MATHEMATICAL MODELS of SYSTEMS

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IUAP Graduate Course **Fall 2002**

Lecture 4

INPUTS and OUTPUTS

THE TRANSFER FUNCTION

THEME

When is a system variable an input? An output?

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Inputs = free, outputs = follow from inputs + intitial conditions.

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OUTLINE

- **Free and bound variables, inputs and outputs: formal def'ns.**
- **Every linear time-invariant differential system admits a I/O partition**
- **The transfer function**
- **Left and right co-prime factorizations, relations with controllability**
- **Time-domain characterization**

FORMAL DEFINITIONS

Intuition

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These concepts (input, output) are strongly domain dependent.
We will discuss them following the usual systems & control setting.

Central is, of course, that the input must in some way *causes* the output.

- In physical systems and in real-time signal processing and control, non-anticipation must be an important feature.
- In non-real-time signal processing problems, or when the independent variable is not time, non-anticipation need not be an issue.
- In many problems (e.g. computing, signal processing) inputs may have to be structured, in order for machines or algorithms to be able to accept them.
- In control, it is customary to assume that inputs are free, and that outputs are bound (determined by the inputs and the initial conditions). We will follow this tradition.

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$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be *memoryless* if

$$[w_1, w_2 \in \mathfrak{B}] \wedge [t \in \mathbb{T}\,] \Rightarrow [w_1 \underset{t}{\wedge} w_2 \in \mathfrak{B}],$$

where $\underset{t}{\wedge}$ denotes *concatenation* at time t, defined by

$$(w_1 \underset{t}{\wedge} w_2)(t') := \begin{cases} w_1(t') & t' < t \\ w_2(t') & t' \geq t \end{cases}$$

Memoryless:= the past and the future are unrelated

(except by the system laws).

$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ **is said to be *trim* if**

$$\forall \mathbf{w} \in \mathbb{W} \text{ and } \forall\, t \in \mathbb{T}\ \ \exists w \in \mathfrak{B} : w(t) = \mathbf{w}.$$

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<u>Note</u>: trim + memoryless ≅ 'free'

(modulo niceties as measurability, integrability, ...)

Recall $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$ is said to be *autonomous* if

$$[w_1, w_2 \in \mathfrak{B}] \wedge [t \in \mathbb{T}] \wedge [w_1(t') = w_2(t') \ \ \forall t' < t] \Rightarrow [w_1 = w_2].$$

Autonomous:= the past implies the future.

Let $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$, $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, be a dynamical system. Then w_1 is said to be *input/output system* with w_1 the *input* and w_2 the *output* if

1. **$\Sigma_1 := (\mathbb{T}, \mathbb{W}_1, \mathfrak{B}_1)$ is free := trim and memoryless,**
   where $\mathfrak{B}_1$ denotes the w_1 behavior
   (i.e., the manifest behavior with w_2 viewed as a latent variable).
2. **for all $w_1 \in \mathfrak{B}_1$ $\Sigma_2^{w_1} := (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B}_2^{w_1})$ is autonomous,**
   where $\mathfrak{B}_2^{w_1}$ denotes the w_2 behavior with fixed w_1,
   i.e., $\mathfrak{B}_2^{w_1} := \{w_2 \mid (w_1, w_2) \in \mathfrak{B}\}$.

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$$\textbf{free} := \mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) = \text{‘}\mathfrak{C}^\infty\text{-free’}.$$

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In keeping with tradition

$$w_1 \to u; \quad w_2 \to y; \quad \mathbb{W}_1 \to \mathbb{U}, \quad \mathbb{W}_2 \to \mathbb{Y}.$$

For linear systems:

$\mathbb{U} = \mathbb{R}^{\mathtt{m}}$ (**$\mathtt{m}$ input variables),** $\quad \mathbb{Y} = \mathbb{R}^{\mathtt{p}}$ (**$\mathtt{p}$ output variables).**

Let $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}$, **with** $\mathbb{R}^{\mathtt{w}} = \mathbb{R}^{\mathtt{m}} \times \mathbb{R}^{\mathtt{p}}, \mathtt{w} = \mathtt{m} + \mathtt{p}$.

