MATHEMATICAL MODELS of SYSTEMS

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Lecture 2

LINEAR DIFFERENTIAL SYSTEMS

THEME

Description, notation, and main mathematical structure of dynamical systems described by linear constant coefficient differential equations.

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Description, notation, and main mathematical structure of dynamical systems described by linear constant coefficient differential equations.

- **Formal definitions, notation**
- **Polynomial matrices**
- **3 theorems:**
  1. **$\exists$ one-to-one relation between linear differential systems and polynomial modules**
  2. **Structure of kernel representations**
  3. **Elimination theorem**

GENERAL PROPERTIES

of

DYNAMICAL SYSTEMS

LINEARITY

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be

linear

if $\mathbb{W}$ is a vector space (over a field $\mathbb{F}$),

and $\mathfrak{B}$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$

(viewed as a vector space over $\mathbb{F}$ with respect to pointwise addition and pointwise multiplication).

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Hence linearity $:\Leftrightarrow$ the *superposition principle* holds:

$$((w_1, w_2 \in \mathfrak{B}) \wedge (\alpha, \beta \in \mathbb{F})) \Rightarrow (\alpha w_1 + \beta w_2 \in \mathfrak{B}).$$

TIME-INVARIANCE

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ (assume $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$)

is said to be

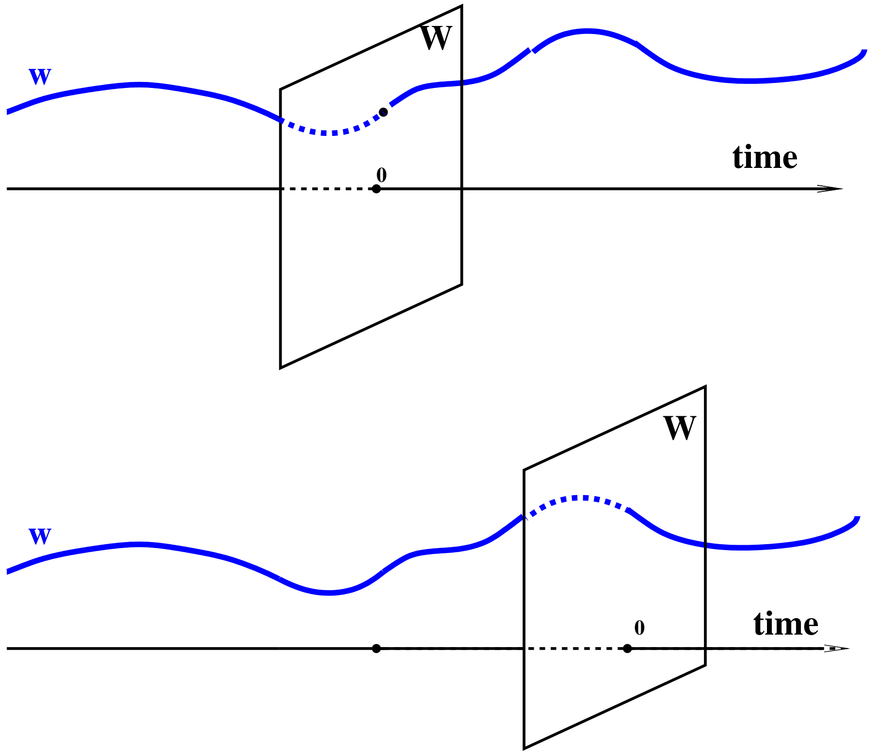
time-invariant

if

$$((w \in \mathfrak{B}) \wedge (t \in \mathbb{T})) \Rightarrow (\sigma^t w \in \mathfrak{B})),$$

where σ^t denotes the *backwards $t-$shift*, defined by

$$\sigma^t w(t') := w(t + t').$$



Time-invariance

DIFFERENTIAL SYSTEMS

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ (assume say $\mathbb{T} = \mathbb{R}$ and $\mathbb{W} = \mathbb{R}^{\mathrm{n}}$) is said to be a

differential system

if its behavior $\mathfrak{B}$ consists of the solutions of a system of differential equations,

$$f(w(t), \frac{d}{dt}w(t), \frac{d^2}{dt^2}w(t), \ldots, \frac{d^{\mathrm{n}}}{dt^{\mathrm{n}}}w(t), t) = 0.$$

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It must be made clear what it means that $w : \mathbb{R} \to \mathbb{R}^n$ satisfies a differential equation. We glance over it.

