# MATHEMATICAL MODELS of SYSTEMS

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# Lecture 3

## CONTROLLABILITY and OBSERVABILITY

# THEME

**Central notions in system theory,**

**controllability and observability,**

**in the setting and language of behavioral models.**

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**Central notions in system theory,**

**controllability and observability,**

**in the setting and language of behavioral models.**

- **Formal definitions**
- **Tests for controllability and observability**
- **Image representations**
- **Autonomous systems**

# CONTROLLABILITY

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The time-invariant system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be

**controllable**

if for all $w_1, w_2 \in \mathfrak{B}$ there exists $w \in \mathfrak{B}$ and $T \geq 0$ such that

$$w(t) = \begin{cases} w_1(t) & t < 0 \\ w_2(t-T) & t \geq T \end{cases}$$

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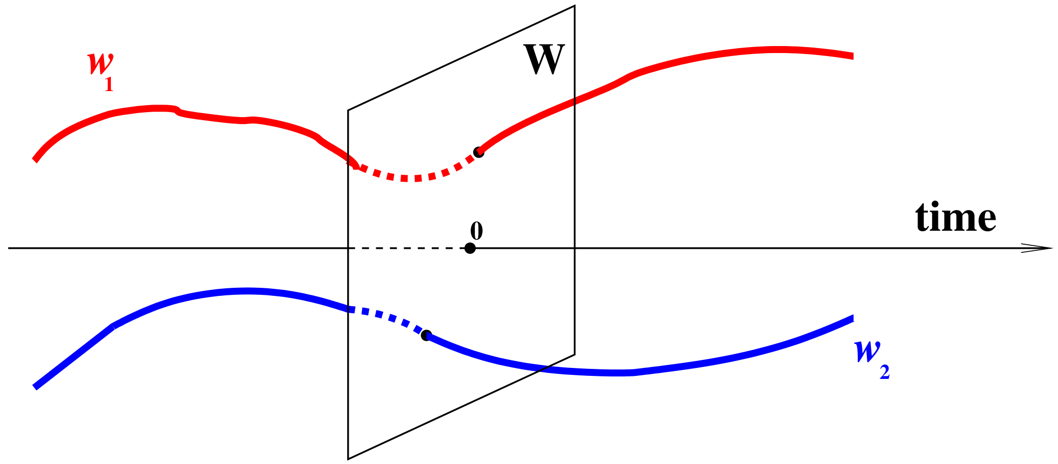
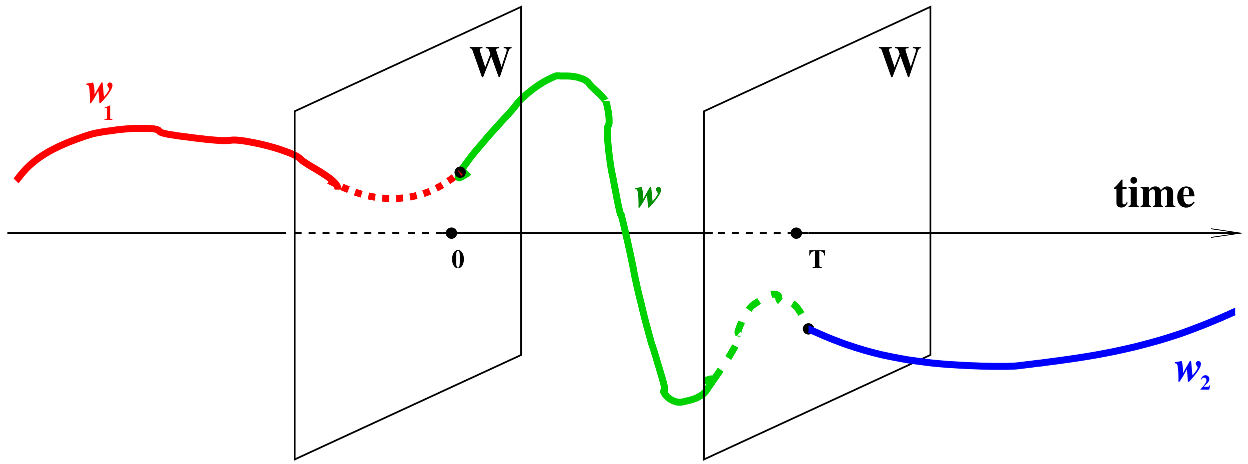
Controllability :$\Leftrightarrow$

legal trajectories must be 'patch-able', 'concatenable'.

