Solutions Exercises Set 5

Exercise 1 (Inputs and outputs)

The aim of this exercise is (i) to develop a more liberal notion of input and output and obtain corresponding representations, and (ii) to make you reflect a bit on the slippery notion of 'non-anticipation'.

Conceptualizing the notion of input and output is more difficult than it seems at first sight. In physical systems, 'causality', the darling notion of the philosophically inclined, is a *red herring*. In physical systems, variables occur together, simultaneously, 'there can be not one without the other', and it makes little sense to say that one variable causes another. Newton's second law states that a point-mass accelerates if and only if a force acts on it. Granted, thou, that it seems to make more sense to say that it is the force that causes motion, than that the motion causes the force to act. But is absurd to insist on viewing the voltage or the current as the input to a resistor.

During the lectures, we have chosen to consider inputs as 'free'. But there are many situations, for example in computer science and in signal processing, where the externally imposed inputs need to be structured. With this we mean not just niceties like measurability, integrability, smoothness, but instead constraints on inputs as periodicity, piecewise constancy, etc. Also, many algorithms used in signal processing are not 'real time', but nevertheless the input/structure is very appropriate (see exercise 2 of set 2).

In this exercise we discuss a more liberal approach to the concepts of input and output, in the context of linear differential systems.

Consider $(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1} \times \mathbb{R}^{\mathtt{w}_2}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$. Call w_1 *input* and w_2 *output* if $(w_1, w_2'), (w_1, w_2'') \in \mathfrak{B}$ and $w_2'(t) = w_2''(t)$ for $t < 0$ implies $w_2' = w_2''$.

1. (a) Prove that if w_2 is observable from w_1, then w_1 is input and w_2 is output.

   w_2 is observable from w_1 iff $(w_1, w_2'), (w_1, w_2'') \in \mathfrak{B}$ implies $w_2' = w_2''$. Obviously, then w_1 is input and w_2 output.

   (b) Prove that w_1 is input and w_2 is output iff $\{w_2 \mid (0, w_2) \in \mathfrak{B}\}$ is autonomous. Conclude that w_1 is input and w_2 is output iff a kernel representation $R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2$ of $\mathfrak{B}$ has $\text{rank}(R_2) = \mathtt{w}_2$.

   By linearity, w_1 is input and w_2 is output iff $(0, w_2) \in \mathfrak{B}$ and $w_2(t) = 0$, for $t < 0$, implies $w_2 = 0$. In other words, iff $\{ w_2 \mid (0, w_2) \in \mathfrak{B} \} \in \mathfrak{L}^{\mathtt{w}_2}$ is autonomous.
   Let

   $$R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2$$

   be a kernel representation of $\mathfrak{B}$. Then $R_2(\frac{d}{dt})w_2 = 0$ is a kernel representation of

   $$\{ w_2 \mid (0, w_2) \in \mathfrak{B} \}.$$

   During the lectures we have seen that this kernel representation defines an autonomous system iff R_2 has rank $\mathtt{w}_2$.

   (c) Prove that w_1 is input and w_2 is output iff there exists a minimal kernel representation of the form

$$R_1(\frac{d}{dt})w_1 = 0, \quad P(\frac{d}{dt})w_2 = Q(\frac{d}{dt})w_1, \tag{1}$$

with $P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_2}[\xi]$ and $\det(P) \neq 0$.

Using a unimodular pre-multiplication, if necessary, R_2 can be brought into the form $\left[\begin{smallmatrix} P \\ 0 \end{smallmatrix}\right]$, with P square and $\det(P) \neq 0$ iff R_2 has full column rank. The representation follows from (b).

(d) Prove that in an autonomous system *any* partition of the variables qualifies as an input/output partition.

Follows immediately from the definitions of "autonomous" and "input/output".

(e) Use the representation (1) to define a notion of *transfer function* from w_1 to w_2. Formulate and prove a theorem which expresses the extent of uniqueness of the transfer function. Prove that the transfer function is unique iff the input is 'free'.

Assume that the system has representation (1). Define the transfer function $G \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_1}(\xi)$ by

$$G := P^{-1}Q.$$

But for a given system $\mathfrak{B}$ and a given input/output partition $w = (w_1, w_2)$, the representation is not uniquely defined. The transfer function, therefore, is also not uniquely defined, but only up to the transformation

$$P^{-1}Q \rightarrow P^{-1}(Q + FR_1),$$

with $F \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ arbitrary. If, however, w is free, then $R_1 = 0$, and the transfer function is uniquely defined.