EXTENSIONS

- The notions of linearity, time-invariance, and differential system can in an obvious way be generalized to systems with latent variables. The latter leads to behavioral equations of the form

$$f\left(w(t), \ldots, \frac{d^{\mathrm{n}}}{dt^{\mathrm{n}}} w(t), \ell(t), \ldots, \frac{d^{\mathrm{n}}}{dt^{\mathrm{n}}} \ell(t), t\right) = 0.$$

- Easy to see: latent variable system linear / time-invariant

  $\Rightarrow$ same for manifest system.

- More difficult, and a most interesting question:

  Latent variable system differential $\Rightarrow$?

  manifest behavior described by differential equations?

We discuss the fundamentals of the theory of dynamical systems

$$\boxed{\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})}$$

that are

1. ***linear,*** **meaning**

$$((w_1, w_2 \in \mathfrak{B}) \wedge (\alpha, \beta \in \mathbb{R})) \Rightarrow (\alpha w_1 + \beta w_2 \in \mathfrak{B});$$

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where σ^t denotes the backwards $t-$shift;

3. ***differential,*** **meaning**

$\mathfrak{B}$ consists of the solutions of a system of differential equations.

NOTATION

LINEAR CONSTANT COEFFICIENT DIFFERENTIAL EQN'S.

Variables: $w_1, w_2, \ldots w_{\mathtt{w}}$, **up to** $\mathtt{n}$**-times differentiated,** $\mathtt{g}$ **equations.**

$$
\begin{array}{rcl}
\Sigma_{j=1}^{\mathtt{w}} R_{1,j}^{0} w_j + \Sigma_{j=1}^{\mathtt{w}} R_{1,j}^{1} \frac{d}{dt} w_j + \cdots + \Sigma_{j=1}^{\mathtt{w}} R_{1,j}^{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w_j & = & 0 \\
\Sigma_{j=1}^{\mathtt{w}} R_{2,j}^{0} w_j + \Sigma_{j=1}^{\mathtt{w}} R_{2,j}^{1} \frac{d}{dt} w_j + \cdots + \Sigma_{j=1}^{\mathtt{w}} R_{2,j}^{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w_j & = & 0 \\
\vdots \qquad\qquad \vdots & & \vdots \\
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Coefficients $R_{i,j}^{k}$**: 3 indices!**

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- $\mathtt{k} = 1, \ldots, \mathtt{n}$ **: for the order** $\frac{d^{\mathtt{k}}}{dt^{\mathtt{k}}}$ **of differentiation.**

In vector/matrix notation:

$$w = \begin{bmatrix} w_1 \\ w_2, \\ \vdots \\ w_{\mathtt{w}} \end{bmatrix}, \quad R_k = \begin{bmatrix} R^k_{1,1} & R^k_{1,2} & \cdots & R^k_{1,\mathtt{w}} \\ R^k_{2,1} & R^k_{2,2} & \cdots & R^k_{2,\mathtt{w}} \\ \vdots & \vdots & \cdots & \vdots \\ R^k_{g,1} & R^k_{g,2} & \cdots & R^k_{g,\mathtt{w}} \end{bmatrix}.$$

Yields

$$\boxed{R_0 w + R_1 \frac{d}{dt} w + \cdots + R_n \frac{d^n}{dt^n} w = 0,}$$

with $R_0, R_1, \cdots, R_n \in \mathbb{R}^{g \times \mathtt{w}}$.

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Including latent variables $\rightsquigarrow$

$$\boxed{R(\frac{d}{dt})w = M(\frac{d}{dt})\ell}$$

with $R, M \in \mathbb{R}^{\bullet\times\bullet}[\xi]$.