If the corresponding $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{m}} \times \mathbb{R}^{\mathtt{p}}, \mathfrak{B})$ **is an input/output system, then we call** $w = (u, y)$ **an** **input/output partition** **of** w.

Proposition: Consider the linear differential system with kernel repr.

$$P(\frac{d}{dt})u = Q(\frac{d}{dt})y, w = (u, y).$$

u is $\mathfrak{C}^\infty$-free $\Leftrightarrow$ $\mathrm{rank}([P \;\; Q]) = \mathrm{rank}(P)$,

y is bound by u $\Leftrightarrow$ P is of full column rank, i.e. $\mathrm{rank}(P) = \dim(y)$.

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Call $\boxed{G := P^{-1}Q}$ $\in \mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{m}}$ its transfer function.

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If one can choose the basis, even with G strictly proper.

We will recall the def's of proper, strictly proper, later.

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1. For a given $\mathfrak{B} \in \mathfrak{L}^\bullet$, which variables are input variables, and which are input variables, is not fixed.

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Our RLC circuit. Since here the t'f f'n is bi-proper, it may be viewed as

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etc., etc.

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Notation: Define the 3 maps $\mathtt{w}, \mathtt{m}, \mathtt{p} : \mathfrak{L}^\bullet \to \mathbb{Z}_+$ **by**

$$\mathtt{w}(\Sigma) = \mathtt{w}(\mathfrak{B}) \quad := \quad \text{the number of variables of } \Sigma = (\mathbb{R}, \mathbb{R}^\bullet, \mathfrak{B}) \in \mathfrak{L}^\bullet$$

$$\mathtt{m}(\Sigma) = \mathtt{m}(\mathfrak{B}) \quad := \quad \text{the number of input variables of } \Sigma = (\mathbb{R}, \mathbb{R}^\bullet, \mathfrak{B}) \in \mathfrak{L}^\bullet$$

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When Σ has the kernel representation $R(\frac{d}{dt})w = 0$, we hence have

$$\mathtt{w}(\Sigma) = \text{coldim}(R), \mathtt{m}(\Sigma) = \text{coldim}(R) - \text{rank}(R), \mathtt{p}(\Sigma) = \text{rank}(P)$$

In particular, $\mathtt{m} + \mathtt{p} = \mathtt{w}$.

RATIONAL FUNCTIONS

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What is a rational function?

The field of rationals is an important mathematical structure that is constructed from a commutative ring R without zero divisors $ab = 0 \Rightarrow a = 0$ or $b = 0$ and with an identity $\exists 1 \in R : 1 * a = a \forall a \in R$. Examples: $\mathbb{Z}$, $\mathbb{R}[\xi]$, $\mathbb{C}[\xi]$.

The arch-typical example is $R = \mathbb{Z}$, but we are mainly interested in $R = \mathbb{R}[\xi]$. The 'rationals' over $\mathbb{R}[\xi]$ are called rational 'functions', denoted $\mathbb{R}(\xi)$.

$\mathbb{R}(\xi)$ **is constructed as follows. Consider**

$$S = \{(a, b) \in \mathbb{R}[\xi] \times \mathbb{R}[\xi] \mid b \neq 0\}.$$

Define an equivalence relation on S **by**

$$[(a', b') \sim (a", b")] :\Leftrightarrow [a'b" = a"b'].$$

Now verify that the set of equivalence classes, $S(\text{mod} \sim)$ **becomes a field under the following definitions of addition and multiplication:**

$$\begin{aligned} (a', b')(\text{mod} \sim) + (a", b")(\text{mod} \sim) &:= (a'b" + a"b', b'b")(\text{mod} \sim); \\ (a', b')(\text{mod} \sim) * (a", b")(\text{mod} \sim) &:= (a'a", b'b")(\text{mod} \sim). \end{aligned}$$

This field is $\mathbb{R}(\xi)$**.**

Think intuitively of $\mathbb{R}(\xi)$ as the ratios of two polynomials, $\dfrac{a(\xi)}{b(\xi)}$,

with common factors in a and b

disregarded or cancelled,

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Of course, the $*$ is usually not written.