Note: examination of the autonomous case (e.g., $w_2 = w_1; p(\frac{d}{dt})w_1 = 0$) indicates that there is more to the notion of transfer function than we have suggested here.

Comment: One way to define the transfer function from first principles is to consider the exponential responses, *i.e.*, to consider the responses in $\mathfrak{B}$ of the form

$$t \in \mathbb{R} \mapsto (a_1, a_2)e^{\lambda t} \in \mathbb{C}^{\mathtt{w}_1 + \mathtt{w}_2}.$$

Now, if such a response exists for a given $a_1 \in \mathbb{C}^{\mathtt{w}_1}$, the associated $a_2 \in \mathbb{C}^{\mathtt{w}_2}$ is unique. Thus we obtain for each $\lambda \in \mathbb{C}$ a map from a subspace $\mathbb{E}_\lambda \subset \mathbb{C}^{\mathtt{w}_1}$ (in fact, the subspace $\ker(R_1(\lambda))$) to $\mathbb{C}^{\mathtt{w}_2}$. In fact,

$$a \in \mathbb{E}_\lambda \mapsto a_2 = G(\lambda)a_1 \in \mathbb{C}^{\mathtt{w}_2}.$$

The issue is that, as in the case of autonomous systems (for example, $w_2 = w_1$, $p(\frac{d}{dt})w_1 = 0$)
1. there may be very few λ's for which $\mathbb{E}_\lambda \neq \{0\}$,
2. the exponential responses do not determine the system completely, due to the lack of controllability.

2. When discussing inputs and outputs, it is hard to restrain from talking of 'causality' and 'non-anticipation'. We now willingly submit to this temptation.

Consider $(\mathbb{T}, \mathbb{R}^{\mathtt{w}_1} \times \mathbb{R}^{\mathtt{w}_2}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}_1 + \mathtt{w}_2}$, with $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$. Refer to exercise set 2, exercise 2 for the basics in the case $\mathbb{T} = \mathbb{Z}$. Denote by $\mathfrak{B}_1$ the projection of $\mathfrak{B}$ on the w_1 variable, i.e., with w_2 eliminated.

We say that w_2 *does not anticipate* w_1 if $(w_1, w_2) \in \mathfrak{B}, w_1' \in \mathfrak{B}_1$, and $w_1(t) = w_1'(t)$ for $t < 0$ implies that there exists w_2' with $w_2'(t) = w_2(t)$ for $t < 0$ such that $(w_1', w_2') \in \mathfrak{B}$. Equivalently, using linearity, if $w_1 \in \mathfrak{B}_1$ and $w_1(t) = 0$ for $t < 0$ implies that there exists w_2 such that (i) $(w_1, w_2) \in \mathfrak{B}$ and (ii) $w_2(t) = 0$ for $t < 0$.

(a) Assume $\mathbb{T} = \mathbb{R}$. Prove that w_2 does not anticipate w_1. Hence *no variable anticipates another one in linear differential systems.* Hence, in particular, in any input/output partition, the output does not anticipate the input. Notwithstanding this fact, properness of transfer functions has often be brought in connection with non-anticipation. Some authors go as far as to argue that in a differentiator the output anticipates the input, since $y(t) = \lim_{\delta \searrow 0} \frac{u(t+\delta)-u(t)}{\delta}$ suggests anticipation. By the same token, however, $y(t) = \lim_{\delta \searrow 0} \frac{u(t)-u(t-\delta)}{\delta}$ suggests non-anticipation.

