MATHEMATICAL MODELS of SYSTEMS

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Lecture 5

MODELLING by TEARING and ZOOMING

THEME

We present a language for modeling interconnected systems. When systems are interconnected what really happens? How do we obtain a model from modeling the components and the interconnections? What are the basic ideas?

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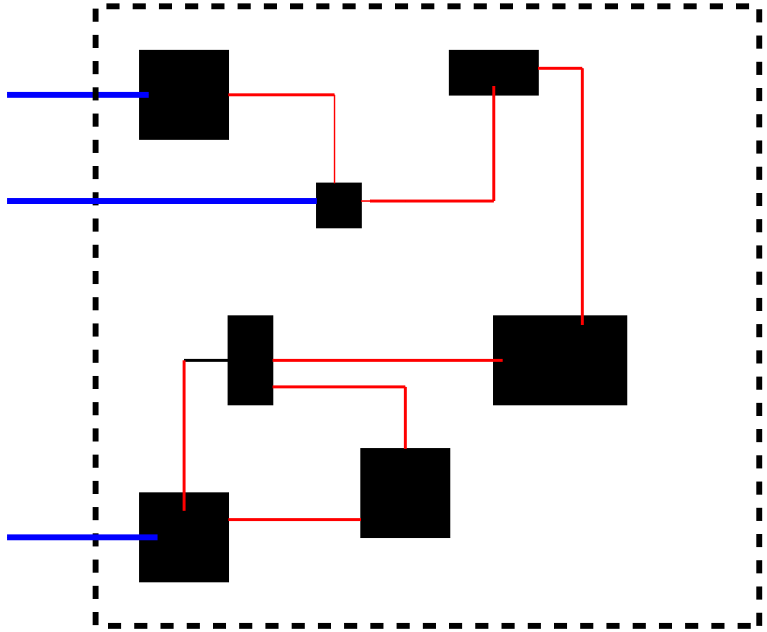
We present a language for modeling interconnected systems. When systems are interconnected what really happens? How do we obtain a model from modeling the components and the interconnections? What are the basic ideas?

- **Terminals**
- **Modules**
- **Interconnection architecture**
- **Examples**
- **RLCT circuits**

GENERAL IDEAS

How do we model a complex interconnected system?



Interconnected system

The ingredients of the language and methodology that we propose:

1. ***Modules*** **: the subsystems**
2. ***Terminals*** **: the physical links between subsystems**
3. **The** ***interconnection architecture*** **:**

   the layout of the modules and their interconnection
4. **The** ***manifest variable assignment*** **:**

   which variables does the model aim at?

Features:

- **Reality — 'physics' — based**
- **Uses behavioral systems concepts**

  more akin to bond-graphs and across/through variables,

  than to input/output thinking and SIMULINK©.
- **Hierarchical: allows new systems to be build from old**
- **Models are reusable, generalizable & extendable**
- **Assumes that accurate and detailed modeling is the aim**

TERMINALS

A ***terminal*** **is specified by its** ***type*****.**

The ***type*** **implies an ordered set of** ***terminal variables*****.**

Examples:

Type of terminal	Variables	Signal space
electrical	(voltage, current)	$\mathbb{R}^2$
mechanical (1-D)	(force, position)	$\mathbb{R}^2$
mechanical (2-D)	((position, attitude), (force, torque))	$(\mathbb{R}^2 \times S^1) \times (\mathbb{R}^2 \times T^* S^1)$
mechanical (3-D)	((position, attitude), (force, torque))	$(\mathbb{R}^2 \times S^2) \times (\mathbb{R}^2 \times T^* S^2)$
thermal	(temp., heat flow)	$\mathbb{R}^2$
fluidic	(pressure, flow)	$\mathbb{R}^2$
thermal - fluidic	(pressure, temp., mass flow, heat flow)	$\mathbb{R}^4$

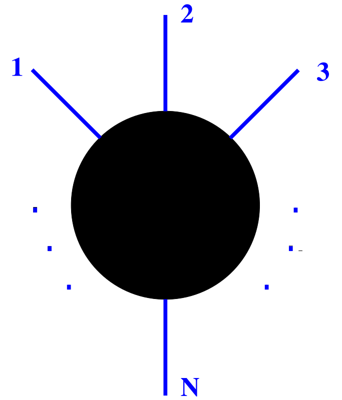
Type of terminal	Variables	Signal space
chemical		
input	u	$\mathbb{U} \subseteq \mathbb{R}$
output	y	$\mathbb{Y} \subseteq \mathbb{R}$
m-dim input	$(u_1, u_2, \ldots, u_m)$	$\mathbb{U} \subseteq \mathbb{R}^m$
p-dim output	$(y_1, y_2, \ldots, y_p)$	$\mathbb{Y} \subseteq \mathbb{R}^p$
etc.	etc.	etc.

MODULES

A *module* is specified by its *its type* , its *parametrization* , and its *parameter values* .

The module *type* specifies an ordered set of terminals

$$(t_1, t_2, \ldots, t_n).$$



Together with the terminal types, this specifies an ordered set of variables

$$((w_{t_1,1}, w_{t_1,2}, \ldots), \ldots, (w_{t_n,1}, w_{t_n,2}, \ldots)),$$

taking values in the product space of the terminal signal spaces.

The module type also specifies a set $\mathbb{B}$ of possible behaviors of the terminal variables of the module.

We assume that this set $\mathbb{B}$ is *parameterized,* (typically by something like a set of integers, and a set of real numbers).

The *parameter values* specify these parameters.

By specifying a module, we thus obtain the *behavior* of the variables

$$(w_1, w_2, \ldots, w_n)$$

on the terminals of the module.

This way we obtain a dynamic model of the interaction of the module with its environment.

Examples:

ELECTRICAL MODULES

Module	Parametrization	Parameter value
2-terminal Ohmic resistors	$\mathbb{R}$	R in ohms
2- terminal Ohmic conductors	$\mathbb{R}$	G in mhos
2- terminal current driven resistors	all maps: $\mathbb{R} \to \mathbb{R}$	$\rho : \mathbb{R} \to \mathbb{R}$
capacitor	$\mathbb{R}$	C in farads
inductor	$\mathbb{R}$	L in henrys

Module	Parametrization	Parameter value
linear impedances	$\mathbb{N}$ (number of ports) $\times\ \mathbb{R}^{n\times n}(\xi)$	$Z \in \mathbb{R}^{n\times n}[\xi]$
resistive $\triangle$	$\mathbb{R}$	R in ohms
Y with linear diff. systems	$(\mathbb{R}^2[\xi])^3$	$R_1, R_2, R_3 \in \mathbb{R}^2[\xi]$
transformer	$\mathbb{R}$	$n \in \mathbb{R}$
transmission line	$(\mathbb{R}_+)^5$	L, ℓ, c, r_s, r_p
transistor		
etc.	etc.	etc.

MECHANICAL MODULES

Module	Parametrization	Parameters
mass	$\mathbb{R}$	m in kgr
solid bar	$\mathbb{R}^2$	L, m
spring		
damper		
multi-terminal mass		geometry
flexible bar		
etc.	etc.	etc.

OTHER DOMAINS

Module	Representation	Parameters
servo joint		$m_r, m_s, J_r, J_s,$ L, R, K
2 inlet tank		geometry
etc.	etc.	etc.

