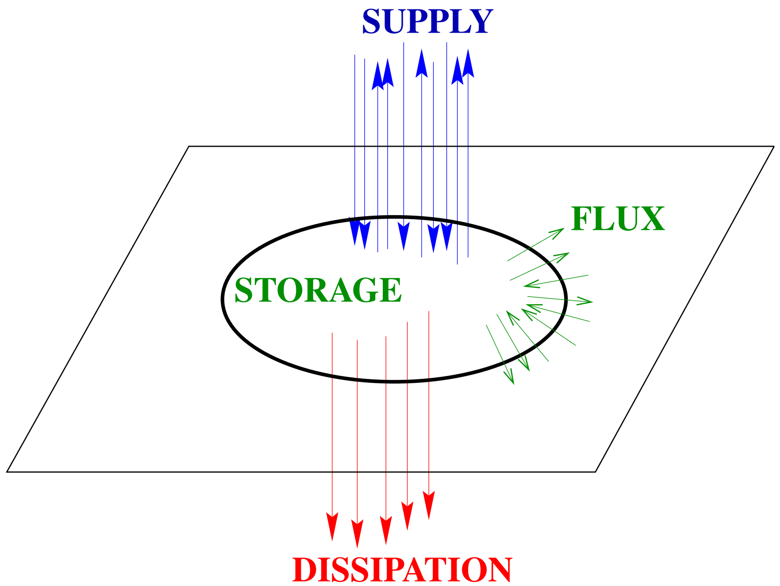
rate of change in storage + spatial flux $\leq$ supply rate



rate of change in storage + spatial flux

= supply rate + (non-negative) dissipation rate ??

Multi-index notation:

$x = (x_1, \ldots, x_n)$,

$k = (k_1, \ldots, k_n), \ell = (\ell_1, \ldots, \ell_n)$,

$\xi = (\xi_1, \cdots, \xi_n), \zeta = (\zeta_1, \ldots, \zeta_n), \eta = (\eta_1, \ldots, \eta_n)$,

$\frac{d}{dx} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}), \frac{d^k}{dx^k} = (\frac{\partial^{k_1}}{\partial x_1^{k_1}}, \ldots, \frac{\partial^{k_n}}{\partial x_n^{k_n}})$,

$dx = dx_1 dx_2 \ldots dx_n$,

$R(\frac{d}{dx})w = 0$ for $R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = 0$,

$w = M(\frac{d}{dx})\ell$ for $w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell$,

etc.

QDF's

The quadratic map in w and its derivatives, defined by

$$w \mapsto \sum_{k,\ell} \left(\frac{d^k}{dx^k} w\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dx^\ell} w\right)$$

is called *quadratic differential form* (QDF) on $\mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$.

$\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$; WLOG: $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$.

Introduce the 2$\mathtt{n}$-variable polynomial matrix Φ

$$\Phi(\zeta, \eta) = \sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell.$$

Denote the QDF as Q_Φ.

We consider only controllable linear differential systems and supply rates that are QDF's.

Definition: $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$, **controllable, is said to be** ***dissipative***

with respect to the supply rate Q_{Φ} **(a QDF) if**

$$\boxed{\int_{\mathbb{R}^{\mathtt{n}}} Q_{\Phi}(w)\, dx \geq 0}$$

for all $w \in \mathfrak{B}$ **of compact support, i.e., for all** $w \in \mathfrak{B} \cap \mathfrak{D}$**.**

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If equality 'conservative'.

Assume $\mathtt{n} = 4$: independent variables $x, y, z; t$: space and time.

<u>Idea</u>: $Q_\Phi(w)(x, y, z; t)\ dxdydz\ dt$:

rate of ‘energy’ delivered to the system.

<u>Dissipativity</u> :⇔

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} Q_\Phi(w)\ dxdydz \right) dt \geq 0 \qquad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

A dissipative system absorbs net energy.

Example: **Maxwell's eq'ns:**

dissipative (in fact, conservative) w.r.t. the QDF $-\vec{E}\cdot\vec{j}$.