Let $(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}, \mathfrak{B}) \in \mathbb{L}^{\mathtt{w}_1+\mathtt{w}_2}$. It admits a kernel representation of the form

$$P(\frac{d}{dt})w_{21} = Q(\frac{d}{dt})w_{22} + M_1(\frac{d}{dt})w_1, \qquad 0 = R_1(\frac{d}{dt})w_1$$

with $w_2 = (w_{21}, w_{22})$ a suitable partition of w_2 and P square and nonsingular. We prove that for any $w_1 \in \mathbb{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2})$, such that $w_1(t) = 0$ for $t < 0$, there exists a solution $w_2 = (w_{21}, w_{22})$ of

$$P(\frac{d}{dt})w_{21} = Q(\frac{d}{dt})w_{22} + M_1(\frac{d}{dt})w_1,$$

with $w_2(t) = 0$, for $t < 0$. By linearity, this implies then that w_2 does not anticipate w_1. In order to see this, take $w_{22} = 0$ and use the Smith form to reduce the problem to the scalar case. Then prove that if $0 \neq p \in \mathbb{R}[\xi]$, then for any $f \in \mathbb{C}^\infty(\mathbb{R}, \mathbb{R})$ with $f(t) = 0$, for $t < 0$, there exists a solution $x \in \mathbb{C}^\infty(\mathbb{R}, \mathbb{R})$ to

$$p(\frac{d}{dt})x = f,$$

with $x(t) = 0$, for $t < 0$. This easily follows by suitably adapting the proof(s) of part 1 of exercise 2 of set 4.

(b) The situation is significantly different in the discrete time case. There, phenomena do not happen instantly, simultaneously, the delays are more than infinitesimal, *etc.* Assume $\mathbb{T} = \mathbb{Z}$. The analogue of (1) becomes

$$R_1(\sigma, \sigma^{-1})w_1 = 0, \quad P(\sigma, \sigma^{-1})w_2 = Q(\sigma, \sigma^{-1})w_1, \tag{2}$$

with $P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_2}[\xi, \xi^{-1}]$ and $\det(P) \neq 0$ Prove that the output does not anticipate the input iff there exists a transfer function from w_1 to w_2 that is a matrix of proper rational functions.

Consider the equations (2). It is hence assumed that w_1 is input and w_2 is output. We prove that the following are equivalent:

A. w_2 does not anticipate w_1;

B. there is $F \in \mathbb{R}^{\bullet \times \bullet}[\xi, \xi^{-1}]$ such that the transfer function $P^{-1}(Q + FR_1)$ is proper;

C. there exists a kernel representation of the form

$$\begin{aligned} P(\sigma^{-1})w_2 &= Q'(\sigma^{-1})w_{11} + Q''(\sigma^{-1})w_{12} \\ P'(\sigma^{-1})w_{12} &= Q'''(\sigma^{-1})w_{11} \end{aligned}$$

with $w_1 = (w_{11}, w_{12})$ a partition of w_1, $P, P', Q', Q'', Q''' \in \mathbb{R}[\xi]$, P and P' square, and $\det(P(0)) \neq 0$, $\det(P'(0)) \neq 0$.

In order to prove this, we use, without further proof, the following preliminary facts:
1. If $P \in \mathbb{R}^{\bullet\times\bullet}[\xi,\xi^{-1}]$ and $\det(P) \neq 0$, then there exists a matrix $U \in \mathbb{R}^{\bullet\times\bullet}[\xi,\xi^{-1}]$ that is unimodular (over $\mathbb{R}^{\bullet\times\bullet}[\xi,\xi^{-1}]$!) such that $P' = UP$ is of the form

$$P'(\xi,\xi^{-1}) = P'_0 + P'_1\xi^{-1} + \cdots + P'_L\xi^{-L},$$

with $\det(P'_0) \neq 0$ (in fact, $P'_0 = I$ is always possible).
2. If $(\mathbb{Z},\mathbb{R}^{\mathtt{w}},\mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}$, then there exists a componentwise partition $w = (w_1, w_2)$ and a kernel representation of the form

$$P(\sigma^{-1})w_2 = Q(\sigma^{-1})w_1,$$

with $P, Q \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ and $\det(P(0)) \neq 0$. Note that this partition is an input/output partition with w_1 the free input and w_2 an output which does not anticipate w_2.

