Solutions Exercises Set 2

Exercise 1 (Time-reversibility)

The aim of this exercise is (i) to let you think of the nature of differential systems and (ii) use the powerful theorem on the structure of minimal kernel representations in a simple but meaningful application.

The time-invariant dynamical system $\Sigma = (\mathbb{R}, \mathbb{W}, \mathfrak{B})$ is said to be *time-reversible* if $w \in \mathfrak{B}$ implies $\text{rev}(w) \in \mathfrak{B}$, where $\text{rev}(w)$ is defined by $\text{rev}(w)(t) := w(-t)$.

1. Do Kepler's laws define a time-reversible system?

Kepler's laws define the system $(\mathbb{R}, \mathbb{R}^3, \mathfrak{B})$ with $w \in \mathfrak{B}$ iff w is periodic and satisfies:

K.1 the set $\{ v \in \mathbb{R}^3 \mid \exists\, t \in \mathbb{R} : v = w(t) \}$ is an elliptic with the sun (at a fixed point, say $0 \in \mathbb{R}^3$), in one of the foci,

K.2 the vector $w(t) \in \mathbb{R}^3$ from the sun to the planet sweeps out equal areas in equal times,

K.3 $\dfrac{(\text{period})^2}{(\text{major axis of the ellipse}^3}$ is a universal constant (i.e., the same for all planets).

Comment: It is hard not to become filled with awe every time one writes this down: Kepler deduced these laws - highly accurate and exact under very reasonable idealizations - from the mere observation of about half a dozen cases!

Now consider $\text{rev}(w)$. Obviously $w(-t)$ also sweeps out the same ellipse, but in opposite direction, with equal areas in equal times, and with the same period. Whence, $\text{rev}(w)$ satisfies K.1, K.2, and K.3. Therefore Kepler's laws define a time-reversible system.

Let $f : \mathbb{R}^{\mathtt{w}(\mathtt{n}+1)} \to \mathbb{R}^{\mathtt{m}}$ and consider the behavioral differential equation

$$f \circ (w, \frac{d}{dt}w, \ldots, \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}}w) = 0.$$

Precisely,

$$\mathfrak{B} = \{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid f(w(t), \frac{d}{dt}w(t), \ldots, \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}}w(t)) = 0 \ \ \forall\, t \in \mathbb{R} \}.$$

2. Prove that this defines a time-reversible system if 'f contains only even derivatives'. Make precise what ' ' means. Use mathematical language, not prose.

The question occurs: *Is this also necessary?* But this is asking the impossible, even for linear differential systems, in view of the highly non-uniqueness of behavioral equations. A better question is therefore: *Can a time-reversible system always be represented by a system of differential equations which contains only even order derivatives?* It turns out that this a very good question, and that the answer is in the affirmative for controllable linear differential systems. In this exercise, we tip the curtain for systems described by *one* linear differential equation.

What does it mean that the differential equations defined by

$$f : (a_0, a_1, \dots, a_{\mathtt{n}}) \in \mathbb{R}^{\mathtt{w}(\mathtt{n}+1)} \to \mathbb{R}^{\mathtt{w}}$$

contains only even derivatives? The easiest way to answer is: there must exist

$$g : (a_0, a_1, \dots, a_{\mathbf{int}(\mathtt{n}/2)}) \in \mathbb{R}^{\mathtt{w}(\mathbf{ent}(\mathtt{n}/2)+1)} \to \mathbb{R}^{\mathtt{w}}$$

(ent(x):= the largest integer less than or equal to x), such that

$$f(a_0, a_1, \dots, a_{\mathtt{n}}) = g(a_0, a_1, \dots, a_{\mathbf{ent}(\mathtt{n}/2)})$$

for all $(a_0, a_1, \dots, a_{\mathtt{n}}) \in \mathbb{R}^{\mathtt{w}}$.