In other words, if $\vec{E},\vec{j}$ **is of compact support and satisfies**

$$\begin{array}{rcl} \varepsilon_0 \frac{\partial}{\partial t}\nabla\cdot\vec{E} + \nabla\cdot\vec{j} & = & 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2 \nabla\times\nabla\times\vec{E} + \frac{\partial}{\partial t}\vec{j} & = & 0, \end{array}$$

then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} (-\vec{E}\cdot\vec{j}) \; dxdydz \right) dt = 0.$$

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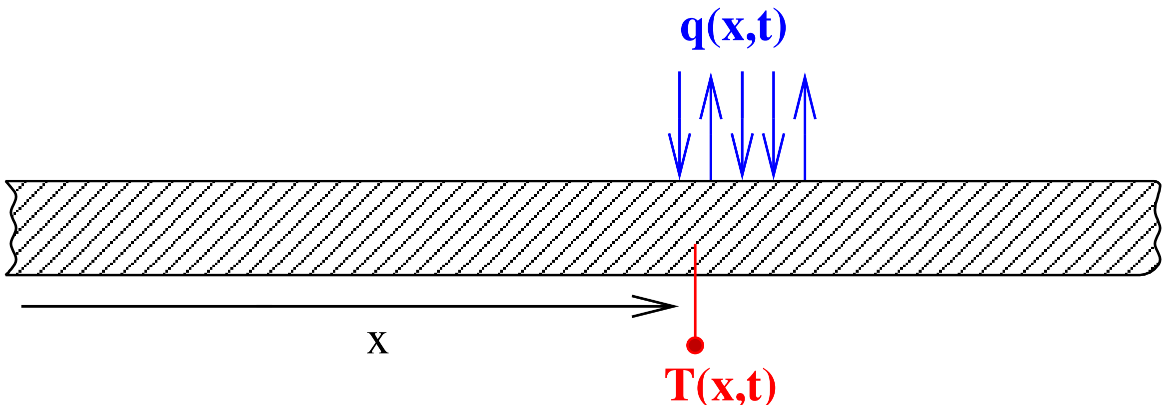
then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} (-\vec{E}\cdot\vec{j}) \; dxdydz \right) dt = 0.$$

Can this be reinterpreted as: As the system evolves,

energy is locally stored, and redistributed over time and space?

First principles motivating example: *Heat diffusion*



The PDE

$$\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q$$

describes the evolution of the temperature $T(x,t)$ ($x \in \mathbb{R}$ position, $t \in \mathbb{R}$ time) in a medium and the heat $q(x,T)$ supplied to / radiated away from it.

For all sol'ns T, q with $T(x,t) = \text{constant} > 0$ (and therefore $q = 0$) outside a compact set, there holds:

First law:

$$\boxed{\int_{\mathbb{R}^2} q(x,t)\, dx\, dt \;=\; 0,}$$

Second law:

$$\boxed{\int_{\mathbb{R}^2} \frac{q(x,t)}{T(x,t)}\, dx\, dt \;\leq\; 0.}$$

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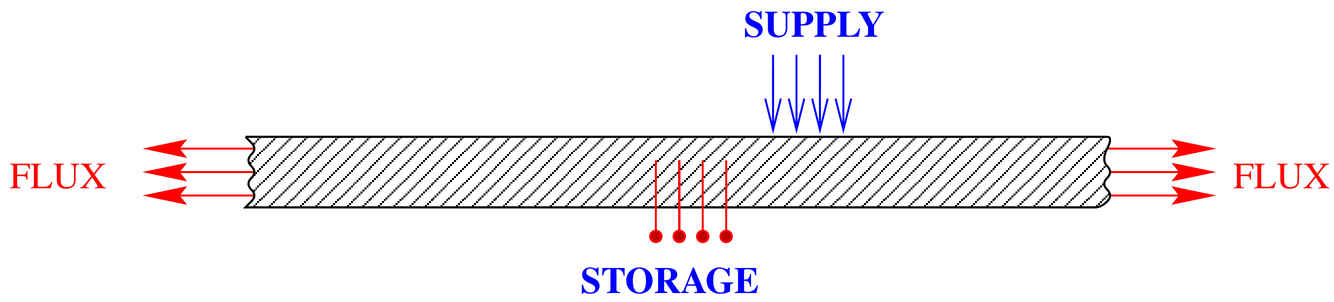
$\Rightarrow$

$$\boxed{\max_{x,t}\{T(x,t) \mid q(x,t) \geq 0\} \geq \min_{x,t}\{T(x,t) \mid q(x,t) \leq 0\}.}$$

It is impossible to transport heat from a 'cold source' to a 'hot sink'.