Now consider (2). Use the input/output partition of w_1 and the unimodular pre-multiplication to obtain the representation

$$\begin{aligned} P(\sigma^{-1})w_2 &= Q'(\sigma,\sigma^{-1})w_{11} + Q''(\sigma,\sigma^{-1})w_{12} \\ P'(\sigma^{-1})w_{11} &= Q'''(\sigma^{-1})w_{12} \end{aligned}$$

with $P, P', Q''' \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, $\det(P(0)) \neq 0$, and $\det(P'(0)) \neq 0$. Now, using repeatedly the second equation, we obtain

$$P(\sigma^{-1})w_2 = Q'(\sigma^{-1})w_{11} + Q''(\sigma,\sigma^{-1})w_{12} \tag{3}$$

$$P'(\sigma^{-1})w_{11} = Q'''(\sigma^{-1})w_{12} \tag{4}$$

with $Q' \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ having $Q'(0) = 0$.
We now prove the equivalence of statements A, B, and C.
(A $\Rightarrow$ C) We prove that A implies that in (3), $Q''(\xi,\xi^{-1})$ does not contain positive powers of ξ. Assume, to the contrary, that the highest power is $L > 0$, with coefficient matrix Q_L. Now consider the impulse input $w_{12} = a\delta$ ($\delta : \mathbb{Z} \to \mathbb{R}$ denotes the pulse: $\delta(0) = 1$ and $\delta(t) = 0$, for $t \neq 0$) with a such that $Q_L a \neq 0$. Then (4) allows a response w_{11} such that $w_{11}(t) = 0$, for $t < 0$. Now consider (3) and show that the assumption $w_2(t) = 0$, for $t < 0$, leads to a contradiction by evaluating both sides of (3) at $t = -L$.
(C $\Rightarrow$ B) is trivial.
(B $\Rightarrow$ A) Pre-multiply P by a suitable unimodular matrix U, such that $P' = UP$ is of the form

$$P'(\xi^{-1}) = P_0 + P_1\xi^{-1} + \cdots$$

with $\det(P_0) \neq 0$. Define $Q' := U(Q + FR_1)$. Since, $(P')^{-1}Q'$ is proper, $Q'(\xi,\xi^{-1})$ does not contain positive powers of ξ, leading to the following law satisfied by (w_1, w_2)

$$P_0w_2(t) + P_1w_2(t-1) + \cdots = Q_0w_1(t) + Q_1w_1(t-1) + \cdots$$

It is easy to conclude that this equation implies that w_2 does not anticipate w_1.

Exercise 2 (Control in a behavioral setting)

Because of time limitations, we have been unable to cover control questions during the lectures. The aim of this exercise is to motivate the behavioral setting of control problems, to illustrate it by means of a simple example, and to present an interesting result, the controller implementability theorem.

Please brace yourself for a lengthy introduction.

The behavioral point of view is compelling as an approach to modeling dynamical systems, but this may seem to be much less so when it comes to control. There is something natural in viewing control variables as inputs and measured variables as outputs. When subsequently a controller is regarded as a feedback processor that accepts the sensor outputs as its inputs and produces as its outputs the actuator inputs, one ends up with the feeling that the input/output structure is in fact an essential feature of control.

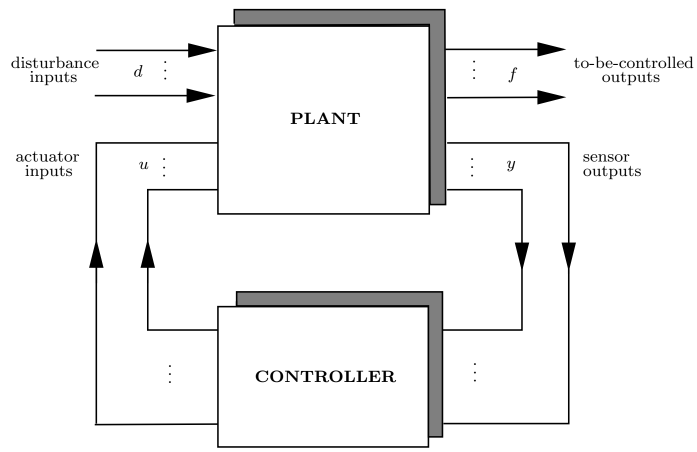

Figure 1: Intelligent control

Present-day control theory centers around the signal flow graph shown in figure 1. The plant has four type of terminals (supporting variables which will typically be vector-valued). There are two input terminals, one for the control, one for the other exogenous variables (in figure 1 these are called 'disturbances', but they may also be set-points, reference signals, tracking inputs, etc.) and there are two output terminals, one for the measurements, and one for the to-be-controlled variables. The control inputs are generated by means of actuators and the measurements are made available through sensors. We call this structure *intelligent control*. The basic idea of a controller is that of an *anthropomorphic* supervisor reacting in an intelligent way to observed events, and directing on this basis the controllable events.