Notation for vectors and matrices of rational functions:

$\mathbb{R}(\xi)^{\mathtt{n}}, \mathbb{R}(\xi)^{\bullet}, \mathbb{R}(\xi)^{\mathtt{n}_1 \times \mathtt{n}_2}, \mathbb{R}(\xi)^{\bullet \times \mathtt{n}}, \mathbb{R}(\xi)^{\mathtt{n} \times \bullet}, \mathbb{R}(\xi)^{\bullet \times \bullet}.$

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A rational function $\frac{a}{b} \in \mathbb{R}(\xi)$ **is said to be**

proper **if** $\text{degree}(a) \leq \text{degree}(b)$,

and *strictly proper* **if** $\text{degree}(a) < \text{degree}(b)$.

$\leadsto$ **vectors, matrices of (strictly) proper rational functions.**

We are mainly interested in real rational functions, but (complex) rational f'ns, $\mathbb{C}(\xi)$, are analogously defined. Any element of $\mathbb{R}(\xi)$ is in a natural way an element of $\mathbb{C}(\xi)$.

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> **a zero of $\frac{a}{b}$ (assume no common factor) if it is a root of a**
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Call $\lambda \in \mathbb{C}$ a pole of a vector or matrix of rational functions if it is a pole of one of the elements.

We <u>do not</u> define zeros or multiplicities in the matrix case.
These are important, but application sensitive.

We consider ξ again as an indeterminate.

We can substitute for ξ real numbers, complex numbers, etc. (square matrices are OK for $\mathbb{R}(\xi)$, but cause problems in the matrix case. Do not mindlessly substitute the differentiation or the shift operator: all kinds of problems!)

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$t \in \mathbb{R}$, not a pole of $F \Rightarrow F(t) \in \mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_2}$.

Hence, there is an induced map $F : \{t \in \mathbb{R} \mid$ not a pole of $F\} \rightarrow \mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_2}$.

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$s \in \mathbb{C}$, not a pole of $F \Rightarrow F(s) \in \mathbb{C}^{n_1 \times n_2}$.

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PRIME POLYNOMIAL MATRICES

$P \in \mathbb{R}^{\mathtt{n_1} \times \mathtt{n_2}}[\xi]$ is said to be *left prime* if $P = P_1 P_2$, with $P_1 \in \mathbb{R}^{\mathtt{n_1} \times \mathtt{n_1}}[\xi]$, $P_2 \in \mathbb{R}^{\mathtt{n_1} \times \mathtt{n_2}}[\xi]$ implies that P_1 must be unimodular.

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Proposition: $P \in \mathbb{R}^{n_1 \times n_2}[\xi]$ is right prime iff $P(\lambda) \in \mathbb{C}^{n_1 \times n_2}$ is of full row rank for all $\lambda \in \mathbb{C}$.

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Every $P \in \mathbb{R}^{n_1 \times n_2}[\xi]$ that is of full row rank (as a polynomial matrix, of course) admits a factorization $P = P_1 P_2$, with $P_1 \in \mathbb{R}^{n_1 \times n_1}[\xi]$, and $P_2 \in \mathbb{R}^{n_1 \times n_2}[\xi]$ left prime.

This factorization is 'essentially unique' (Explain!).

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This factorization is 'essentially unique' (Explain!).

Right prime and right factorization: analogous.

Call $P_1, P_2, \ldots, P_n \in \mathbb{R}^{n_1 \times \bullet}[\xi]$ ***left co-prime*** **if the composite polynomial matrix** $[P_1 P_2 \cdots P_n] \in \mathbb{R}^{n_1 \times \bullet}[\xi]$ **is left prime.**

Call $P_1, P_2, \ldots, P_n \in \mathbb{R}^{\bullet \times n_2}[\xi]$ ***right co-prime*** **if the composite polynomial matrix** $\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} \in \mathbb{R}^{\bullet \times n_2}[\xi]$ **is right prime.**

FACTORIZATION of MATRICES of RATIONAL F'Ns

Consider a matrix of rational f'ns $F \in \mathbb{R}^{n_1 \times n_2}(\xi)$.