Let $\mathfrak{B}$ be governed by

$$g \circ \Big(w, \frac{d^2}{dt^2} w, \dots, \frac{d^{2\mathtt{n}'}}{dt^{2\mathtt{n}'}} w \Big) = 0.$$

Assume that $w \in \mathfrak{B}$. Observe that $\frac{d}{dt}\,\mathbf{rev}(w) = -\mathbf{rev}(\frac{d}{dt}\,w)$, whence $\frac{d^{2\mathtt{k}}}{dt^{2\mathtt{k}}}\,\mathbf{rev}(w) = \mathbf{rev}(\frac{d^{2\mathtt{k}}}{dt^{2\mathtt{k}}}\,w)$. Therefore

$$\begin{aligned} w \in \mathfrak{B} \Leftrightarrow g \circ \Big(w, \frac{d^2}{dt^2} w, \dots, \frac{d^{2\mathtt{n}'}}{dt^{2\mathtt{n}'}} w \Big) = 0 \Leftrightarrow g\Big(w(t), \frac{d^2}{dt^2} w(t), \dots, \frac{d^{2\mathtt{n}'}}{dt^{2\mathtt{n}'}} w(t) \Big) = 0 \quad \forall t \in \mathbb{R} \\ \Leftrightarrow g\Big(w(-t), \frac{d^2}{dt^2} w(-t), \dots, \frac{d^{2\mathtt{n}'}}{dt^{2\mathtt{n}'}} w(-t) \Big) = 0 \quad \forall t \in \mathbb{R} \\ \Leftrightarrow g \circ \Big(\mathbf{rev}(w), \mathbf{rev}(\frac{d^2}{dt^2} w), \dots, \mathbf{rev}(\frac{d^{2\mathtt{n}'}}{dt^{2\mathtt{n}'}} w) \Big) = 0 \Leftrightarrow \mathbf{rev}(w) \in \mathfrak{B}. \end{aligned}$$

Whence, $w \in \mathfrak{B}$ iff $\mathbf{rev}(w) \in \mathfrak{B}$. Hence the system defined by $\mathfrak{B}$ is time-reversible.

3. Let $p \in \mathbb{R}[\xi]$. Prove that the system (in $\mathfrak{L}^1$) described by

$$p(\frac{d}{dt})w = 0$$

is time-reversible if and only if p is either an even or an odd polynomial.

Hint: in the time-reversible case, $p(-\frac{d}{dt})w = 0$ is also a kernel representation.

Let $\mathfrak{B}$ be described by

$$p(\frac{d}{dt})w = 0$$

with $p \neq 0$ (treat $p = 0$ separately). Then (see the proof of 2) rev($\mathfrak{B}$) is described by

$$p(-\frac{d}{dt})w = 0.$$

By the structure theorem for kernel representations, $\mathfrak{B} = \mathbf{rev}(\mathfrak{B})$ iff there exists a unimodular $U \in \mathbb{R}[\xi]$, such that

$$p(-\xi) = U(\xi)p(\xi).$$

But $U \in \mathbb{R}[\xi]$ is unimodular iff U is a nonzero constant, say α. Hence $\mathfrak{B} = \mathbf{rev}(\mathfrak{B})$ iff there exists $\alpha \neq 0$ such that

$$p(-\xi) = \alpha p(\xi).$$

But this implies that $\alpha = +1$ (if p has even degree), p is then even, or $\alpha = -1$ (if p has odd degree), p is then odd.

4. Let $p, q \in \mathbb{R}^{\mathtt{w}}[\xi]$. Prove that the system (in $\mathfrak{L}^2$) described by

$$p(\frac{d}{dt})w_1 = q(\frac{d}{dt})w_2$$

is time-reversible if and only if p and q are either both even or both odd polynomials.

Repeat, *mutatis mutandis,* the proof of 3.

5. Assume in addition that p and q are co-prime. We will see that this means that

$$p(\frac{d}{dt})w_1 = q(\frac{d}{dt})w_2$$

describes a controllable system. Prove that time-reversibility then implies that p and q are both even.

If p and q are co-prime, then they can not be both odd (since ξ is then a common factor of p and q). Therefore, a controllable linear system

$$p(\frac{d}{dt})w_1 = q(\frac{d}{dt})w_2$$

is time-reversible iff this differential equation contains only even derivatives.

Comment: For the general multivariable case, this result becomes: $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is time-reversible iff it admits a (minimal) kernel representation of the form

$$R_+(\frac{d}{dt})w = 0, \qquad R_-(\frac{d}{dt})w = 0,$$

with R_+ even and R_- odd. $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ controllable is time-reversible iff it allows a (minimal) kernel representation of the form

$$R(\frac{d}{dt})w = 0,$$

with R even.

Exercise 2 (Discrete-time systems)

The aim of this exercise is (i) to generalize some of the results on linear differential equations to difference equations and (ii) to illustrate the use of Laurent polynomials and the small difference regarding unimodularity that occurs.