The paradigm embodied in figure 1 has been universally prevalent ever since the beginning of the subject. It is indeed a very appealing paradigm, which will undoubtedly gain in impact as logic devices become ever more prevalent, reliable, and inexpensive. This paradigm has a number of features which are important for considerations which will follow. Some of these are:

1. The intelligent control paradigm tells us to be wary of errors in the measurements. Thus it is considered as being ill-advised to differentiate measurements, presumably, because this will lead to noise amplification.

2. The plant and the controller are dynamical systems which can be interconnected at any moment in time. If for one reason or another the feedback controller temporarily fails to receive an input signal, then the control input can be set to a default value, and, later on, the controller can resume its action.

We will now analyze an example of a very low-tech controller, a wide-spread control mechanism, namely the traditional device which ensures the automatic closing of doors. There is nothing peculiar about this example. Devices based on similar principles are used for the dampers and suspension of automobiles and the points which we make through this example could also be made just as well through many vibration, temperature, or pressure control devices. A typical automatic door-closing mechanism is schematically shown in figure 2.

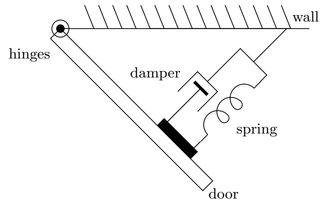

Figure 2: Door-closing mechanism

A door-closing mechanism usually consists of a spring to ensure the closing of the door and a damper in order to make sure that it closes gently. These mechanisms often have considerable weight so that their mass cannot be neglected as compared to the mass of the door itself. The automatic door-closing mechanism can be modelled as a mass/spring/damper combination.

We model the door as a mass M', on which, neglecting friction in the hinges, two forces act. The first force, F_c, is the force exerted by the door-closing device, while the second force, F_e, is the exogenous force exerted for example by a person pushing the door in order to open it. The equation of motion for the door becomes

$$M'\frac{d^2}{dt^2}\theta = F_c + F_e,$$

where θ denotes the opening angle of the door. The automatic door-closing mechanism, modelled as a mass/spring/damper combination, yields

$$M''\frac{d^2}{dt^2}\theta + D\frac{d}{dt}\theta + K\theta = -F_c.$$

Here, M'' denotes the mass of the door-closing mechanism, D its damping coefficient, and K its spring constant. Combining these equations yields

$$(M' + M'')\frac{d^2}{dt^2}\theta + D\frac{d}{dt}\theta + K\theta = F_e.$$

In order to ensure proper functioning of the door-closing device, the designer can to some extent choose M'', D, and K (all of which must, for physical reasons, be positive). The desired response requirements are: small overshoot (to avoid banging of the door), fast settling time, and a reasonably high steady state gain (to avoid having to exert excessive force when opening the door). This is an example of an elementary control design exercise. A good design will be achieved by choosing a light mechanism (M'' small), with a reasonably strong spring (K large), but not too strong so as to avoid having to use excessive force in order to open the door, and with the value of D chosen so as to achieve slightly less than critical damping (ensuring gentle closing of the door).

It is completely natural to view in this example the door as the plant and the door-closing mechanism as the controller. Then, if we insist on interpreting this plant/controller combination in terms of control system configurations as figure 1, we obtain.

$$\text{Plant:} \quad M'\frac{d^2}{dt^2}\theta = u + v; \; y = \theta; \; z = \theta$$

with u the control input ($u = F_c$), v the exogenous input ($v = F_e$), y the measured output, and z the to-be-controlled output.

$$\text{Controller:} \quad u = -M''\frac{d^2}{dt^2}y - D\frac{d}{dt}y - Ky.$$

This yields the controlled system, described by:

$$\text{Controlled system:} \quad (M' + M'')\frac{d^2}{dt^2}z + D\frac{d}{dt}z + Kz = v.$$