Examples:

1. RLC-circuit: **Case 1:** $CR_C \neq \frac{L}{R_L}$.

Then the relation between V and I is

$$\left(\frac{R_C}{R_L} + (1 + \frac{R_C}{R_L})CR_C\frac{d}{dt} + CR_C\frac{L}{R_L}\frac{d^2}{dt^2}\right)V = (1 + CR_C\frac{d}{dt})(1 + \frac{L}{R_L}\frac{d}{dt})R_C I.$$

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$$R(\xi) =$$

$$[\ (\tfrac{R_C}{R_L}+(1+\tfrac{R_C}{R_L})CR_C\xi+CR_C\tfrac{L}{R_L}\xi^2 \ \mid\ -1-(CR_C+\tfrac{L}{R_L})\xi-(CR_C\tfrac{L}{R_L})\xi^2\ ]$$

$$= [\ \tfrac{R_C}{R_L} \ \mid\ -1\ ] + [\ 1+\tfrac{R_C}{R_L} \ \mid\ -CR_C-\tfrac{L}{R_L}\ ]\,\xi + [\ CR_C\tfrac{L}{R_L} \ \mid -CR_C\tfrac{L}{R_L}\ ]\,\xi^2$$

2. Linear systems:

▶ **The ubiquitous**

$$\boxed{P(\tfrac{d}{dt})y = Q(\tfrac{d}{dt})u, \quad w = (u, y)}$$

with $P, Q \in \mathbb{R}^{\bullet \times \bullet}[\xi]$, $\det(P) \neq 0$ **and, perhaps,** $P^{-1}Q$ **proper.**

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We will learn the *raison d'être* of these special representations later.

3. <u>Linearization</u>: **Consider the system described by the systems of nonlinear differential equations**

$$f\left(w(t), \frac{d}{dt}w(t), \ldots, \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}}w(t)\right) = 0$$

with $f : (w_0, w_1, \ldots, w_{\mathtt{n}}) \mapsto \mathbb{R}^\bullet$. **Assume that** $\mathbf{w}^\star \in \mathbb{R}^{\mathtt{w}}$ **is an equilibrium:**

$$f(\mathbf{w}^\star, 0, \ldots, 0) = 0.$$

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Linearize around $\mathrm{w}^{\star}$!

Define $R_{\mathrm{k}} = \frac{\partial}{\partial x_{\mathrm{k}}} f(\mathrm{w}^{\star}, 0, \ldots, 0)$. The system

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathrm{n}} \frac{d^{\mathrm{n}}}{dt^{\mathrm{n}}} w = 0,$$

is called the *linearized system* around $\mathrm{w}^{\star}$. Under reasonable conditions it describes the behavior in the neighborhood of $\mathrm{w}^{\star}$.

POLYNOMIAL MATRICES

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What is a polynomial?

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What is a polynomial?

Let R be a ring with addition + and multiplication $\bullet$

Consider the set $\mathfrak{P}_R$ consisting of the infinite sequences $(r_0, r_1, \ldots, r_n, \ldots)$ with $r_k \in \mathbb{R}$, $k \in \mathbb{Z}_+$, with all but a finite number of the r_k's $\neq 0$.

Define binary operations $\oplus$ (addition) and $*$ (multiplication) on $\mathfrak{P}_R$ by

$$(r'_0, r'_1, \ldots, r'_n, \ldots) \oplus (r''_0, r''_1, \ldots, r''_n, \ldots)$$
$$:= (r'_0 + r''_0, r'_1 + r''_1, \ldots, r'_n + r''_n, \ldots)$$

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$$(r'_0, r'_1, \ldots, r'_n, \ldots) * (r''_0, r''_1, \ldots, r''_n, \ldots)$$
$$:= (r'_0 \bullet r''_0, r'_0 \bullet r''_1 + r'_1 \bullet r''_0, \ldots, \Sigma_{k=0}^{n} r'_k \bullet r''_{n-k}, \ldots)$$

$*$ is, of course, *convolution*.

It is easy to see that $(\mathfrak{P}_R, \oplus, *)$ is also a ring. Basically this is the ring of polynomials with coefficients in R. Indeed, code

$$(r_0, r_1, \ldots, r_{\mathrm{n}}, \ldots) \text{ as } r(\xi) := r_0\xi^0 + r_1\xi^1 + \cdots + r_{\mathrm{n}}\xi^{\mathrm{n}} + \cdots$$

and verify that addition and multiplication corresponds to 'collecting equal order powers' of ξ.