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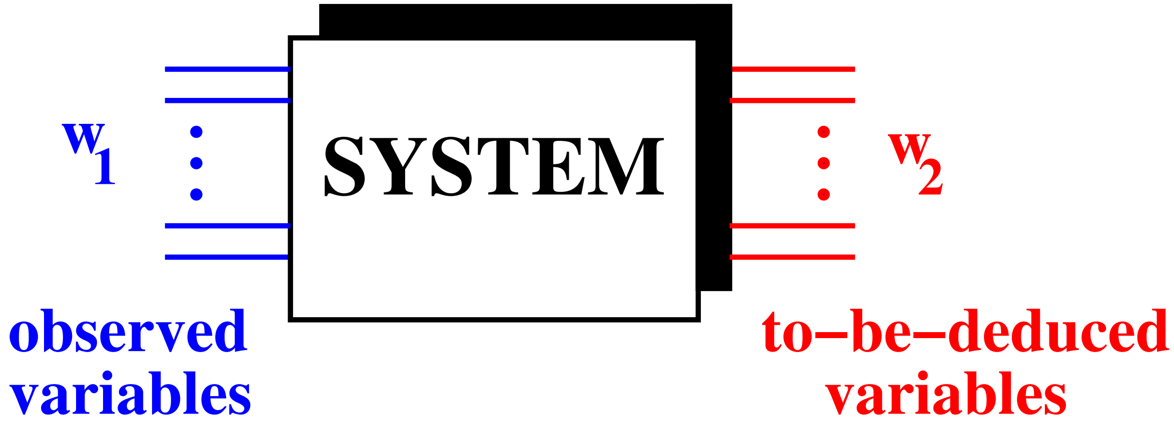
# OBSERVABILITY




# Observability

**Consider the system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$.**

**Each element of the behavior $\mathfrak{B}$ hence consists of**

**a pair of trajectories $(w_1, w_2)$.**

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**Definition: $w_2$ is said to be**

***observable* from $w_1$**

**if $((w_1, w_2') \in \mathfrak{B}$, and $(w_1, w_2'') \in \mathfrak{B}) \Rightarrow (w_2' = w_2'')$,**

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Very often manifest = observed, latent = to-be-deduced.

We then speak of an observable latent variable system.


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**If a system is not (state) controllable, why is it?**

**Insufficient influence of the control?**

**Or bad choice of the state?**


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Why is it so interesting to try to deduce the state, of all things?

The state is a derived notion, not a 'physical' one.


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**Kalman definitions address rather special situations.**

# TESTS for CONTROLLABILITY and OBSERVABILITY

**Consider the system defined by**

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**Under what conditions on $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ does it define a controllable system?**

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**<u>Theorem</u>:** $R(\frac{d}{dt})w = 0$ **defines a controllable system**

**if and only if**

**rank$(R(\lambda)) =$ constant**

**over $\lambda \in \mathbb{C}$.**

**Notes:**

**1. If $R(\frac{d}{dt})w = 0$ is minimal, then**

$$\text{controllability} \Leftrightarrow R(\lambda) \text{ is of full row rank } \forall \ \lambda \in \mathbb{C}.$$

**Equivalently, $R$ is right-invertible as a polynomial matrix.**

$P \in \mathbb{R}^{n_1 \times n_2}[\xi]$ is *right-invertible* $:\Leftrightarrow \exists \ Q \in \mathbb{R}^{n_2 \times n_1}[\xi]$ such that $PQ = I_{n_1}$

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2. $\frac{d}{dt}x = Ax + Bu, w = (x, u)$ is controllable iff

$$\text{rank}([A - \lambda I \quad B]) = \dim(x)\ \forall\, \lambda \in \mathbb{C}$$

*Hautus' test* for controllability. Of course,

$$\Leftrightarrow \text{rank}([B \ \ AB \ \ \cdots \ \ A^{\dim(x)-1}B]) = \dim(x).$$

**3. When is**

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**controllable?** $p, q \in \mathbb{R}[\xi]$**, not both zero.**

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**4. Example: Our electrical circuit is controllable unless**

$$CR_C = \frac{L}{R_L} \text{ and } R_C = R_L.$$

**Reasonable physical systems can be uncontrollable.**

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**same conditions as on $R$....**

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if and only if

$\text{rank}(R_2(\lambda)) = \dim(w_2)$

for all $\lambda \in \mathbb{C}$.