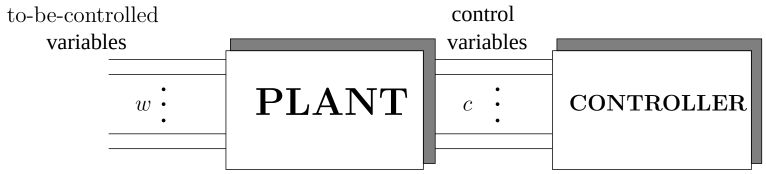

Figure 3: Control as interconnection

Observe that in the control law, the measurement y should be considered as the input, and the control u should be considered as the output. This suggests that we are using what would be called a *PD^2-controller* (a proportional and twice differentiating controller), a singular controller which would be thought of as causing high noise amplification. Of course, no such noise amplification occurs in reality. Further, the plant is second order, the controller is second order, and the closed loop system is also second order (unequal to the sum of the order of the plant and the controller). Hence, in order to connect the controller to the plant, we will have to 'prepare' the initial states of the controller and the plant. Indeed, in attaching the door-closing mechanism to the door, we will make sure that at the moment of attachment the initial values of θ and $\frac{d}{dt}\theta$ are zero *both* for the door and the door-closing mechanism.

We now come to our most important point concerning this example. Let us analyze the signal flow graph. In the plant, it is natural to view the forces F_c and F_e as inputs and θ as output. This input/output choice is logical both from the physical and from the cybernetic, control theoretic point of view. In the controller, on the other hand, the physical and the cybernetic points of view clash. From the cybernetic, control theoretic point of view, it is logical to regard the opening angle θ as input and the control force F_c as output. From the physical point of view, however, it is logical to regard (just as in the plant) the force F_c as input and θ as output. It is evident that as an interconnection of two mechanical systems, the door and the door-closing mechanism play completely symmetric roles. However, the cybernetic, control theoretic point of view obliges us to treat the situation as asymmetric, making the force the cause in one mechanical subsystem, and the effect in another.

This simple but realistic example permits to draw the following conclusions. Notwithstanding all its merits, the intelligent control paradigm of figure 1 gives an unnecessarily restrictive view of control theory. In many practical control problems, the signal-flow-graph interpretation of figure 1 is untenable. There are no measurements, there are no actuators, and signal flows are a figment of our imagination. We will therefore abandon the intelligent control signal flow graph as a paradigm for control. We will abandon the distinction between control inputs and measured outputs. Instead, we will view *interconnection of a controller to a plant* as the central paradigm in control theory. In particular, 'physical' interconnections through terminals which carry more than one variable simultaneously, variable sharing (as described extensively in lecture 5) should feature prominently in control theory.

We are not insinuating that the intelligent control paradigm is without merits. To the contrary, it is extremely useful and important. Claiming that the input/output framework is *not always* the suitable framework to approach a problem does not mean that one claims that it is *never* the suitable framework. However, a good universal framework for control should be able to deal both with interconnection, with designing subsystems, with the basic examples as the one given above, as well as with intelligent control. The behavioral framework does, the intelligent control framework does not.

The importance as a controller of precisely the mass/spring/damper/ type of device has recently been discussed by Dennis Bernstein in *What makes some control problems hard?*, IEEE Control Systems Magazine, volume 24, number 4, pages 8-19, August 2002. In this article, this type of controller is called 'a controller with build in actuators and sensors'. *Absurd!*

The view of control which will now be pursued is depicted in figure 3. The figure contains two systems, shown as black-boxes with terminals. It is through their terminals that systems interact with their environment. The black-box imposes relations on the variables that 'live' on its terminals. These relations are formalized by the behavior of the system in the black-box. The system to the left in figure 3 is called the *plant*, the one to the right the *controller*. The terminals of the plant consist of *to-be-controlled variables* w, and *control variables* c. The controller has only terminals with the control variables c. Before interconnection, the variables w and c

of the plant have to satisfy the laws imposed by the plant behavior. But, after interconnection, the variables c also have to satisfy the laws imposed by the controller. Thus, after interconnection, the restrictions imposed on the variables c by the controller will be transmitted to the variables w. Choosing the black-box to the right so that the variables w have a desirable behavior in the interconnected black-box is, in our view, the basic problem of control.