In the land of difference equations some people like to use forward differences, others use backward differences, and some think that using z- transforms is the solution. The most convenient thing (when working on the time-axis $\mathbb{Z}$) is to use both forward and backward lags.

A (real) *Laurent polynomial* is a 'polynomial' that contains both positive and negative powers of ξ, i.e., an expression of the type

$$\Sigma_{\mathtt{k}\in\mathbb{Z}} p_{\mathtt{k}}\xi^{\mathtt{k}}$$

with the $p_{\mathtt{k}}$'s $\in \mathbb{R}$, and all but a finite number of them zero. The set of real Laurent polynomials is denoted by $\mathbb{R}[\xi,\xi^{-1}]$. Under the obvious definitions of addition and multiplication, $\mathbb{R}[\xi,\xi^{-1}]$ becomes a ring.

Note this instance of the the strange habit of mathematicians to associate the names of their heros with trivialities. It stands to reason that if Laurent would have wanted to be remembered, it would have been for more that the suggestion that he introduced polynomials with negative powers.

1. An element u of a ring R with an identity 1 is said to be *unimodular* if there exists $v \in R$ such that $uv = vu = 1$. Which elements of $\mathbb{R}[\xi,\xi^{-1}]$ are unimodular? Which elements of $\mathbb{R}^{\mathtt{n}\times\mathtt{n}}[\xi,\xi^{-1}]$ are unimodular? Contrast this with the unimodular elements of $\mathbb{R}[\xi], \mathbb{R}^{\mathtt{n}\times\mathtt{n}}[\xi]$.

Assume that (in the obvious notation)

$$(u_\ell\xi^\ell + u_{\ell+1}\xi^{\ell+1} + \cdots + u_L\xi^L)(v_{\ell'}\xi^{\ell'} + v_{\ell'+1}\xi^{\ell'+1} + \cdots + v_{L'}\xi^{L'}) = 1,$$

with u_ℓ, u_L, $v_{\ell'}$, $v_L \neq 0$. Then (equate degrees)

$$\ell + \ell' = 0 \qquad \textbf{and} \qquad L + L' = 0.$$

Whence, since $\ell \leq L, \ell' \leq L', \ell = L$ and $\ell' = L'$. Therefore, $u \in \mathbb{R}[\xi,\xi^{-1}]$ is unimodular iff it is of the form

$$u(\xi,\xi^{-1}) = \alpha\xi^k$$

with $0 \neq \alpha \in \mathbb{R}$ and $k \in \mathbb{Z}$.

For $U, V \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}[\xi,\xi^{-1}]$, observe that

$$UV = I \quad \Rightarrow \quad \det(U)\det(V) = 1.$$

Therefore $U \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}[\xi,\xi^{-1}]$unimodular implies $\det(U) \in \mathbb{R}[\xi,\xi-1]$ unimodular. Conversely, if $\det(U) \in \mathbb{R}[\xi,\xi-1]$ is unimodular, then

$$V = \big(\det(U)\big)^{-1}\mathbf{cof}(U)$$

(cof(U) denotes the matrix of co-factors, defined as in the case of real matrices) is its inverse.

<u>Conclusion</u>: $U \in \mathbb{R}^{n\times n}[\xi,\xi^{-1}]$ is unimodular iff $\det(U(\xi)) = \alpha\xi^k$ with $0 \neq \alpha \in \mathbb{R}$ and $k \in \mathbb{Z}$. In contrast, $U \in \mathbb{R}^{n\times n}[\xi]$ is unimodular iff $\det(U(\xi)) = \alpha$ with $0 \neq \alpha \in \mathbb{R}$. The ring $\mathbb{R}^{n\times n}[\xi,\xi^{-1}]$ has many more unimodular elements than $\mathbb{R}^{n\times n}[\xi]$.