LINEAR SYSTEMS

Module	Parametrization	Parameters
$\Sigma \in \mathfrak{L}^\bullet$	$\mathbb{N} \times \{\text{ker}, \text{im}, \text{etc.}\}$ $\times \mathbb{R}^{\bullet\times\bullet}[\xi]$, or $\cdots$	$(\mathtt{w}, \text{ker}, R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi])$ $\cdots$
$\Sigma \in \mathfrak{L}^\bullet_{\text{cont}}$	$\mathbb{N} \times \{\text{im}, \ldots\}$	$(\mathtt{w}, M \in \mathbb{R}^{\mathtt{w}\times\bullet}[\xi])$, $\cdots$
$\Sigma \in \mathfrak{L}^{\text{i/o}}_{\text{cont}}$	$\mathbb{N} \times \mathbb{N} \times \{\text{tf. fn.}, \ldots\} \times \mathbb{R}^{\bullet\times\bullet}(\xi), \ldots$	$\mathtt{m}, \mathtt{p}, G \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}[\xi]$ $\cdots$
$\Sigma \in \mathfrak{L}^{\text{i/s/o}}$	$\mathbb{N}^3, \ldots$	$\mathtt{m}, \mathtt{n}, \mathtt{p}, (A, B, C, D)$
etc.	etc.	etc.

INTERCONNECTION ARCHITECTURE

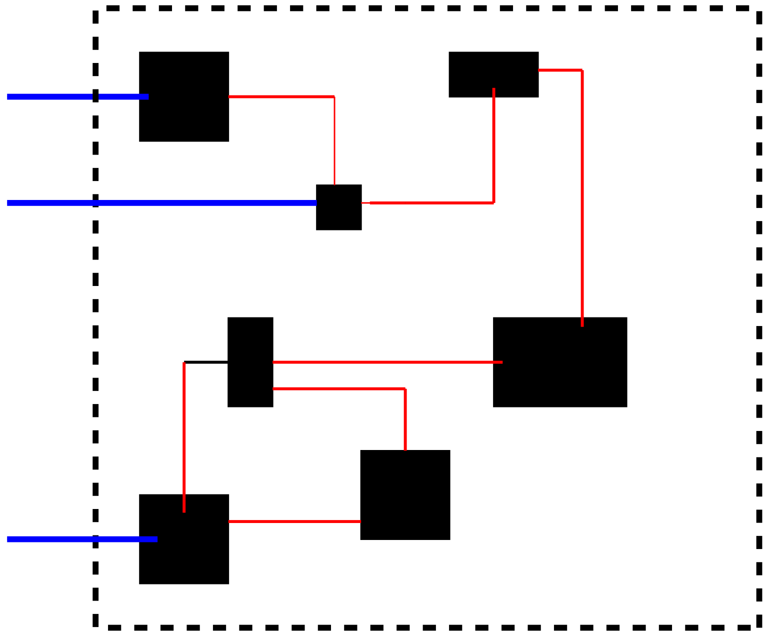
Let $T = \{t_1, t_2, \ldots, t_{|T|}\}$ be a set of terminals.

The interconnection architecture is a set of *terminal pairs* (unordered, disjoint, and with distinct elements), denoted by $\mathbb{I}$.

If $\{t_{\mathtt{i}}, t_{\mathtt{k}}\} \in \mathbb{I}$, then we say that these terminals are connected.
We impose that connected terminals must be adapted.

In the case of physical terminals, this means that they must be of the same type (both electrical, 2-D mechanical, thermal, etc.).

In the case of logical terminals (input or output terminals), this means that if one is an `m`-dimensional input terminal, the other must be an `m` dimensional output terminal.



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Graph with leaves

INTERCONNECTION CONSTRAINTS

Pairing of terminals imposes an *interconnection law*.

Pair of terminals	Terminal 1	Terminal 2	Law
electrical	(V_1, I_1)	(V_2, I_2)	$V_1 = V_2, I_1 + I_2 = 0$
1-D mech.	(F_1, q_1)	(F_2, q_2)	$F_1 + F_2 = 0, q_1 = q_2$
2-D mech.			
thermal	(Q_1, T_1)	(Q_2, T_2)	$Q_1 + Q_2 = 0, T_1 = T_2$
fluidic	(p_1, f_1)	(p_2, f_2)	$p_1 = p_2, f_1 + f_2 = 0$
info processing	$\mathbb{m}$-input u	$\mathbb{m}$-output y	$u = y$
etc.	etc.	etc.	etc.

MANIFEST VARIABLE ASSIGNMENT

We finally assume that the modeler assigns the variables at which the model aims. These are the *manifest variables*.

The latent variables in the ultimate model are

either

interconnection variables,

or

latent variables used to describe the behavior of the modules.

MODEL GENERATION

So, in order to obtain a model of an interconnected system, specify:

- **Modules** $M_1, M_2, \cdots, M_{\mathtt{m}}$

  $\rightsquigarrow$ **type & parametrization & parameter values.**

  This yields a list of terminals $T = \{t_1, t_2, \ldots, t_{|T|}\}$ **and, for each module, the behavior** $\mathfrak{B}_{\mathtt{i}}, \mathtt{i} = 1, \ldots, \mathtt{m}$ **for the variables living on the terminals.**

- **Denote** $\mathfrak{B}' = \mathfrak{B}_1 \times \cdots \times \mathfrak{B}_{\mathtt{m}}$.

- Interconnection architecture $\mathbb{I}$ on $T = \{t_1, t_2, \ldots, t_{|T|}\}$

  $\rightsquigarrow$ interconnection laws,

  and a behavior $\mathfrak{B}''$

  for the variables living on the terminals.

- The manifest variable assignment identifies certain of the variables as latent variables.

- The yields $\mathfrak{B}' \cap \mathfrak{B}'' =$

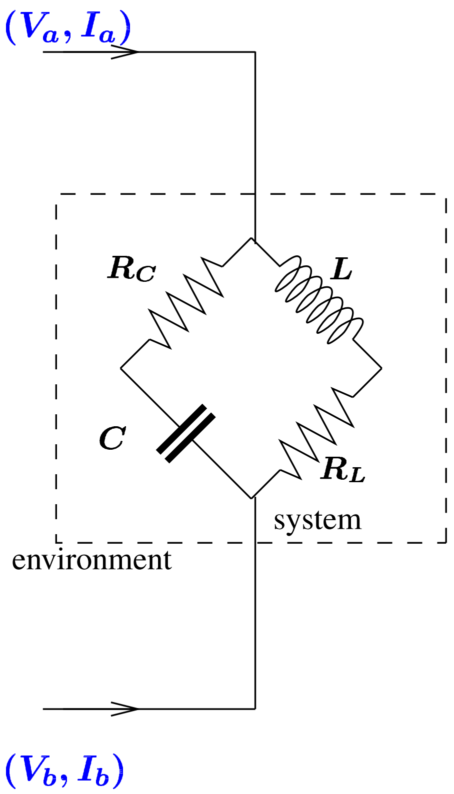
  the full behavior of the interconnected system

  contains latent variables and manifest variables.

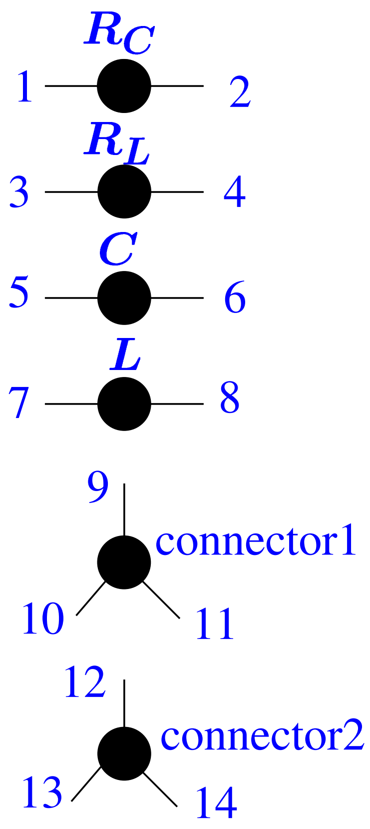
- Elimination of latent variables $\rightarrow$ the manifest behavior $\mathfrak{B}$.