This leads to the following mathematical formulation. The *plant* and the *controller* are both dynamical systems $\Sigma_{\text{plant}} = (\mathbb{T}, \mathbb{W} \times \mathbb{V}, \mathfrak{B}_{\text{plant}})$ and $\Sigma_{\text{controller}} = (\mathbb{T}, \mathbb{V}, \mathfrak{B}_{\text{controller}})$ where $\mathbb{W}$ denotes the signal space of the to-be-controlled variables, $\mathbb{V}$ denotes the signal space of control variables, and both systems are assumed to have the same time axis $\mathbb{T}$. The interconnection of Σ_{plant} and $\Sigma_{\text{controller}}$ leads to the system $\Sigma_{\text{full}} = (\mathbb{T}, \mathbb{W}, \mathbb{V}, \mathfrak{B}_{\text{full}})$ which is a system with latent variables ($\mathbb{V}$) and full behavior defined by

$$\mathfrak{B}_{\text{full}} = \{(w, c) : \mathbb{T} \to \mathbb{W} \times \mathbb{V} \mid (w, c) \in \mathfrak{B}_{\text{plant}} \text{ and } c \in \mathfrak{B}_{\text{controller}}\}$$

The manifest system obtained by Σ_{full} is the *controlled system* and is hence defined as $\Sigma_{\text{controlled}} = (\mathbb{T}, \mathbb{W}, \mathfrak{B}_{\text{controlled}})$ with

$$\mathfrak{B}_{\text{controlled}} = \{w : \mathbb{T} \to \mathbb{W} \mid \text{ there exists } c : \mathbb{T} \to \mathbb{V} \text{ such that} (w, c) \in \mathfrak{B}_{\text{plant}} \text{ and } c \in \mathfrak{B}_{\text{controller}}\}$$

A nice question that arises in this context is the following. Assume that Σ_{plant} is given.

Which systems $\Sigma_{\text{controlled}}$ can be obtained by suitably choosing $\Sigma_{\text{controller}}$?

This question can be answered very explicitly for linear time-invariant differential systems. Assume that the plant is given by $\Sigma_{\text{plant}} = (\mathbb{R}, \mathbb{R}^{\mathtt{w}} \times \mathbb{R}^{\mathtt{c}}, \mathfrak{B}_{\text{plant}}) \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. Let and assume that $\mathfrak{B}_{\text{controller}}$ is similarly described in kernel representation by a system of linear constant coefficient differential equations, i.e., $\Sigma_{\text{controller}} = (\mathbb{R}, \mathbb{R}^{\mathtt{c}}, \mathfrak{B}_{\text{controller}}) \in \mathfrak{L}^{\mathtt{c}}$. Then, by the elimination theorem, $\Sigma_{\text{controlled}} = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}_{\text{controlled}}) \in \mathfrak{L}^{\mathtt{w}}$, i.e., it has also a behavior that is described as the set of solutions of a system of linear constant coefficient differential equations. It turns out that the possible behaviors $\mathfrak{B}_{\text{controlled}} \in \mathfrak{L}^{\mathtt{w}}$ that can be obtained this way can be characterized in a very nice, simple, and explicit way.

Define therefore two behaviors, $\mathfrak{P}$ and $\mathfrak{N}$; $\mathfrak{P}$ is called the *realizable (plant) behavior* and $\mathfrak{N}$ the *hidden behavior*. They are defined as follows: $\mathfrak{P}$ is the manifest behavior of the system, i.e.,

$$\boxed{\mathfrak{P} = \{w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}} \mid \text{ there exists } c : \mathbb{R} \to \mathbb{R}^{\mathtt{c}} \text{ such that } (w, c) \in \mathfrak{B}_{\text{plant}}\},}$$

and $\mathfrak{N}$ is defined as

$$\boxed{\mathfrak{N} = \{w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}} \mid (w, 0) \in \mathfrak{B}_{\text{plant}}\}.}$$

Hence $\mathfrak{P}$ is the behavior consisting of all to-be-controlled signals that can occur (without a controller acting), and $\mathfrak{N}$ is the behavior of the to-be-controlled variables that are compatible with the control variables put equal to zero. We say that $\mathfrak{B}_{\text{controller}}$ *implements* $\mathfrak{B}_{\text{controlled}}$ if there exists a controller such that the controlled behavior after interconnecting the controller with behavior $\mathfrak{B}_{\text{controller}}$ to the plant, yields $\mathfrak{B}_{\text{controlled}}$ as the controlled behavior.