A factorization of F as $F = P^{-1}Q$ with $P \in \mathbb{R}^{n_1 \times n_1}[\xi], \det(P) \neq 0$ and $Q \in \mathbb{R}^{n_1 \times n_2}[\xi]$ is said to be *left co-prime factorization* of F

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Consider a matrix of rational f'ns $F \in \mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_2}(\xi)$.

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A factorization of F as $F = ND^{-1}$ with $N \in \mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_2}[\xi]$ and $D \in \mathbb{R}^{\mathtt{n}_2 \times \mathtt{n}_2}[\xi], \det(D) \neq 0$ is said to be *right co-prime factorization* of F

if N and D are right co-prime.

THE TRANSFER FUNCTION

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Theorem: Consider

$$P_1(\frac{d}{dt})y = Q_1(\frac{d}{dt})u, \quad w = (u, y),$$

with $P_1 \in \mathbb{R}^{\mathtt{p}(\mathfrak{B}) \times \mathtt{p}(\mathfrak{B})}[\xi]$, $\det(P_1) \neq 0$, and transfer function $G_1 = P_1^{-1} Q_1$.

$$P_2(\frac{d}{dt})y = Q_2(\frac{d}{dt})u, \quad w = (u, y),$$

with $P_2 \in \mathbb{R}^{\mathtt{p}(\mathfrak{B}) \times \mathtt{p}(\mathfrak{B})}[\xi]$, $\det(P_2) \neq 0$, and transfer function $G_2 = P_2^{-1} Q_2$.

?? When do two systems have the same transfer function ??

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$G_1 = G_2$: same transfer function

iff these systems have the same controllable part.

Hence:

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1. **Two controllable systems with the same transfer function are equal.**

2. **The transfer function determines only the controllable part of a system.**

!!! Watch out in stability considerations !!

Relations primeness with differential systems

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iff R is left prime.

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Consider the system with minimal kernel representation $R(\frac{d}{dt})w = 0$.

Factor $R = FR'$, with $R' \in \mathbb{R}^{\mathtt{p}(\mathfrak{B})\times\mathtt{m}(\mathfrak{B})}[\xi]$, left prime, $F \in \mathbb{R}^{\mathtt{p}(\mathfrak{B})\times\mathtt{p}(\mathfrak{B})}[\xi]$.

Then $R'(\frac{d}{dt})w = 0$ determines the controllable part

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$w = M(\frac{d}{dt})\ell$ is an observable latent variable system iff M is right prime.

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A left co-prime factorization $G = P^{-1}Q \quad \rightsquigarrow$

a kernel repr. of the (unique!) controllable system with transfer f'n G.

A factorization $G = ND^{-1}$ $\rightsquigarrow$ an image representation

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} D(\frac{d}{dt}) \\ N(\frac{d}{dt}) \end{bmatrix}$$

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A right co-prime factorization $G = ND^{-1}$ $\rightsquigarrow$

an observable image repr. of the controllable system with transfer f'n G.

THE EXPONENTIAL RESPONSE

Let $\lambda \in \mathbb{C}$. Denote by $\exp_\lambda$ the ***exponential map*** **$t \in \mathbb{R} \mapsto e^{\lambda t} \in \mathbb{C}$.**

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Define, for each $\lambda \in \mathbb{C}$ the set

$$\mathfrak{E}_\lambda^{\mathfrak{B}} := \{a \in \mathbb{C}^{\mathtt{w}} \mid \exp_\lambda a \in \mathfrak{B}\}.$$

Easy: $\mathfrak{E}_\lambda^{\mathfrak{B}}$ is a linear subspace of $\mathbb{C}^{\mathtt{w}}$.