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It is important to realize that ξ in $r(\xi)$ is an *'indeterminate'*, nothing more than a 'place marker'.

We henceforth write $\oplus$ as $+$, $r' * r''$ and $r'r''$, and $\mathfrak{P}_R$ as $R[\xi]$.
This means that the indeterminate is denoted by ξ and that the coefficients of the polynomials are in the ring R.

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It is a 'p.i.d.': a *principal ideal domain*

$\Rightarrow$ the greatest common divisor and the least common multiple

of a set of polynomials are well defined notions in $\mathbb{R}[\xi]$.

In fact, $\mathbb{R}[\xi]$ is an *Euclidean domain*,

meaning that the *degree* of a real polynomial

is well-defined (and satisfies a number of properties required of a degree function in a ring).

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<u>Notation</u>: $\mathbb{R}^{n}[\xi], \mathbb{R}^{\bullet}[\xi], \mathbb{R}^{n_1 \times n_2}[\xi], \mathbb{R}^{\bullet \times n}[\xi], \mathbb{R}^{n \times \bullet}[\xi], \mathbb{R}^{\bullet \times \bullet}[\xi]$.

We may view $P \in \mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_2}[\xi]$ both as a polynomial with matrix coefficients:

$$P(\xi) = P_0 + P_1\xi + \cdots + P_{\mathtt{n}}\xi^{\mathtt{n}},$$

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with $P_0, P_1, \ldots, P_n \in \mathbb{R}^{n_1 \times n_2}$,

or as a matrix of polynomials:

$$P(\xi) = \begin{bmatrix} P_{1,1}(\xi) & P_{1,2}(\xi) & \cdots & P_{1,n_2}(\xi) \\ P_{2,1}(\xi) & P_{2,2}(\xi) & \cdots & P_{2,n_2}(\xi) \\ \vdots & \vdots & \cdots & \vdots \\ P_{n_1,1}(\xi) & P_{n_1,2}(\xi) & \cdots & P_{n_1,n_2}(\xi) \end{bmatrix},$$

with the $P_{i,j}$'s elements of $\mathbb{R}[\xi]$.

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$P(\frac{d}{dt})$ can act on maps $w : \mathbb{R} \to \mathbb{R}^{n_2}$ and produces maps $P(\frac{d}{dt})w : \mathbb{R} \to \mathbb{R}^{n_1}$ (assuming enough differentiability).

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THE BEHAVIOR OF $R(\frac{d}{dt})w = 0$

What do we mean by the behavior of this system of differential equations?

When shall we define $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ to be a solution of $R(\frac{d}{dt})w = 0$?

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Possibilities:

Strong solutions?

Weak solutions?

$\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ (infinitely differentiable) solutions?

Distributional solutions?

$\mathfrak{C}^\infty$**-solution:** $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ **is a** $\mathfrak{C}^\infty$**-solution of** $R(\frac{d}{dt})w = 0$ **if**

1. w **is infinitely differentiable** ($:= w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$), **and**
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Weak solution: $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ is a **weak solution** of $R(\frac{d}{dt})w = 0$ if

1. $\int_{t_0}^{t_1} ||w(t)||\, dt < \infty$ for all $t_0, t_1 \in \mathbb{R}$, and
2. $\int_{-\infty}^{+\infty} (R^\top(-\frac{d}{dt})a)^\top(t) w(t)\, dt = 0$
   for all $a \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{rowdim}(R)})$ of compact support
   (i.e., a is zero outside some finite interval).

Since for $w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$,

$$\int_{-\infty}^{+\infty} (R^{\top}(-\frac{d}{dt})a)(t)w(t)\, dt = \int_{-\infty}^{+\infty} a^{\top}(t)(R(\frac{d}{dt})w(t))\, dt,$$

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Example: Consider $\frac{d}{dt}w_2 = w_1$. Take $w_1(t) = w_2(t) = 0$ for $t < 0$, and $w_1(t) = 1, w_2(t) = t$ for $t \geq 0$.