The controller implementability problem asks what behaviors $\mathfrak{B}_{\text{controlled}}$ can be obtained this way. For linear time-invariant systems it is possible to prove that $\mathfrak{B}_{\text{controlled}}$ is implementable if and only if

$$\boxed{\mathfrak{N} \subseteq \mathfrak{B}_{\text{controlled}} \subseteq \mathfrak{P}.}$$

This result shows that the behaviors that are implementable by means of a controller are precisely those that are wedged in between the hidden behavior $\mathfrak{N}$ and the realizable plant $\mathfrak{P}$. It reduces the controller design problem (assuming no further restrictions on the controller than $\Sigma_{\text{controller}} \in \mathfrak{L}^{\mathtt{c}}$) to finding a suitable a behavior that is wedged in between two given behaviors.

1. Prove that this implementability condition is necessary.

$$\mathfrak{B}_{\textbf{controlled}} = \{\, w \mid \exists\, c : (w,c) \in \mathfrak{B}_{\textbf{plant}} \text{ and } c \in \mathfrak{B}_{\textbf{controlled}} \,\}.$$

Since $0 \in \mathfrak{B}_{\textbf{controller}}$**,**

$$\mathfrak{B}_{\textbf{controlled}} \supset \{\, w \mid (w,0) \in \mathfrak{B}_{\textbf{plant}} \,\} = \mathfrak{N}.$$

Further,

$$\mathfrak{B}_{\textbf{controlled}} \subset \{\, w \mid \exists\, c : (w,c) \in \mathfrak{B}_{\textbf{plant}} \,\} = \mathfrak{P}.$$

2. Prove that $\mathfrak{N} = 0$ iff w is observable from c in $\mathfrak{B}_{\text{plant}}$. Prove that in this case the implementable behaviors are the same as those that can be obtained when $c = w$ (this is called 'full control'). More concretely, define a new plant $\Sigma'_{\text{plant}} = (\mathbb{R}, \mathbb{R}^{\mathtt{w}} \times \mathbb{R}^{\mathtt{w}}, \mathfrak{B}'_{\text{plant}})$ with $\mathfrak{B}'_{\text{plant}} = \{(w,c) \mid w \in \mathfrak{P} \text{ and } c = w\}$, and compare the controlled behaviors that can be obtained from attaching a controller to Σ_{plant} with those that can be obtained from attaching a controller to Σ'_{plant}.

We prove that any $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, $\mathfrak{B} \subset \mathfrak{P}$ can be implemented as the controlled behavior both for $\mathfrak{B}_{\textbf{plant}}$ and $\mathfrak{B}'_{\textbf{plant}}$. For $\mathfrak{B}'_{\textbf{plant}}$, choose the controller $\mathfrak{B}_{\text{controller}} = \mathfrak{B}$. Note that since w is observable from c, $\mathfrak{B}_{\textbf{plant}}$ admits a kernel representation of the form $w = M(\frac{d}{dt})c$, $N(\frac{d}{dt})c = 0$. Let $R(\frac{d}{dt})w = 0$ be a kernel representation of $\mathfrak{B}$. Use the controller $C(\frac{d}{dt})c = 0$ with $C = RM$ on $\mathfrak{B}_{\text{plant}}$. The controlled behavior is described by $w = M(\frac{d}{dt})c$, $N(\frac{d}{dt})c = 0$, $R(\frac{d}{dt})M(\frac{d}{dt})c = 0$. Eliminate c, and use the inclusion $\mathfrak{B} \subset \mathfrak{P}$ to obtain $R(\frac{d}{dt})w = 0$ as the kernel representation of the controlled behavior, which hence equals $\mathfrak{B}$.

3. Assume that the plant is described by

$$\frac{d}{dt}x = Ax + Bu + Gd_1, y = Cx + d_2, z = Hx, c = (u,y), w = (d_1, d_2, u, z).$$

Note that, as in this example, often some of control variables (in casu the u's) will also be to-be-controlled variables: as in LQ-control, they will appear in the performance functional.

$\mathfrak{N}$ has the representation

$$\frac{d}{dt}x = Ax + Gd_1, \qquad 0 = Cx + d_2, \qquad z = Hx, \qquad w = (d_1, d_2, 0, x).$$

x is a latent variable (in fact, this is a state representation of $\mathfrak{N}$).

Give a representation of $\mathfrak{N}$ and $\mathfrak{P}$ for this system.

$\mathfrak{P}$ has the representation

$$\frac{d}{dt}x = Ax + Bu + Gd_1, \qquad z = Hx, \qquad w = (d_1, d_2, u, x).$$

x is again a latent variable (in fact, this is again a state representation, this time of $\mathfrak{P}$).