Define the ***exponential response*** **of $\mathfrak{B}$ as the set of all exponentials in $\mathfrak{B}$:**

$$\mathfrak{E}^{\mathfrak{B}} = \{\exp_\lambda a \mid a \in \mathfrak{E}_\lambda^{\mathfrak{B}}\} \subset \mathfrak{B}.$$

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For the system described by $R(\frac{d}{dt})w = 0$ we obviously have $\mathfrak{E}_\lambda^{\mathfrak{B}} = \ker(R(\lambda))$.

Proposition:

1. $\dim(\mathfrak{E}_{\lambda}^{\mathfrak{B}}) = \mathtt{m}(\mathfrak{B})$ **for all but a finite number of elements of** $\mathbb{C}$**.**
This dimension can be larger than $\mathtt{m}(\mathfrak{B})$ **at a finite number of points.**

2. $\dim(\mathfrak{E}_{\lambda}^{\mathfrak{B}}) =$ **constant** $(= \mathtt{m}(\mathfrak{B}))$ **iff** $\mathfrak{B}$ **is controllable.**

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2. $\dim(\mathfrak{E}_\lambda^{\mathfrak{B}}) =$ constant $(= \mathtt{m}(\mathfrak{B}))$ iff $\mathfrak{B}$ is controllable.

3. If $\mathfrak{B}$ is controllable, then the exponential response determines $\mathfrak{B}$ uniquely.

The exponential response is closely related to the transfer function. Consider

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u, w = (u, y),$$

with $P \in \mathbb{R}^{\mathtt{p}(\mathfrak{B}) \times \mathtt{p}(\mathfrak{B})}[\xi]$, $\det(P) \neq 0$, **and transfer function** $G = P^{-1}Q$.

For $\lambda \in \mathbb{C}$, **not a root of** $\det(P)$, **we have**

$$\mathfrak{E}_{\lambda}^{\mathfrak{B}} = \{(a, G(\lambda)a) \mid a \in \mathbb{C}^{\mathtt{m}(\mathfrak{B})}\}.$$

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By continuity, this determines an $\mathtt{m}(\mathfrak{B})$-dimensional subspace of $\mathbb{C}^{\mathtt{m}(\mathfrak{B})}$ also at the roots of $\det(P)$. $\rightsquigarrow$ the controllable part of the system.

Let $i\omega, \omega \in \mathbb{R}$. Define the subspace

$$\mathfrak{F}_\omega^{\mathfrak{B}} := \{a \in \mathbb{C}^{\mathtt{w}} \mid \exp_{i\omega} a \in \mathfrak{B}\}.$$

Define the *frequency response* of $\mathfrak{B}$ as the set

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The frequency response is the exponential response restricted to the imaginary axis.

Once again: the frequency response determines again the controllable part of a system.

Note: No stability considerations required for the exponential or the frequency response.

In particular, the frequency transfer $u \mapsto y$:

$$u = \exp_{i\omega} a \mapsto y = \exp_{i\omega} G(i\omega) a$$

is well-defined for all $i\omega$'s that are not roots of $\det(P)$.

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It is the notion of exponential response (more so than Laplace transform considerations) that is the origin of the transfer function.

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In continuous time: SMOOTHNESS

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The t'f f'n G is (strictly) proper iff

$(u, y) \in \mathfrak{B}$ **and** $u \in \mathfrak{C}^{\mathrm{k}}(\mathbb{R}, \mathbb{R}^{\mathrm{m}(\mathfrak{B})})$ **imply**

$$y \in \mathfrak{C}^{\mathrm{k}}(\mathbb{R}, \mathbb{R}^{\mathrm{p}(\mathfrak{B})}) \ (y \in \mathfrak{C}^{\mathrm{k}+1}(\mathbb{R}, \mathbb{R}^{\mathrm{p}(\mathfrak{B})})).$$

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<u>In discrete time</u>: NON-ANTICIPATION

The output (lags) does not anticipate that the input. cfr. the exercises.