Can these 'global' versions be expressed as 'local' laws?



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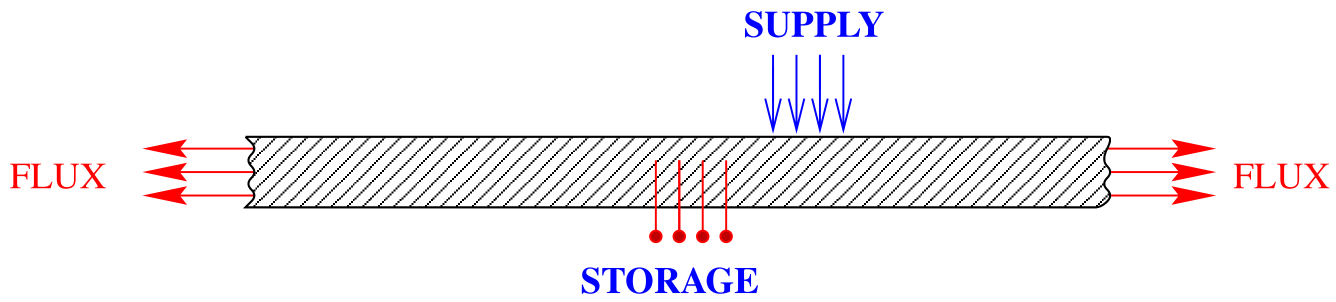


rate of change in storage + spatial flux $\leq$ supply rate

To be invented:

an 'extensive' quantity for the first law: internal energy

an 'extensive' quantity for the second law: entropy

Define the following variables:

$$E = T \quad : \text{ the stored energy density,}$$

$$S = \ln(T) \quad : \text{ the entropy density,}$$

$$F_E = -\frac{\partial}{\partial x} T \quad : \text{ the energy flux,}$$

$$F_S = -\frac{1}{T}\frac{\partial}{\partial x} T \quad : \text{ the entropy flux,}$$

$$D_S = \left(\frac{1}{T}\frac{\partial}{\partial x} T\right)^2 \quad : \text{ the rate of entropy production.}$$

Local versions **of the first and second law:**

rate of change in storage + spatial flux $\leq$ supply rate

Conservation of energy:

$$\boxed{\frac{\partial}{\partial t}E + \frac{\partial}{\partial x}F_E = q,}$$

Entropy production:

$$\frac{\partial}{\partial t}S + \frac{\partial}{\partial x}F_S = \frac{q}{T} + D_S. \qquad \text{Since} \quad (D_S \geq 0) \quad \Rightarrow$$

$$\boxed{\frac{\partial}{\partial t}S + \frac{\partial}{\partial x}F_S \geq \frac{q}{T}.}$$

Our problem:

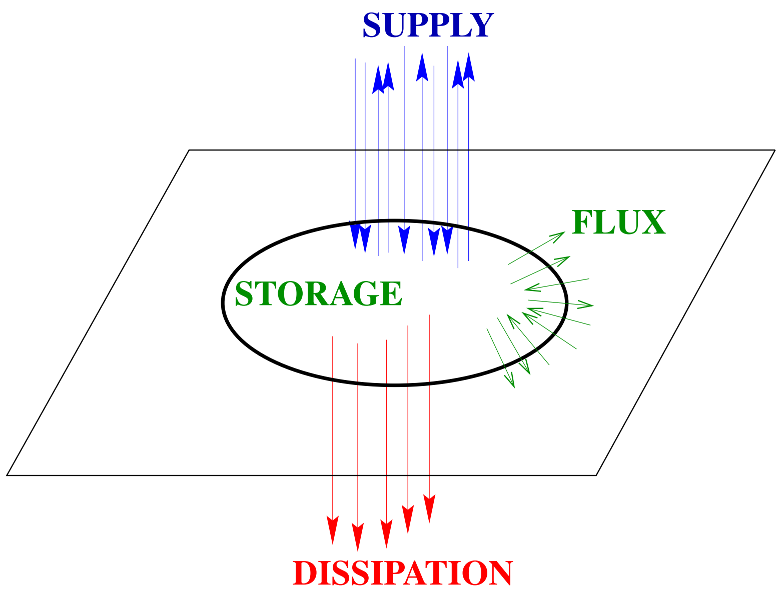
theory behind these ad hoc constructions of E, F_E and S, F_S.