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**1. In $R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2$, $w_2$ is observable from $w_1$ if and only if $R_2(\lambda)$ is of full column rank $\forall\ \lambda \in \mathbb{C}$.**

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Equivalently, iff $\exists$ an equivalent behavioral equation representation

$$\begin{aligned} R(\frac{d}{dt})w_1 &= 0 \\ w_2 &= M(\frac{d}{dt})w_1 \end{aligned}$$

This representation puts observability into evidence.

**2. In $\frac{d}{dt}x = Ax + Bu, y = Cx, w_1 = (u, y), w_2 = x$ the state is observable from the input/output $(u, y)$ iff**

$$\text{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = \dim(x) \ \forall \lambda \in \mathbb{C}$$

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**Iff $q$ is a non-zero constant. No zeros!**

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***observable*** **(as a system!)**

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**Conditions, e.g. $\exists$ equivalent repr. of full behavior**

$$R(\frac{d}{dt})w = 0 \qquad \ell = R'(\frac{d}{dt})w$$

**$R(\frac{d}{dt})w = 0$ hence specifies the manifest behavior.**

**We can therefore speak of a controllable & observable state system.**

**5. The RLC circuit is observable iff**

$$CR_C \neq \frac{L}{R_L}$$

# Image representations

**Representations of $\mathfrak{L}^\bullet$:**

$$\boxed{R(\tfrac{d}{dt})w = 0}$$

**called a *'kernel' representation* of $\mathfrak{B} = \ker(R(\frac{d}{dt}))$.**

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$$\boxed{R(\tfrac{d}{dt})w = M(\tfrac{d}{dt})\ell}$$

**called a *'latent variable' representation* of the manifest behavior**

$\mathfrak{B} = (R(\frac{d}{dt}))^{-1} M(\frac{d}{dt}) \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\ell)$.

**Elimination theorem $\Rightarrow \; \in \mathfrak{L}^\bullet$.**

**Missing link:**

$$w = M(\tfrac{d}{dt})\ell$$

**called an** ***'image' representation*** **of** $\mathfrak{B} = \mathrm{im}(M(\tfrac{d}{dt}))$**.**

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**¿¿ Which kernels are also images ??**

**Controllability!**

**Theorem**: **(Controllability and image repr.)**:

**The following are equivalent for $\mathfrak{B} \in \mathfrak{L}^\bullet$ :**

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**The following are equivalent for $\mathfrak{B} \in \mathfrak{L}^\bullet$ :**

1. $\mathfrak{B}$ is **controllable,**
2. $\boxed{\mathfrak{B} \text{ admits an image representation,}}$
3. for any $a \in \mathbb{R}^{\mathtt{w}}[\xi]$,

   $a^\top[\frac{d}{dt}]\mathfrak{B}$ equals 0 or all of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$,
4. $\mathbb{R}^{\mathtt{w}}[\xi]/\mathfrak{N}_{\mathfrak{B}}$ is **torsion free,**
5. etc.

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**Controllability $\Leftrightarrow < R^{\top} > = < R' >$ .**

**$\Rightarrow$ Numerical test for contr. on coefficients of $R$.**

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**3. $\exists$ similar results for time-varying systems.**

**4. $\exists$ partial results for nonlinear systems.**

# AUTONOMOUS SYSTEMS

**The time-invariant system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be *autonomous* if**

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**i.e. when the past implies the future.**

## Examples:

- **Kepler's laws**
- $\frac{d^n}{dt^n} w = f(\frac{d^{n-1}}{dt^{n-1}} w, \ldots, w)$, **reasonable** $f$
- $\frac{d}{dt} x = f(x), w = h(x)$, **reasonable** $f, h$**?**
- **Discrete-time counterparts**
- **Most (deterministic) models studied in mathematics, physics, (not engineering)**

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5. $\mathfrak{B}$ has a latent variable repr. $\frac{d}{dt}x = Ax, w = Cx$.