EXAMPLES

RLC circuit



TEARING



ZOOMING

The list of the modules & the associated terminals:

Module	Type	Terminals	Parameter
R_C	resistor	(1, 2)	R in ohms
R_L	resistor	(3, 4)	R in ohms
C	capacitor	(5, 6)	C in farad
L	inductor	(7, 8)	L in henry
connector1	3-terminal connector	(9, 10, 11)	
connector2	3-terminal connector	(12, 13, 14)	

The interconnection architecture:

Pairing
$\{10, 1\}$
$\{11, 7\}$
$\{2, 5\}$
$\{8, 3\}$
$\{6, 13\}$
$\{4, 14\}$

Manifest variable assignment:

the variables on the external terminals $\{9, 12\}$.

The internal terminals are $\{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14\}$.

The variables on these terminals are latent variables.

Equations for the full behavior:

Modules	Constitutive equations	
R_C	$I_1 + I_2 = 0$	$V_1 - V_2 = R_C I_1$
R_L	$I_7 + I_8 = 0$	$V_7 - V_8 = R_L I_7$
C	$I_5 + I_6 = 0$	$C \frac{d}{dt}(V_5 - V_6) = I_5$
L	$I_7 + I_8 = 0$	$V_7 - V_8 = L \frac{d}{dt} I_7$
connector1	$I_9 + I_{10} + I_{11} = 0$	$V_9 = V_{10} = V_{11}$
connector2	$I_{12} + I_{13} + I_{14} = 0$	$V_{12} = V_{13} = V_{14}$

Interconnection pair	Interconnection equations	
$\{10, 1\}$	$V_{10} = V_1$	$I_{10} + I_1 = 0$
$\{11, 7\}$	$V_{11} = V_7$	$I_{11} + I_7 = 0$
$\{2, 5\}$	$V_2 = V_5$	$I_2 + I_5 = 0$
$\{8, 3\}$	$V_8 = V_3$	$I_8 + I_3 = 0$
$\{6, 13\}$	$V_6 = V_{13}$	$I_6 + I_{13} = 0$
$\{4, 14\}$	$V_4 = V_{14}$	$I_4 + I_{14} = 0$

These define a latent variable system in the manifest variables

$$w = (V_9, I_9, V_{12}, I_{12})$$

with latent variables

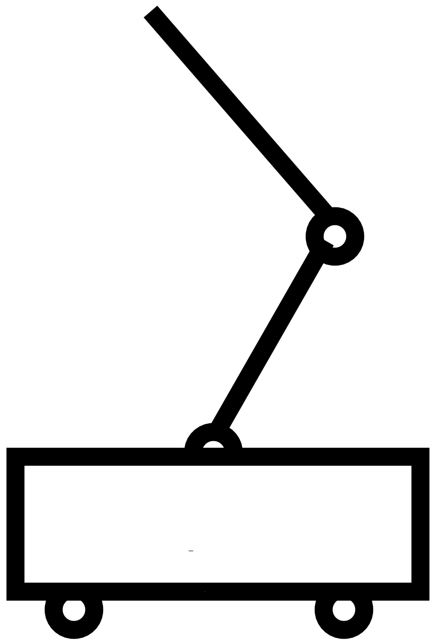
$$\ell = (V_1, I_1, V_2, I_2, V_3, I_3, V_4, I_4, V_5, I_5, V_6, I_6, V_7, I_7, V_8, I_8, V_{10}, I_{10}, V_{11}, I_{11}, V_{13}, I_{13}, V_{14}, I_{14}).$$

The manifest behavior $\mathfrak{B}$ is given by

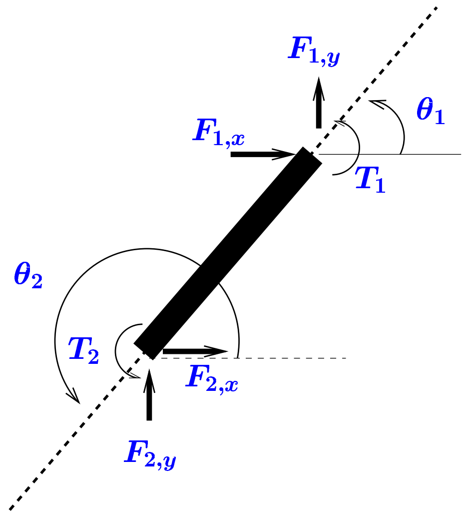
$$\mathfrak{B} = \{(V_9, I_9, V_{12}, I_{12}) : \mathbb{R} \rightarrow \mathbb{R}^4 \mid \exists\ \ell : \mathbb{R} \rightarrow \mathbb{R}^{24} \ldots\}$$

<u>Elimination</u>: for example, using Gröbner bases.

CART with DOUBLE PENDULUM



Required modules: Solid bars, servo's.



Solid bar

Terminals: 2 mechanical 2-D terminals.

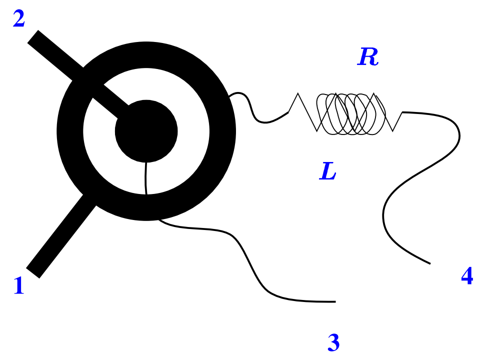
Parameters: $L \in \mathbb{R}_+$ (length),

$m \in \mathbb{R}_+$ (mass per unit length).

Behavioral equations:

$$mL\frac{d^2}{dt^2}x_c = F_{x_1} + F_{x_2},$$
$$mL\frac{d^2}{dt^2}y_c = F_{y_1} + F_{y_2} - mLg,$$
$$m\frac{L^3}{12}\frac{d^2}{dt^2}\theta_c = T_1 + T_2 - \frac{L}{2}F_{x_1}\sin(\theta_1) + \frac{L}{2}F_{y_1}\cos(\theta_1) - \frac{L}{2}F_{x_2}\sin(\theta_2) + \frac{L}{2}F_{y_2}\cos(\theta_2),$$
$$\theta_1 = \theta_c,$$
$$\theta_2 = \theta_1 + \pi,$$
$$x_1 = x_c + \frac{L}{2}\cos(\theta_c),$$
$$x_2 = x_c - \frac{L}{2}\cos(\theta_c),$$
$$y_1 = y_c + \frac{L}{2}\sin(\theta_c),$$
$$y_2 = y_c - \frac{L}{2}\sin(\theta_c).$$

Note: Contains latent variables.