Assume that a system is 'globally' dissipative.

¿¿ Can this dissipativity be expressed through a 'local' law??

Such that in every spatial domain there holds:

$$\frac{d}{dt}\ \text{Storage} + \text{Spatial flux} \leq \text{Supply.}$$



Supply = Stored + radiated + dissipated.

Main Theorem:

$\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$, controllable, is dissipative w.r.t. the supply rate Q_Φ **iff**

$\exists$ an image representation $w = M(\frac{d}{dx})\ell$ of $\mathfrak{B}$,

an $\mathtt{n}-$vector of QDF's $Q_\Psi = (Q_{\Psi_1}, \ldots, Q_{\Psi_\mathtt{n}})$

on $\mathfrak{C}^\infty(\mathbb{R}^\mathtt{n}, \mathbb{R}^{\dim(\ell)})$, called the *flux*,

such that the *local dissipation law*

$$\nabla \cdot Q_\Psi(\ell) \leq Q_\Phi(w)$$

holds for all (w, ℓ) that satisfy $w = M(\frac{d}{dx})\ell$.

As usual $\nabla \cdot Q_\Psi := \frac{\partial}{\partial x_1} Q_{\Psi_1} + \cdots + \frac{\partial}{\partial x_\mathtt{n}} Q_{\Psi_\mathtt{n}}$.

$\leftrightarrow$ the QDF induced by $(\zeta + \eta)^\top \Psi(\zeta, \eta)$

Assume $\mathtt{n} = 4$: independent variables $x, y, z; t$: space and time.

Let $\mathfrak{B} \in \mathfrak{L}_4^{\mathtt{w}}$ be controllable. Then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} Q_\Phi(w)\, dxdydz \right) dt \geq 0 \qquad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

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$\exists$ an image representation $w = M(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t})\ell$ of $\mathfrak{B}$,

and QDF's S, the *storage*, and

F_x, F_y, F_z, the *spatial flux*,

such that the *local dissipation law*

$$\boxed{\frac{\partial}{\partial t} S(\ell) + \frac{\partial}{\partial x} F_x(\ell) + \frac{\partial}{\partial y} F_y(\ell) + \frac{\partial}{\partial z} F_z(\ell) \leq Q_\Phi(w)}$$

holds for all (w, ℓ) that satisfy $w = M(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t})\ell$.

Note: the local law involves

(possibly unobservable, - i.e., hidden!)

latent variables (the ℓ's).

EXAMPLE: ENERGY STORED IN EM FIELDS

Maxwell's equations are dissipative (in fact, conservative) with respect to

$-\vec{E}\cdot\vec{j}$, **the rate of energy supplied.**

Introduce the *stored energy density,* S, and

the *energy flux density* (the *Poynting vector*), $\vec{F}$,

$$S(\vec{E},\vec{B}) := \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{\varepsilon_0 c^2}{2}\vec{B}\cdot\vec{B},$$

$$\vec{F}(\vec{E},\vec{B}) := \varepsilon_0 c^2 \vec{E}\times\vec{B}.$$

The following is a local conservation law for Maxwell's equations:

$$\boxed{\frac{\partial}{\partial t}S(\vec{E},\vec{B}) + \nabla\cdot\vec{F}(\vec{E},\vec{B}) = -\vec{E}\cdot\vec{j}.}$$

Local version involves $\vec{B}$, unobservable from $\vec{E}$ and $\vec{j}$,

the variables in the rate of energy supplied.