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**Each such $\mathfrak{B}$ is parametrized by**

$\mathtt{m} \in \mathbb{N}$

$\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_{\mathtt{m}} \in \mathbb{C}$, **all distinct**

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and

$$\mathfrak{B} = \{w : \mathbb{R} \to \mathbb{C} \mid \exists\, r_1, r_2, \ldots, r_{\mathtt{k}} \text{ with } \mathrm{degree}(r_{\mathtt{k}}) < \mathtt{n}_{\mathtt{k}}$$
$$\text{such that } w(t) = \Sigma_{\mathtt{k}=1,\ldots,\mathtt{m}} r_{\mathtt{k}}(t) e^{\lambda_{\mathtt{k}} t}\}$$

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**Exercise: Do the real case. Get sum of products of polynomials, exponentials, and trigonometric functions.**

**In the multivariable autonomous case, all trajectories are still vectors of sums of products of polynomial/exponentials/(trigonometric) functions, but more structure on the coefficients of the polynomials (more that just the degree).**

**Theorem**: $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ admits a direct sum decomposition:

$$\boxed{\mathfrak{B} = \mathfrak{B}_{\text{controllable}} \oplus \mathfrak{B}_{\text{autonomous}},}$$

with $\mathfrak{B}_{\text{controllable}} \in \mathfrak{L}^{\mathtt{w}}$ controllable,

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**Theorem**: $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ admits a direct sum decomposition:

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We now define the controllable part and isomorphic.

## The controllable part

**There are a number of equivalent definitions of the controllable part of a behavior.**

**Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Define**

$$\mathfrak{B}_{\text{controllable part}} := \{w \in \mathfrak{B} \mid \exists\, w' \in \mathfrak{B} \text{ such that } w'(t) = w(t) \text{ for } t \geq 0 \text{ and } \exists\, t_0 \in \mathbb{R} : w'(t) = 0 \text{ for } t < t_0\}$$

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$\mathfrak{B}_{\text{controllable part}}$ **is also the largest controllable behavior $\in \mathfrak{L}^{\mathtt{w}}$ contained in $\mathfrak{B}$.**

## Isomorphic systems

$\mathfrak{B}, \mathfrak{B}' \in \mathfrak{L}^{\mathtt{w}}$ are said to be *isomorphic* if $\exists$ a unimodular $U \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ such that

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Differential bijection between behaviors.

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Controllable systems are isomorphic iff $\text{rank}(R) = \text{rank}(R')$:

isomorphy is very weak relation.

For autonomous systems: isomorphy is a very strong relation.

**Whence there is a very tight relationship between $\mathfrak{B}'_{\text{autonomous}}$ and $\mathfrak{B}''_{\text{autonomous}}$ in two different controllable/autonomous decompositions**

$$\mathfrak{B} = \mathfrak{B}_{\text{controllable}} \oplus \mathfrak{B}'_{\text{autonomous}},$$

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## STABILITY

**The autonomous $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be *asymptotically stable* :⇔**

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**and *stable* :$\Leftrightarrow$**

$$w \in \mathfrak{B} \Rightarrow w|_{\mathbb{R}_+} \text{ is bounded.}$$

**The system defined by $R(\frac{d}{dt})w = 0$ is asymptotically stable iff**

**$\mathrm{rank}(R(\lambda)) = \mathtt{w}$ for $\lambda \in \mathbb{C}^+ := \{\lambda \in \mathbb{C} \mid \mathrm{Real}(\lambda) \geq 0\}$.**

**All singularities of $R$ ($\lambda$'s where $R(\lambda)$ drops rank) in closed left half of the complex plane.**

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The system defined by $R(\frac{d}{dt})w = 0$ with $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is stable iff

1. $R(\lambda) = \mathtt{w}$ for $\lambda \in \mathbb{C}^{++} := \{\lambda \in \mathbb{C} \mid \text{Real}(\lambda) > 0\}$, and
2. $\mathtt{w} - \text{rank}(R(\lambda))$ = the multiplicity of $\lambda$ as a root of $\det(R)$ for $\lambda \in i\mathbb{C} := \{\lambda \in \mathbb{C} \mid \text{Real}(\lambda) = 0\}$.

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More about stability (Routh-Hurwitz, Lyapunov): later in the course.

## STABILIZABILITY

**The time-invariant system $\Sigma = (\mathbb{T}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ is said to be**

**stabilizable**

**if for all $w \in \mathfrak{B}$ there exists $w' \in \mathfrak{B}$ such that $w(t) = w'(t)$ for $t < 0$ and $w'(t) \underset{t \to \infty}{\longrightarrow} 0$.**

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**Stabilizability :$\Leftrightarrow$**

**legal trajectories can be steered to a desired point (0).**

**Consider the system defined by**

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**Equivalently, iff ‘the autonomous part’ is stable**

# RECAP

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- **Stabilizability** := all sol'ns can be steered to 0
- These central concepts in control take a much more intrinsic meaning in the context of behavioral systems

**End of Lecture 3**