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We will be 'pragmatic', and take the easy way out: $\rightsquigarrow$ $\boxed{\mathfrak{C}^\infty \text{ soln's!}}$

Transmits main ideas, easier to handle, easy theory,

sometimes (too) restrictive (step-response, etc.).

Whence, $R(\frac{d}{dt})w = 0$ **defines the system** $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ **with**

$$\boxed{\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{d}{dt})w = 0\}.}$$

NOTATION

$\mathfrak{L}^\bullet$: **all such systems (with any - finite - number of variables)**

$\mathfrak{L}^{\mathtt{w}}$: **with** $\mathtt{w}$ **variables**

$\mathfrak{B} = \ker(R(\frac{d}{dt}))$

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NOMENCLATURE

Elements of $\mathfrak{L}^\bullet$: *linear differential systems*

$R(\frac{d}{dt})w = 0$: a *kernel representation* of the corresponding $\Sigma \in \mathfrak{L}^\bullet$ or $\mathfrak{B} \in \mathfrak{L}^\bullet$

$R(\frac{d}{dt})w = 0$ 'has' behavior $\mathfrak{B}$

Σ or $\mathfrak{B}$: the system *induced* by $R \in \mathbb{R}^{\bullet\times\bullet}[\xi]$

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Some other properties of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$:

$$(w \in \mathfrak{B}) \Rightarrow (\frac{d}{dt}w \in \mathfrak{B});$$

$$(w \in \mathfrak{B} \quad \text{and} \quad p \in \mathbb{R}[\xi]) \Rightarrow (p(\frac{d}{dt})w \in \mathfrak{B});$$

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$(w \in \mathfrak{B}$ and $p \in \mathbb{R}[\xi]) \Rightarrow (p(\frac{d}{dt})w \in \mathfrak{B})$;

Further niceties:

$(w \in \mathfrak{B}$ and $f \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R})) \Rightarrow (f * w \in \mathfrak{B})$,

$*$ denotes convolution;

$\mathfrak{C}^{\infty}$-solutions of $R(\frac{d}{dt})w = 0$ are dense in the set of

weak (or distributional) solutions.

ALGEBRAIZATION of $\mathfrak{L}^{\bullet}$

An important type of square polynomial matrix:

Definition: $P \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}[\xi]$ **is said to be** ***unimodular*** **if there exists**

$Q \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}[\xi]$ **such that** $QP = I_{\mathtt{n}\times\mathtt{n}}$**.**

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Proposition: $P \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}[\xi]$ ***is unimodular iff*** $\det(P) = \alpha$,

with $0 \neq \alpha \in \mathbb{R}$.

R defines $\mathfrak{B} = \ker(R(\frac{d}{dt}))$, but not vice-versa!

Obviously, $R(\frac{d}{dt})w = 0$ and $U(\frac{d}{dt})R(\frac{d}{dt})w = 0$ define the same behavior whenever U is unimodular.

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Define the ***annihilators*** **of** $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ **by**

$$\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{\mathtt{w}}[\xi] \mid n^{\top}(\tfrac{d}{dt})\mathfrak{B} = 0\}.$$

An intermezzo about the structure of $\mathbb{R}^{\mathrm{n}}[\xi]$.

Many properties of matrices generalize to polynomial matrices, e.g., the rank:

The polynomial vectors $r_1, \ldots, r_{\mathrm{n}} \in \mathbb{R}^{\mathrm{n}}[\xi]$ are said to be *independent* if

$$(\alpha_1, \ldots, \alpha_{\mathrm{n}} \in \mathbb{R}[\xi], \text{ and } \sum_{\mathrm{j}=1}^{\mathrm{n}} \alpha_{\mathrm{j}} r_{\mathrm{j}} = 0)$$

$$\Leftrightarrow \ (\alpha_1 = \cdots = \alpha_{\mathrm{n}} = 0).$$

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Let $P \in \mathbb{R}^{\bullet\times\bullet}[\xi]$. The *row rank* of R is defined as the maximal number of independent rows. It equals the *column rank* of P, defined as the maximal number of independent columns $\quad\rightsquigarrow\quad \text{rank}(P)$.