Hinge with servo

Terminals: **2 mechanical 2-D terminals, 2 electrical.**

Parameters: **rotor mass m_r, the stator mass m_s, the rotor inertia J_r, the stator inertia J_s, the inductance L, the resistance R of the motor circuit, the motor torque constant K.**

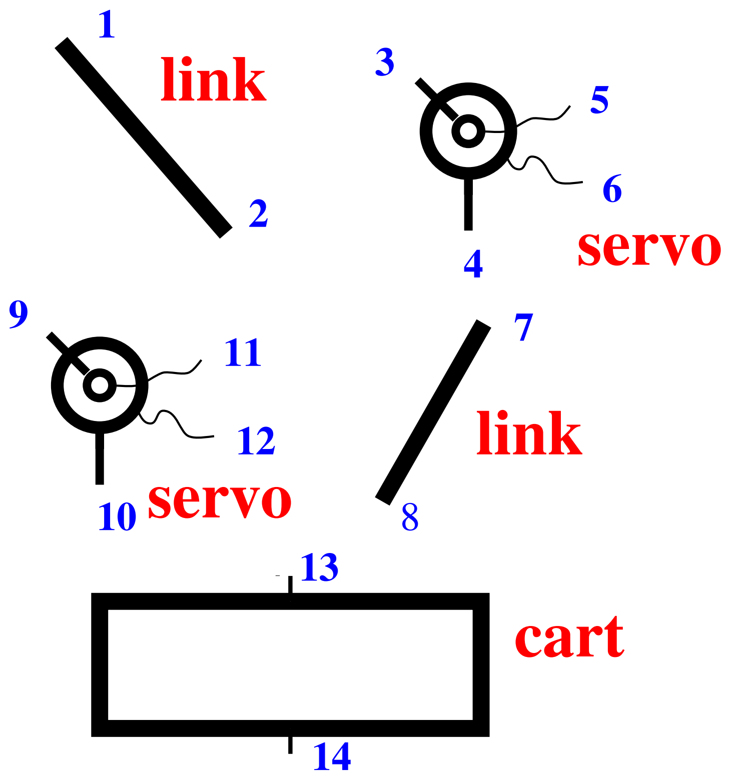
Behavioral equations:

$$(m_r + m_s)\frac{d^2}{dt^2}x_1 = F_{x_1} + F_{x_2}$$
$$(m_r + m_s)\frac{d^2}{dt^2}y_1 = F_{x_1} + F_{x_2}$$
$$J_r\frac{d^2}{dt^2}\theta_1 = T_1 + T_m$$
$$J_s\frac{d^2}{dt^2}\theta_2 = T_2 - T_m$$
$$V_3 - V_4 = L\frac{d}{dt}I_3 + RI_3 + K\frac{d}{dt}(\theta_1 - \theta_2)$$
$$KI_3 = T_m$$
$$x_1 = x_2$$
$$y_1 = y_2$$
$$I_3 = -I_4$$

Terminal variables: $(x_1, y_1, \theta_1, F_{x_1}, F_{y_1}, T_1,$
$x_2, y_2, \theta_2, F_{x_2}, F_{y_2}, T_2, V_3, I_4, V_4, I_4).$

The motor torque T_m is a latent variable.

TEARING



ZOOMING

The list of the modules & the associated terminals:

Module	Type	Terminals	Parameter
Link 1	bar	(7,8)	L_1, m_1
Link 2	bar	(1,2)	L_2, m_2
Support	bar	(13,14)	L_3, m_3
Hinge 1	servo	(9,10,11,12)	$m_{r_1}, m_{s_1}, J_{r_1}, J_{r_1}, L_1, R_1, K_1$
Hinge 2	servo	(3,4,5,6)	$m_{r_2}, m_{s_2}, J_{r_2}, J_{r_2}, L_2, R_2, K_2$

The interconnection architecture:

Pairing
$\{2, 3\}$
$\{4, 7\}$
$\{8, 9\}$
$\{10, 13\}$

Manifest variable assignment:

the variables on the external terminals $\{1, 5, 6, 11, 12, 14\}$.

All other variables are latent variables.

Equations for the full behavior:

$$
\begin{aligned}
m_1 L_1 \tfrac{d^2}{dt^2} x_{c_1} &= F_{x_1} + F_{x_2}, \\
m_1 L_1 \tfrac{d^2}{dt^2} y_{c_1} &= F_{y_1} + F_{y_2} - m_1 L_1 g, \\
m_1 \tfrac{L_1^3}{12} \tfrac{d^2}{dt^2} \theta_{c_1} &= T_1 + T_2 - \\
&\quad \tfrac{L_1}{2} F_{x_1} \sin(\theta_1) + \tfrac{L_1}{2} F_{y_1} \cos(\theta_1) - \tfrac{L_1}{2} F_{x_2} \sin(\theta_2) + \tfrac{L_1}{2} F_{y_2} \cos(\theta_2),
\end{aligned}
$$

$$
\begin{aligned}
&\theta_1 = \theta_{c_1}, \\
&\theta_2 = \theta_1 + \pi, \\
&x_1 = x_{c_1} + \tfrac{L_1}{2} \cos(\theta_{c_1}), \\
&x_2 = x_{c_1} - \tfrac{L_1}{2} \cos(\theta_{c_1}), \\
&y_1 = y_{c_1} + \tfrac{L_1}{2} \sin(\theta_{c_1}), \\
&y_2 = y_{c_1} - \tfrac{L_1}{2} \sin(\theta_{c_1}), \\
&m_2 L_2 \tfrac{d^2}{dt^2} x_{c_2} = F_{x_7} + F_{x_8}, \\
&m_2 L_2 \tfrac{d^2}{dt^2} y_{c_2} = F_{y_7} + F_{y_8} - m_2 L_2 g, \\
&m_2 \tfrac{L_2^3}{12} \tfrac{d^2}{dt^2} \theta_{c_2} = T_7 + T_8 - \tfrac{L_2}{2} F_{x_7} \sin(\theta_7) + \tfrac{L_2}{2} F_{y_7} \cos(\theta_7), \\
&\qquad - \tfrac{L_2}{2} F_{x_8} \sin(\theta_8) + \tfrac{L_2}{2} F_{y_8} \cos(\theta_8), \\
&\theta_7 = \theta_{c_2}, \\
&\theta_8 = \theta_7 + \pi,
\end{aligned}
$$

$$
\begin{aligned}
& x_7 = x_{c_2} + \frac{L_1}{2}\cos(\theta_{c_2}), \\
& x_8 = x_{c_2} - \frac{L_1}{2}\cos(\theta_{c_2}), \\
& y_7 = y_{c_2} + \frac{L_1}{2}\sin(\theta_{c_2}), \\
& y_8 = y_{c_2} - \frac{L_1}{2}\sin(\theta_{c_2}), \\
& m_3 L_3 \frac{d^2}{dt^2} x_{c_3} = F_{x_{13}} + F_{x_{14}}, \\
& m_3 L_3 \frac{d^2}{dt^2} y_{c_3} = F_{y_{13}} + F_{y_{14}} - m_3 L_3 g, \\
& m_3 \frac{L_3^3}{12} \frac{d^2}{dt^2}\theta_{c_3} = T_{13} + T_{14} - \frac{L_3}{2} F_{x_{13}} \sin(\theta_{13}) + \frac{L_3}{2} F_{y_{13}} \cos(\theta_{13}) - \\
& \qquad \frac{L_3}{2} F_{x_{14}} \sin(\theta_{14}) + \frac{L_3}{2} F_{y_{14}} \cos(\theta_{14}), \\
& \theta_{13} = \theta_{c_3}, \\
& \theta_{14} = \theta_{c_3} + \pi, \\
& x_{13} = x_{c_3} + \frac{L_1}{2}\cos(\theta_{c_3}), \\
& x_{14} = x_{c_3} - \frac{L_1}{2}\cos(\theta_{c_3}), \\
& y_{13} = y_{c_3} + \frac{L_1}{2}\sin(\theta_{c_3}), \\
& y_{14} = y_{c_3} - \frac{L_1}{2}\sin(\theta_{c_3}), \\
& (m_{r_1} + m_{s_1}) \frac{d^2}{dt^2} x_3 = F_{x_3} + F_{x_4}, \\
& (m_{r_1} + m_{s_1}) \frac{d^2}{dt^2} y_3 = F_{y_3} + F_{y_4}, \\
& J_{r_1} \frac{d^2}{dt^2}\theta_3 = T_3 + T_m, \\
& J_{s_1} \frac{d^2}{dt^2}\theta_4 = T_4 - T_m, \\
& V_5 - V_6 = L_1 \frac{d}{dt} I_5 + R_1 I_5 + K \frac{d}{dt}(\theta_3 - \theta_4),
\end{aligned}
$$