TIME-DOMAIN CHARACTERIZATIONS

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Assume that an I/O partition $w = (u, y)$ has been made. $\rightsquigarrow$

$$y(t) = y_{\text{autonomous}}(t) + \Sigma_{\mathrm{k} \in \mathbb{Z}_+} H_{\mathrm{k}} \frac{d^{\mathrm{k}}}{dt^{\mathrm{k}}} u(t) + \int_0^t H(t - t') u(t') dt'$$

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with

1. $y_{\text{autonomous}} \in \mathfrak{B}_{\text{autonomous}} \in \mathfrak{L}^{\mathtt{p}(\mathfrak{B})}$, an **autonomous** system,
2. $H_{\mathtt{k}} \in \mathbb{R}^{\mathtt{p}(\mathfrak{B})\times\mathtt{m}(\mathfrak{B})}$, matrices, only a finite number $\neq 0$,
3. $H : \mathbb{R} \to \mathbb{R}^{\mathtt{p}(\mathfrak{B})\times\mathtt{m}(\mathfrak{B})}$ a matrix with each column $\in \mathfrak{B}_{\text{autonomous}}$.

Autonomous intermezzo

Fact:

Consider the autonomous systems, behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, kernel representation $P(\frac{d}{dt})w = 0, \det(P) \neq 0$.

There is a one-to-one relation

$$w \in \mathfrak{B} \overset{1:1}{\leftrightarrow} f \in \mathbb{R}^{\mathtt{w}}[\xi] : P^{-1}f \text{ strictly proper}$$

Relation with the transfer function

In terms of the minimal kernel representation

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u, w = (u, y),$$

and the transfer function $G = P^{-1}Q$, we have:

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$P(\frac{d}{dt})y = 0$ is a kernel representation of the autonomous system $\mathfrak{B}_{\text{autonomous}}$, and

$$G(\xi) = \Sigma_{\mathtt{k}\in\mathbb{Z}_+} H_{\mathtt{k}}\xi^{\mathtt{k}} + G'(\xi)$$

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There is a great deal more that can be said, for example, related to image representations, Laplace transforms, partial fraction expansion of the transfer function, but ...

RECAP

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No unique input/output partition.

In applications often no natural choice.

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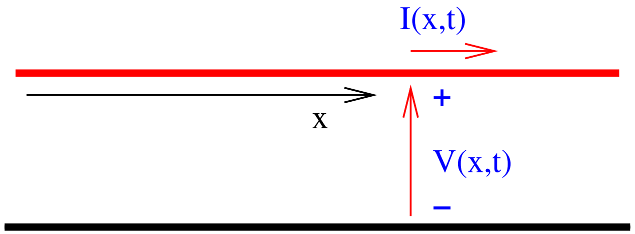
- **(Co-prime) factorizations of the transfer functions: a way of obtaining (controllable/observable) kernel and image representations from the transfer function.**

- **(Strict) properness of the transfer f'n $\cong$ Does the system smooth inputs?**

A final example: Coaxial cable



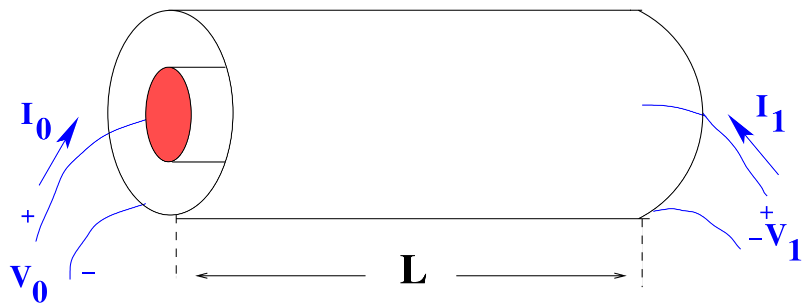
⇝ **The PDE's:**

$$\boxed{\begin{aligned} \frac{\partial}{\partial x} V &= -L_0 \frac{\partial}{\partial t} I, \\ \frac{\partial}{\partial x} I &= -C_0 \frac{\partial}{\partial t} V. \end{aligned}}$$