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$\operatorname{rank}(P)$ = the dimension of the largest square submatrix with a non-zero determinant (i.e., the non-zero minors).

SMITH FORM

The following result is very useful in proofs. It shows that, by pre- and postmultiplying by unimodular matrices, polynomial matrices can be brought in *Smith form,* a simple, diagonal like form.

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$$UPV = \begin{bmatrix} \mathrm{diag}(p_1, p_2, \ldots, p_r) & 0_{r \times (n_2 - r)} \\ 0_{(n_1 - r) \times r} & 0_{(n_1 - r) \times (n_2 - r)} \end{bmatrix}$$

where $r = \mathrm{rank}(P)$ and p_{k+1} is a factor of p_k for $k = 1, 2, \ldots k - 1$.

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where $r = \mathrm{rank}(P)$ and p_{k+1} is a factor of p_k for $k = 1, 2, \ldots k - 1$.

The polynomials $p_1, p_2, \ldots, p_r$ are called the *invariant factors* of P.

$\mathbb{R}^{\mathtt{n}}[\xi]$ **has the structure of a** ***module*** **over the ring of 'scalars'** $\mathbb{R}[\xi]$**.**

$\mathbb{R}^n[\xi]$ has the structure of a *module* over the ring of 'scalars' $\mathbb{R}[\xi]$.

This means that the obvious

1. **binary operation, *addition*, $(+)$ on $\mathbb{R}^n[\xi]$,**
2. **and *scalar multiplication*, $(\bullet)$, the map from $\mathbb{R}[\xi] \times \mathbb{R}^n[\xi]$ to $\mathbb{R}^n[\xi]$**
3. **satisfy the required obvious axioms.**

Usually scalar multiplication, $p \bullet v$ is simply written as $p\,v$.

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An important property of submodules of $\mathbb{R}^{\mathtt{n}}[\xi]$ is that they are all *finitely generated*, meaning that $\mathfrak{M} \subset \mathbb{R}^{\mathtt{n}}[\xi]$ is a submodule iff there exists $g_1, g_2, \ldots, g_{\mathtt{k}}$ such that

$$\mathfrak{M} = \{m \in \mathbb{R}^{\mathtt{n}}[\xi] \mid \exists\, p_1, p_2, \ldots, p_{\mathtt{k}} \text{ such that } m = p_1 \bullet g_1 + p_2 \bullet g_2 + \cdots + p_{\mathtt{k}} \bullet g_{\mathtt{k}}\}$$

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In fact, one can always take $\boxed{\mathtt{k} \leq \mathtt{n}}$

$\exists$ a nice system theoretic proof of this result.

Back to $\mathfrak{L}^{\mathtt{w}}$:

Proposition: $\mathfrak{N}_{\mathfrak{B}}$ **is a** $\mathbb{R}[\xi]$ **sub-module of** $\mathbb{R}^{\mathtt{w}}[\xi]$**.**

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Let $< R^\top >$ denote the sub-module of $\mathbb{R}^{\mathtt{w}}[\xi]$ spanned by the transposes of the rows of R. Obviously $< R^\top > \subset \mathfrak{N}_{\mathfrak{B}}$.

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But, indeed:

$$\boxed{\mathfrak{N}_{\mathfrak{B}} = < R^\top >!}$$

Note: Depends on $\mathfrak{C}^\infty$. $\subset$ may be false for compact support soln's.

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Theorem: $\mathfrak{L}^{\mathtt{w}} \xleftrightarrow{1:1}$ **sub-modules of** $\mathbb{R}^{\mathtt{w}}[\xi]$

KERNEL REPRESENTATIONS

MINIMAL KERNEL REPRESENTATIONS

Definition: $R(\frac{d}{dt})w = 0$ is said to be a *minimal* kernel representation of $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ if, whenever $R'(\frac{d}{dt})w = 0$ is another kernel representation of this $\mathfrak{B}$, i.e., whenever $\ker(R(\frac{d}{dt})) = \ker(R'(\frac{d}{dt}))$, there holds:

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Nomenclature: $R(\frac{d}{dt})w = 0$ 'is' ***minimal***.

minimal :$\Leftrightarrow$ number of equations 'as small as possible'.