$$K_1 I_5 = T_{m_1},$$
$$x_3 = x_4,$$
$$y_3 = y_4,$$
$$I_5 = -I_6,$$
$$(m_{r_2} + m_{s_2})\frac{d^2}{dt^2}x_9 = F_{x_9} + F_{x_{10}},$$
$$(m_{r_2} + m_{s_2})\frac{d^2}{dt^2}y_9 = F_{y_9} + F_{y_{10}},$$
$$J_{r_2}\frac{d^2}{dt^2}\theta_9 = T_9 + T_m,$$
$$J_{s_2}\frac{d^2}{dt^2}\theta_{10} = T_{10} - T_m,$$
$$V_{11} - V_{12} = L_2\frac{d}{dt}I_{11} + R_2 I_{11} + K\frac{d}{dt}(\theta_9 - \theta_{10}),$$
$$K_2 I_{11} = T_{m_2},$$
$$x_{10} = x_{11}, y_{10} = y_{11},$$
$$I_{11} = -I_{12},$$
$$F_{x_2} + F_{x_3} = 0,\ F_{y_2} + F_{y_3} = 0,\ x_2 = x_3,\ y_2 = y_3,\ \theta_2 = \theta_3 + \pi,\ T_2 + T_3 = 0,$$
$$F_{x_4} + F_{x_7} = 0,\ F_{y_4} + F_{y_7} = 0,\ x_4 = x_7,\ y_4 = y_7,\ \theta_4 = \theta_7 + \pi,\ T_4 + T_7 = 0,$$
$$F_{x_8} + F_{x_9} = 0,\ F_{y_8} + F_{y_9} = 0,\ x_8 = x_9,\ y_8 = y_9,\ \theta_8 = \theta_9 + \pi,\ T_8 + T_9 = 0,$$
$$F_{x_{10}} + F_{x_{13}} = 0,\ F_{x_{10}} + F_{x_{13}} = 0, x_{10} = x_{13},\ y_{10} = y_{13},$$
$$\theta_{10} = \theta_{13} + \pi,\ T_{10} + T_{13} = 0.$$

LINEAR RLCT CIRCUITS

BUILDING BLOCKS

Module Types:

Resistors, Capacitors, Inductors, Transformers, Connectors.

All terminals are of the same type: **electrical**

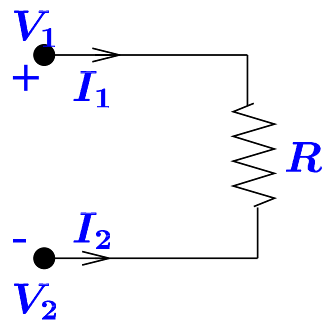
There are 2 variables associated with each terminal, (V, I),

V the *potential*,

I the *current* (counted positive when it flows *into* the module).

$\rightsquigarrow$ terminal signal space $\mathbb{R}^2$.

SPECIFICATION of the BEHAVIOR of the MODULES

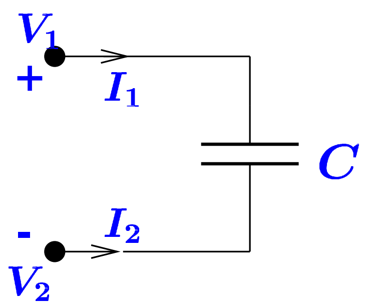


Resistor: **2-terminal module.**

Parameter: R (resistance in ohms, say).

Device laws:

$$V_1 - V_2 = R\, I_1 \; ; \quad I_1 + I_2 = 0.$$

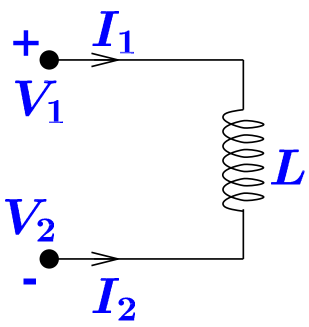


Capacitor: **2-terminal module.**

Parameter: C (capacitance in farads, say).

Device laws:

$$C\,\frac{d}{dt}(V_1 - V_2) = I_1\,; \quad I_1 + I_2 = 0.$$

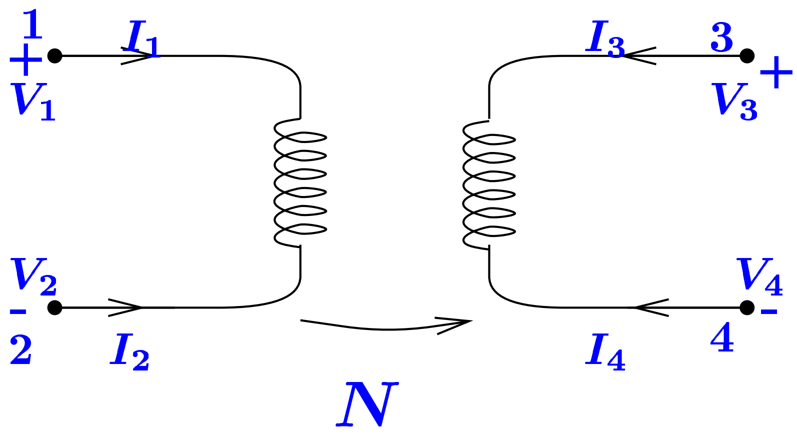


Inductor: **2-terminal module.**

Parameter: L (inductance in henrys, say).

Device laws:

$$L\,\frac{d}{dt}I_1 = V_1 - V_2\,; \quad I_1 + I_2 = 0.$$



<u>Transformer</u>: **4-terminal module; terminals (1,2): primary;**

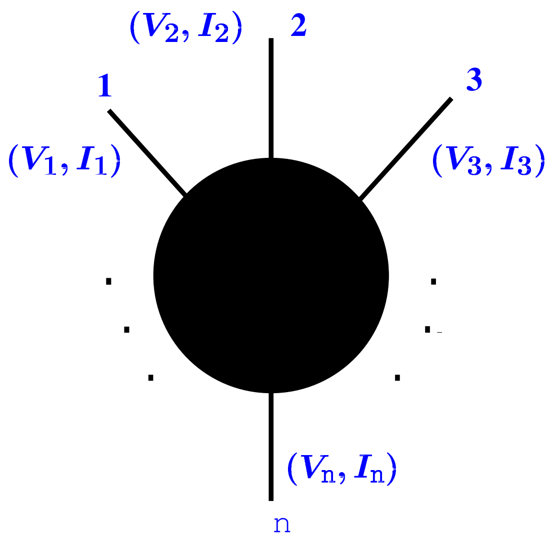
terminals (3,4): secondary.