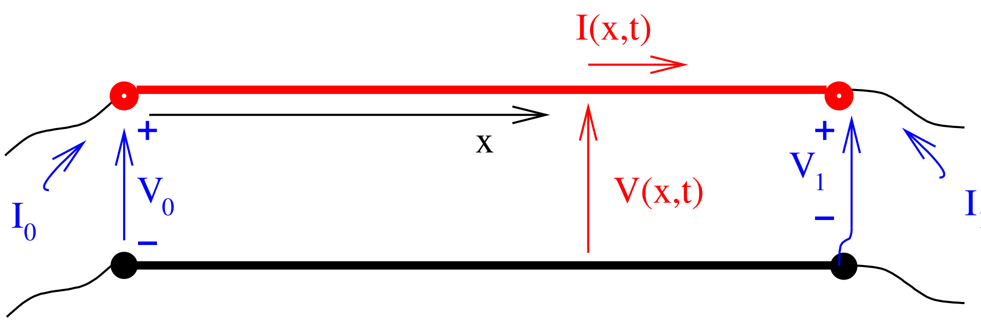
with L_0 **the inductance, and** C_0 **the capacitance per unit length.**

<u>With boundary conditions</u> (cable of length L):

!! Model the relation between the voltages V_0, V_1 and the currents I_0, I_1 at the ends of a uniform cable of length L.



Introduce the voltage $V(x,t)$ and the current flow $I(x,t)$ $0 \le x \le L$ in the cable.

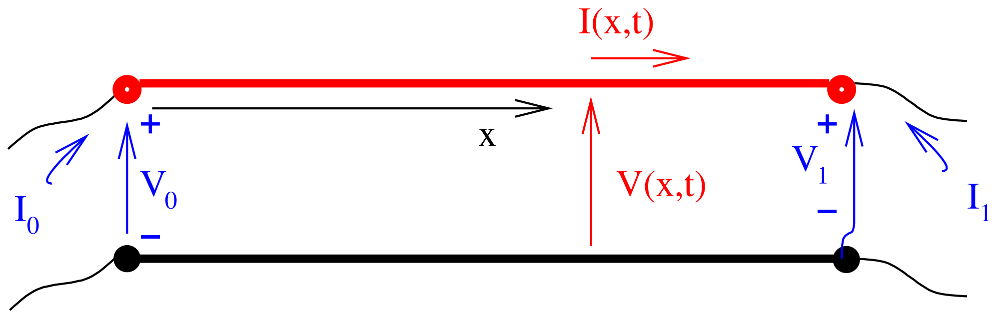


$\leadsto$ **The equations:**

$$\begin{aligned}
\frac{\partial}{\partial x} V &= -L_0 \frac{\partial}{\partial t} I, \\
\frac{\partial}{\partial x} I &= -C_0 \frac{\partial}{\partial t} V,
\end{aligned}$$

$$\begin{aligned}
V_0(t) &= V(0, t), \\
V_1(t) &= V(L, t), \\
I_0(t) &= I(0, t), \\
I_1(t) &= -I(L, t).
\end{aligned}$$

Viewed as a black box



Relation between V_0, V_1:

$$\frac{\partial^2}{\partial x^2} V = L_0 C_0 \frac{\partial^2}{\partial t^2} V, \;\; V_0(\cdot) = V(0,\cdot), \; V_1(\cdot) = V(L,\cdot),$$

and between I_0, I_1:

$$\frac{\partial^2}{\partial x^2} I = L_0 C_0 \frac{\partial^2}{\partial t^2} I, \;\; I_0(\cdot) = I(0,\cdot), \; I_1(\cdot) = I(L,\cdot).$$

Two terminal variables are 'free', the other two are 'bound',

(free = one voltage, one current, bound = one voltage, one current), but

there is no reasonable choice of inputs and outputs!

It breaks the symmetry

End of Lecture 4