Schematic of the proof

Using controllability and image representations, we may assume WLOG:

$$\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$$

Global dissipativity $:\Leftrightarrow \quad \int_{\mathbb{R}^n} Q_\Phi(w) \geq 0$ **for all** $w \in \mathfrak{D}$

$\Updownarrow$ **(Parseval)**

$\Phi(-i\omega, i\omega) \geq 0$ **for all** $\omega \in \mathbb{R}^n$

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$$\Updownarrow \quad \boxed{\text{(Factorization equation)}}$$

$$\exists\ D : \quad \Phi(-\xi, \xi) = D^{\top}(-\xi)D(\xi)$$

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$\Updownarrow$ **(clearly)**

Local diss. $:\Leftrightarrow\ \exists\ \Psi : \quad \nabla \cdot Q_\Psi(w) \leq Q_\Phi(w)$ **for all** $w \in \mathfrak{C}^\infty$

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THE FACTORIZATION EQUATION

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with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ given, and X the unknown. ***Solvable??***

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For $\mathtt{n} = 1$ **and** $Y \in \mathbb{R}[\xi]$**, solvable (for** $X \in \mathbb{R}^2[\xi]$ **!) iff**

$$Y(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}.$$

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For $\mathtt{n} = 1$, and $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, it is well-known (but non-trivial) that this factorization equation is solvable (with $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$!) iff

$$Y(\alpha) = Y^\top(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}.$$

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Solve $p = p_1^2 + p_2^2 + \cdots + p_k^2$, p **given**

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A polynomial $p \in \mathbb{R}[\xi_1, \cdots, \xi_n]$, **with** $p(\alpha_1, \ldots, \alpha_n) \geq 0$ **for all** $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ **can in general not be expressed as a sum of squares of polynomials, with the** p_i**'s** $\in \mathbb{R}[\xi_1, \cdots, \xi_n]$.

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But a rational function (and hence a polynomial) $p \in \mathbb{R}(\xi_1, \cdots, \xi_n)$, **with** $p(\alpha_1, \ldots, \alpha_n) \geq 0$, **for all** $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, **can be expressed as a sum of squares of** ($k = 2^n$) **rational functions, with the** p_i**'s** $\in \mathbb{R}(\xi_1, \cdots, \xi_n)$.

$\Rightarrow$ **solvability of the factorization eq'n**

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^{\mathrm{n}}$$

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The need to introduce

rational functions in this factorization

an image representation of $\mathfrak{B}$ to reduce the pbm to $\mathfrak{C}^\infty$

are the causes of the unavoidable presence of (possibly unobservable, i.e., 'hidden') latent variables in the local dissipation law.

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**For conservative systems, $\Phi(-\xi,\xi) = 0$, whence $D = 0$,
but, when $\mathtt{n} > 1$, the third source of non-uniqueness remains, even when working with a specific image representation.**

It seems to be a very real non-uniqueness, even for EM fields. Cfr.

The ambiguity of the field energy

... There are, in fact, an infinite number of different possibilities for u **[the internal energy]** ***and*** S **[the flux]** ***... It is sometimes claimed that this problem can be resolved using the theory of gravitation ... as yet nobody has done such a delicate experiment ... So we will follow the rest of the world - besides, we believe that it*** **[our choice]** ***is probably perfectly right.***

The Feynman Lectures on Physics,
Volume II, page 27-6.

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► ► $\mathfrak{L}_n^w \overset{1:1}{\leftrightarrow}$ **submodules of** $\mathbb{R}[\xi_1, \cdots, \xi_n]$.

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- **Formalized as systems with a richer set of 'independent' variables, e.g., $\mathbb{R}^n$.**
- **Linear constant coefficient PDE's $\leftrightarrow$ polynomials in many variables**
- **Controllability $:\Leftrightarrow$ patchability**
- **3 central theorems:**
  - $\mathfrak{L}_n^w \overset{1:1}{\leftrightarrow}$ **submodules of** $\mathbb{R}[\xi_1, \cdots, \xi_n]$.
  - **Elimination thm.**

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▶ **Proof ≅ Hilbert's 17-th problem**

End of Lecture 9