Parameter: N **(the turns ratio,** $\in (0, \infty)$ **).**

Device laws:

$$V_3 - V_4 = N(V_1 - V_2)\;; \qquad I_1 = -N I_3\;;$$

$$I_1 + I_2 = 0\;; \qquad I_3 + I_4 = 0.$$



Connector: **many-terminal module.**

Parameter: n **(number of terminals, an integer).**

Device laws:

$$V_1 = V_2 = \cdots = V_n \; ; \quad I_1 + I_2 + \cdots + I_n = 0.$$

MODULES and TERMINAL ASSIGNMENT

Modules Resistors $r_1, r_2, \ldots, r_{n_r}$, parameters $R_1, R_2, \ldots, R_{n_r}$;

Capacitors $c_1, c_2, \ldots, c_{n_c}$, parameters $C_1, C_2, \ldots, C_{n_c}$;

Inductors $\ell_1, \ell_2, \ldots, \ell_{n_\ell}$, parameters $L_1, L_2, \ldots, L_{n_\ell}$;

Transformers $T_1, T_2, \ldots, T_{n_T}$, parameters $N_1, N_2, \ldots, N_{n_T}$;

Connectors $k_1, k_2, \ldots, k_{n_k}$, parameters $n_1, n_2, \ldots, n_{n_k}$.

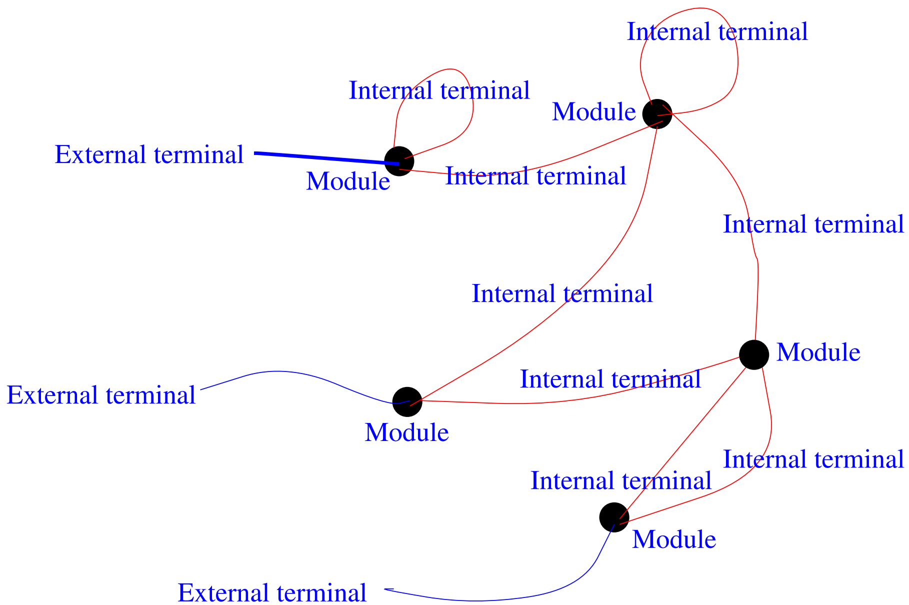
This yields the set of **terminals**

$$\mathbb{T} = \{1, 2, \ldots, 2(n_r + n_c + n_\ell) + 4n_T + n_1 + n_2 + \ldots + n_{n_k}\}.$$

INTERCONNECTION ARCHITECTURE

<u>Interconnection architecture</u> :

$\mathbb{I}$ $=$ a set of disjoint (unordered) pairs of different elements (i.e., doubletons) from $\mathbb{T}$.



MANIFEST VARIABLE ASSIGNMENT

External terminals = $\mathbb{E} := \mathbb{T} - \cup_{\mathbb{I}} \{a, b\}$.

Manifest variables = external terminal voltages and currents

$= \Pi_{k \in \mathbb{I}} (V_k, I_k)$. Denote the manifest variables by $\Pi_{k \in \mathbb{I}} (V_k, I_k)$ as $(V, I) \in \mathbb{R}^{2\mathbb{E}}$.

Manifest behavior: $\mathfrak{B}_{\mathbb{E}} \subseteq (\mathbb{R}^{2\mathbb{E}})^{\mathbb{R}}$.

Denote further the full behavior (the behavior of all the terminal voltages and currents) by $\mathfrak{B}_{\mathbb{T}} \subseteq (\mathbb{R}^{2\mathbb{T}})^{\mathbb{T}}$.

FULL BEHAVIORAL EQUATIONS

1. **Module Laws:**

1.1 Resistors: for each resistor $r_{\mathtt{n}}$, terminals $(t_1^{r_{\mathtt{n}}}, t_2^{r_{\mathtt{n}}})$,

$$\boxed{V_{t_1^{r_{\mathtt{n}}}} - V_{t_2^{r_{\mathtt{n}}}} = R_{\mathtt{n}}\, I_{t_1^{r_{\mathtt{n}}}}\ ; \quad I_{t_1^{r_{\mathtt{n}}}} + I_{t_2^{r_{\mathtt{n}}}} = 0,}$$

for $\mathtt{n} = 1, \ldots \mathtt{n}_r$.

1.2 Capacitors: for each capacitor c_{n}, terminals $(t_1^{c_{\mathrm{n}}}, t_2^{c_{\mathrm{n}}})$,

$$\boxed{\frac{d}{dt} C_{\mathrm{n}} \left(V_{t_1^{c_{\mathrm{n}}}} - V_{t_2^{c_{\mathrm{n}}}} \right) = I_{t_1^{c_{\mathrm{n}}}} \; ; \quad I_{t_1^{c_{\mathrm{n}}}} + I_{t_2^{c_{\mathrm{n}}}} = 0,}$$

for $\mathrm{n} = 1, \ldots \mathrm{n}_c$.

1.3 Inductors: for each inductor ℓ_{n}, terminals $(t_1^{\ell_{\mathrm{n}}}, t_2^{\ell_{\mathrm{n}}})$,

$$\boxed{\frac{d}{dt} L_{\mathrm{n}} \, I_{t_1^{\ell_{\mathrm{n}}}} - V_{t_2^{\ell_{\mathrm{n}}}} \; ; \quad I_{t_1^{\ell_{\mathrm{n}}}} + I_{t_2^{\ell_{\mathrm{n}}}} = 0,}$$

for $\mathrm{n} = 1, \ldots \mathrm{n}_\ell$.

1.4 Transformers: for each transformer $T_{\mathtt{n}}$,

terminals $(t_1^{T_{\mathtt{n}}}, t_2^{T_{\mathtt{n}}}, t_3^{T_{\mathtt{n}}}, t_4^{T_{\mathtt{n}}})$,

$$\boxed{\begin{array}{rr} V_{t_1^{T_{\mathtt{n}}}} - V_{t_2^{T_{\mathtt{n}}}} = N_{\mathtt{n}}(V_{t_3^{T_{\mathtt{n}}}} - V_{t_4^{T_{\mathtt{n}}}})\,; & I_{t_3^{T_{\mathtt{n}}}} = -N_{\mathtt{n}} I_{t_1^{T_{\mathtt{n}}}} \\ I_{t_1^{T_{\mathtt{n}}}} + I_{t_2^{T_{\mathtt{n}}}} = 0\,; & I_{t_3^{T_{\mathtt{n}}}} + I_{t_4^{T_{\mathtt{n}}}} = 0 \end{array}}$$

for $\mathtt{n} = 1, \ldots \mathtt{n}_T$.