$P \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ is said to be of *full row rank* if its rows are linearly independent i.e., iff $\mathrm{rank}(P) = \mathrm{rowdim}(P)$. Equivalently iff $\exists$ a submatrix of size $\mathrm{rowdim}(P) \times \mathrm{rowdim}(P)$ with non-zero determinant.

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Note: Assume $R(\frac{d}{dt})w = 0$ has an $R \in \mathbb{R}^{\bullet \times \mathtt{w}}$ that is not of full row rank.

Can one or more of the equations be removed?

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$$\boxed{R_2 = FR_1.}$$

3. $R(\frac{d}{dt})w = 0$ is minimal iff R is of full row rank.

We may hence use *minimal* representation and *full row rank* kernel representation as synonymous.

Theorem (Structure of kernel representations):

1. Let U be unimodular. Then $R(\frac{d}{dt})w = 0$ and $U(\frac{d}{dt})R(\frac{d}{dt})w = 0$ have the **same** behavior.

2. Let $R_1(\frac{d}{dt})w = 0$ have behavior $\mathfrak{B}_1$, and $R_2(\frac{d}{dt})w = 0$ have behavior $\mathfrak{B}_2$. Then $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ iff $\exists\ F \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ such that

$$\boxed{R_2 = FR_1.}$$

3. $R(\frac{d}{dt})w = 0$ is **minimal** iff R is of **full row rank.**
We may hence use ***minimal*** representation and ***full row rank*** kernel representation as synonymous.

4. Let $R(\frac{d}{dt})w = 0$ be minimal. **All minimal** kernel representations with the same behavior are obtained by **pre-multiplying** R by an **arbitrary unimodular** polynomial matrix.

ELIMINATION

LATENT VARIABLE SYSTEMS

First principle models $\rightsquigarrow$ latent variables. In the case of systems described by linear constant coefficient differential equations:

$$R_0 w + \cdots + R_n \frac{d^n}{dt^n} w = M_0 \ell + \cdots + M_n \frac{d^n}{dt^n} \ell .$$

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This is the natural model class to start a study of finite dimensional linear time-invariant systems!!

But is it(s manifest behavior) really a differential system ??

The full behavior of $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$, **i.e.,**

$$\mathfrak{B}_{\text{full}} = \{(w,\ell) \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}+\ell}) \mid R(\frac{d}{dt})w = M(\frac{d}{dt})\ell.\}$$

belongs to $\mathfrak{L}^{\mathtt{w}+\ell}$**, by definition. Its manifest behavior equals**

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Theorem: It does!

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with P square and $\det(P) \neq 0$. Why: soon!

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! Compute $(E, F, G) \mapsto R$. Dimension minimal kernel representation?

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Why is there *only one* equation? Passivity! ... Later.

Let

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be a kernel representation of the full behavior.

Let

$$R'(\frac{d}{dt})w = 0$$

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be a kernel representation of the manifest behavior.

There exist effective algorithms for

$$(R, M) \mapsto R'$$

incorporating, if desired, minimality of $R'(\frac{d}{dt})w = 0$.

RECAP

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▶ **Minimal kernel representations: those of full row rank**

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▶ **Elimination theorem:**

full behavior linear ODE $\Rightarrow$ manifest behavior linear ODE

It follows from all this that $\mathfrak{L}^\bullet$ is closed under:

- **Intersection**: $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}) \Rightarrow (\mathfrak{B}_1 \cap \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}})$.
- **Addition**: $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}) \Rightarrow (\mathfrak{B}_1 + \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}})$.
- **Projection**: $(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}) \Rightarrow (\Pi_{w_1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1})$.
- **Action of a linear differential operator**:

  $$(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}, P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_1}[\xi]) \Rightarrow (P(\tfrac{d}{dt})\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2}).$$
- **Inverse image of a linear differential operator**:

  $$(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2}, P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_1}[\xi]) \Rightarrow ((P(\tfrac{d}{dt}))^{-1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}).$$

End of Lecture 2