1.5 Connectors: for each connector $k_{\mathtt{n}}$, terminals $(t_1^{k_{\mathtt{n}}}, \ldots, t_{\mathtt{n}_{k_{\mathtt{n}}}}^{k_{\mathtt{n}}})$,

$$\boxed{V_{t_1^{k_{\mathtt{n}}}} = \cdots = V_{t_{\mathtt{n}_{k_{\mathtt{n}}}}^{k_{\mathtt{n}}}}\,; \quad I_{t_1^{k_{\mathtt{n}}}} + \cdots + I_{t_{\mathtt{n}_{k_{\mathtt{n}}}}^{k_{\mathtt{n}}}}}$$

for $\mathtt{n} = 1, \ldots, \mathtt{n}_k$

2. Interconnection Laws:

For each 'connected' terminal pair $\{a, b\} \in \mathbb{I}$:

$$\boxed{V_a = V_b; \quad I_a + I_b = 0.}$$

Solution of behavioral equations $\rightsquigarrow$ $\mathfrak{B}_{\mathbb{T}}$.

After elimination of internal variables $\rightsquigarrow$ $\mathfrak{B}_{\mathbb{E}}$.

PROPERTIES of $\mathfrak{B}_{\mathbb{E}}$

When is $\mathfrak{B}_{\mathbb{E}} \subseteq (\mathbb{R}^{2\mathbb{E}})^{\mathbb{R}}$

the external terminal behavior of a circuit

containing a finite number of positive

R's, L's, C'c, T's, and connectors?

It is possible to derive necessary & sufficient conditions!

1. $\boxed{\mathfrak{B}_{\mathbb{E}} \in \mathfrak{L}^{2\mathbb{E}}.}$

2. **KVL**:

$$\boxed{((V, I) \in \mathfrak{B}_{\mathbb{E}}) \text{ and } (\alpha \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R})) \Rightarrow ((V + \alpha e) \in \mathfrak{B}_{\mathbb{E}})}$$

with

$$e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

3. **KCL**:

$$\boxed{((V, I) \in \mathfrak{B}_{\mathbb{E}}) \Rightarrow (e^{\top} I = 0)}$$

4. **Input cardinality:**

$$\boxed{\mathtt{m}(\mathfrak{B}_{\mathbb{E}}) = \mathbb{E}}$$

5. **Hybridicity:**

There exists an input/output choice such that the input variables $(u_1, u_2, \ldots, u_{\mathbb{E}})$ and output variables $(y_1, y_2, \cdots, y_{\mathbb{E}})$ pair as follows:

$$\boxed{\{u_{\mathtt{i}}, y_{\mathtt{i}}\} = \{V_{\mathtt{i}}, I_{\mathtt{i}}\}}$$

Each terminal is either current controlled or voltage controlled.

6. **Passivity:**

Assume that $\mathfrak{B}_{\mathbb{E}} \in \mathfrak{L}^{2\mathbb{E}}_{\text{cont}}$.

The uncontrollable case is an open pbm!

There holds

$$\boxed{\int_0^{+\infty} V^\top(t) I(t)\, dt \geq 0}$$

for all $(V, I) \in \mathfrak{B}_{\mathbb{E}}$ of compact support.

This states that the net electrical energy goes <u>into</u> the circuit.

7. **Reciprocity:**

Assume again for simplicity $\mathfrak{B}_{\mathbb{E}} \in \mathfrak{L}^{2\mathbb{E}}_{\text{cont}}$. There holds

$$\int_{-\infty}^{+\infty} V_1^\top(t) I_2(-t)\, dt = \int_{-\infty}^{+\infty} I_1^\top(t) V_2(-t)\, dt$$

for all $(V_1, I_1), (V_2, I_2) \in \mathfrak{B}_{\mathbb{E}}$ of compact support.

Equivalently: $\mathfrak{B}_{\mathbb{E}} = \text{rev}(\mathfrak{B}_{\mathbb{E}}^{\perp_\Sigma})$,

where rev denotes time-reversal, and $\Sigma = \begin{bmatrix} O & I \\ -I & O \end{bmatrix}$.

This curious properties may be translated into:

The influence of terminal `i` on terminal `j` is equal

to the influence of terminal `j` on terminal `i`.

Proof of necessity:

Show that the modules satisfy properties (1) to (7).

Show that these properties remain valid after one additional interconnection.

The difficult part here is (4).

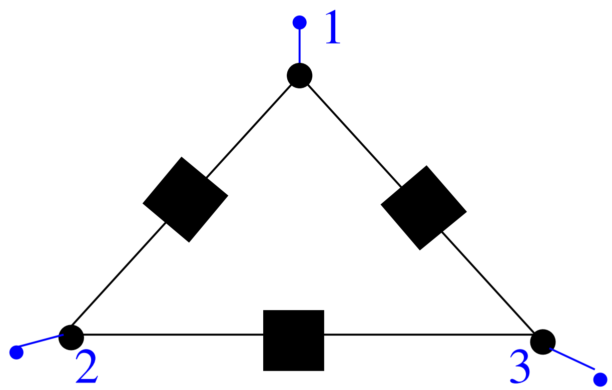
Proof of sufficiency:

‘Synthesis’.

TERMINALS versus PORTS

Note that we have used throughout the terminal description of circuits. It is simply more appropriate and more general (even when using only 'port' devices).

Example:



However, it is more parsimomious in the choice of variables (it halves their number). It is of interest to incorporate this feature.

RECAP

- **Modelling interconnected systems $\cong$ Interplay of**

  modules

  terminals

  interconnection laws

  interconnection architecture

  manifest variable assignment

- **Adapted to computer assisted modeling**
- **hierarchical, reusable, extendable**
- **Many latent variables, many equations (many static relations, i.e., algebraic equations). Far distance from i/o, i/s/o, tf. fns., etc.**

  Stresses the importance of elimination algorithms.

- **Paradigmatic example: RLCT circuits.**

  N.a.s.c. on the terminal behavior:

  1. **linear, time-invariant, differential**
  2. **KVL**
  3. **KCL**
  4. **input cardinality = number of terminals**
  5. **hybridicity**
  6. **passivity**
  7. **reciprocity**

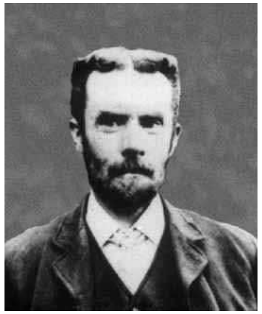
- **Terminal description in circuits is more general than port description.**

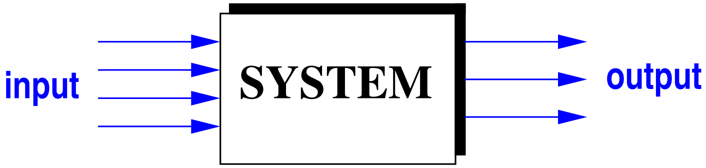
DISCUSSION

INPUT/OUTPUT THINKING

Early 20-th century: emergence of the notion of a transfer function

(Rayleigh, Heaviside).





Since the 1920's: routinely used in circuit theory

(Foster, Brune, Cederbaum, $\cdots$)

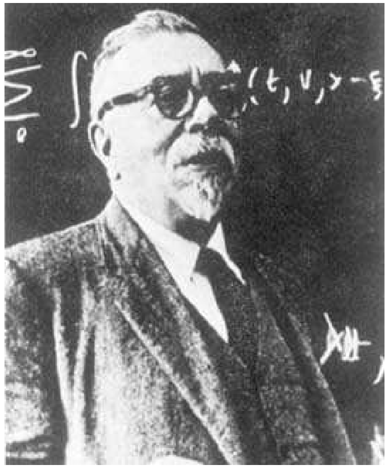
$\leadsto$ impedances, admittances, scattering matrices, etc.

1930's: control embraces transfer functions

(Nyquist, Bode, $\cdots$) $\leadsto$ plots and diagrams, classical control.

Around 1950: Wiener sanctifies the notion of a blackbox,

attempts nonlinear generalization (via Volterra series).



<u>1960's</u>: Kalman's state space ideas (incl. controllability, observability, recursive filtering, state models and representations) come in vogue

$\rightsquigarrow$ **input/state/output systems, and the ubiquitous**

$$\boxed{\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du,}$$

or its nonlinear counterpart

$$\frac{d}{dt}x = f(x,u), \quad y = h(x,u).$$

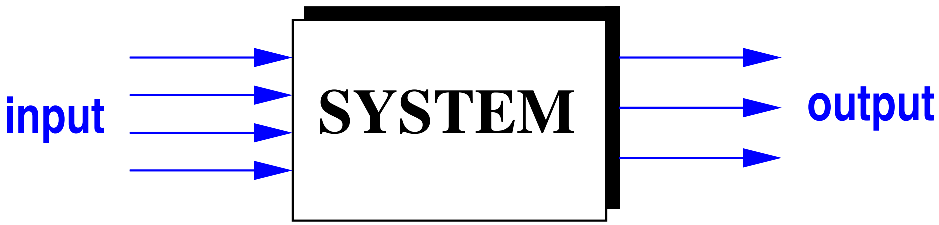
These mathematical structures, transfer functions, + their discrete-time analogs, are nowadays the basic models used in control and signal processing (cfr. MATLAB©).

SIMULINK©

sees interconnected systems as input-to-output connections (only):

very limited!

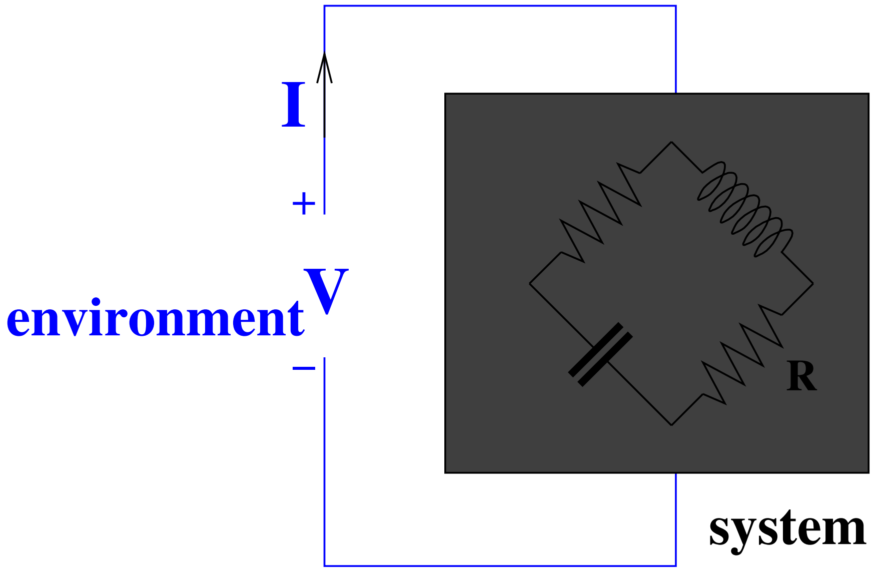
All these theories: input/output; **cause $\Rightarrow$ effect.**



What’s wrong with input/output thinking?

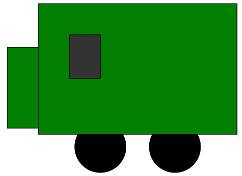
Let’s look at examples:

Our electrical circuit.



Is V the input? Or I ? Or both, or are they both outputs?

An automobile:



External terminals:

wind, tires, steering wheel, gas/brake pedal.

What are the inputs?

at the wind terminal: the force,

at the tire terminals: the forces, or, more likely, the positions?

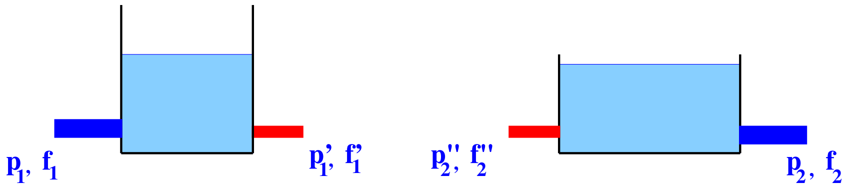
at the steering wheel: the torque or the angle?

at the gas-pedal, or the brake-pedal: the force or the position?

Difficulty: at each terminal there are many (typically paired) interconnection variables

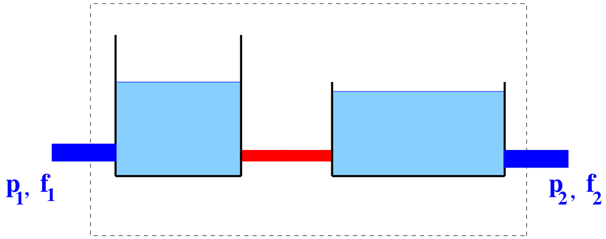
Input/output is awkward in modeling interconnections.

Consider a two-tank example.



Reasonable input choices: the pressures, output choices: the flows.

Assume that we model the interconnection of two tanks.



Interconnection: $p'_1 = p''_2, \quad f'_1 + f''_2 = 0$

input=input; output=output! $\Rightarrow\Leftarrow$ **SIMULINK©**

Interconnections contradicting SIMULINK© are in fact normal, not exceptions, in mechanics, fluidics, heat transfer, etc.

Mathematical difficulties:

Is a system a map $u(\cdot) \mapsto y(\cdot)$?

How to incorporate 'initial conditions'?

Is it a parametrized map $(u(\cdot), \alpha) \mapsto y(\cdot)$?

All sorts of new difficulties...

Construct the state!

But from what?

From the system model!

What system?

Conclusions $*$ **for physical systems (⇒⇐ signal processors)** $*$

- **External variables are basic, but what 'drives' what, is not.**
- **It is impossible to make an a priori, fixed, input/output selection for off-the-shelf modeling.**
- **What can be the input, and what can be the output should be deduced from a dynamical model. Therefore, we need a more general notion of 'system', of 'dynamical model'.**

Interconnection, variable sharing,

rather that input selection,

is the basic mechanism by which a system interacts with its environment.

BONDGRAPHS

Views interconnected systems indeed in terms of

ports, modules, and interconnections.

It is assumed that for each of the terminals the interconnection variables come in pairs:

an effort variable and a flow variable

their (inner) product must be power.

Examples:

- **Electrical ports: effort: voltage, flow: current**
- **Mechanical ports: effort: force, flow: velocity**
- **Thermal ports: effort: T, flow: Q/T**
- **etc. etc.**

- **Bondgraphs ideas very good, brilliant**
- **certainly superior to SIMULINK©**
- **notation very awkward, mathematical notions primitive**
- **terminal variable structure seems limited to linearity**
- **some interconnections fail their assumptions:**
  mechanical terminals equate positions, NOT velocities
- **effort/flow, while apparently deep, remains unexplored**
- **interconnections happen via terminals, not ports.**
- **if anything, there is more structure to interconnection variables than effort/flow**

End of Lecture 5