2. Let σ denote, as usual, the *shift*: $\sigma(f)(t) := f(t+1)$. Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi, \xi^{-1}]$ and consider the system of difference equations

$$R(\sigma, \sigma^{-1})w = 0. \tag{1}$$

This defines the dynamical system $\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$. Define $\mathfrak{B}$ formally.

$$\mathfrak{B} := \{w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} \mid R(\sigma, \sigma^{-1})w = 0\}.$$

Note that $\mathfrak{B} = \ker(R(\sigma, \sigma^{-1}))$ **with** $R(\sigma, \sigma^{-1})$ **is viewed as a map from** $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ **to** $(\mathbb{R}^{\mathrm{rowdim}(R)})^{\mathbb{Z}}$**.**

3. Prove that Σ is linear and time-invariant. Prove that it is also *complete*, meaning that it has the property that a trajectory is 'legal' (i.e. belongs to the behavior) iff all its finite windows are 'legal'. Formally $\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ is said to be *complete* if for $w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}}$ there holds

$$[w \in \mathfrak{B}] \Leftrightarrow [w|_{[\mathtt{k}_1, \mathtt{k}_2]} \in \mathfrak{B}|_{[\mathtt{k}_1, \mathtt{k}_2]} \;\; \forall\, \mathtt{k}_1, \mathtt{k}_2 \in \mathbb{Z}, \mathtt{k}_1 \leq \mathtt{k}_2].$$

As usual, $w|_{[\mathtt{k}_1, \mathtt{k}_2]}$, $\mathfrak{B}|_{[\mathtt{k}_1, \mathtt{k}_2]}$ denotes the map w or maps $\in \mathfrak{B}$ with domain restricted to the interval $[\mathtt{k}_1, \mathtt{k}_2]$ (in $\mathbb{Z}$).

Let $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ **with** $\mathfrak{B} = \ker(R(\sigma, \sigma^{-1}))$**.**

3.1 Σ **is linear. Indeed,** $\mathbb{R}^{\mathtt{w}}$ **is a vector space over** $\mathbb{R}$ **and** $\mathfrak{B}$ **is the kernel of a linear map.**

3.2 Σ **is time-invariant. Indeed,**

$$\begin{aligned}
w \in \mathfrak{B} &\Leftrightarrow R(\sigma, \sigma^{-1})w = 0 && (w \in \mathfrak{B}) \\
&\Leftrightarrow \sigma R(\sigma, \sigma^{-1})w = 0 && (f = 0 \Leftrightarrow \sigma f = 0) \\
&\Leftrightarrow R(\sigma, \sigma^{-1})\sigma w = 0 && (\sigma R(\sigma, \sigma^{-1}) = R(\sigma, \sigma^{-1})\sigma) \\
&\Leftrightarrow \sigma w \in \mathfrak{B}.
\end{aligned}$$

3.3 Σ **is complete. Let** $R(\xi, \xi^{-1}) = R_\ell \xi^\ell + R_{\ell+1}\xi^{\ell+1} + \cdots + R_L \xi^L$**. Define**

$$\mathbb{L} := \{(a_0, a_1, \ldots, a_{L-\ell+1}) \in \mathbb{R}^{\mathtt{w}(L+\ell-1)} \mid R_\ell a_0 + R_{\ell+1} a_1 + \cdots + R_L a_{L-\ell+1} = 0\}.$$

It is trivial to see that

$$w \in \mathfrak{B} \Rightarrow w|_{[k_1, k_2]} \in \mathfrak{B}|_{(k_1, k_2)}.$$

Conversely,

$$\begin{aligned}
w \in \mathfrak{B} &\Leftrightarrow R_\ell w(t+\ell) + R_{\ell+1} w(t+\ell+1) + \cdots + R_L w(t+L) = 0 \quad \textbf{for all } t \in \mathbb{Z} \\
&\Leftrightarrow \big(w(t+\ell), w(t+\ell+1), \ldots, w(t+L)\big) \in \mathbb{L} \quad \textbf{for all } t \in \mathbb{Z}.
\end{aligned}$$

Hence, if

$$w|_{(k_1, k_2)} \in \mathfrak{B}|_{[k_1, k_2]} \quad \textbf{for all } k_1, k_2,$$

then

$$w|_{[t, t+L-\ell+1]} \in \mathfrak{B}|_{[k, k+L-\ell+1]} \quad \textbf{for all } t \in \mathbb{Z},$$

and

$$\begin{aligned}
&\big(w(t+\ell), w(t+\ell+1), \ldots, w(t+L)\big) \in \mathbb{L} \quad \textbf{for all } t \in \mathbb{Z} \\
\Leftrightarrow \quad & R(\sigma, \sigma^{-1}) = 0 \Leftrightarrow w \in \mathfrak{B}.
\end{aligned}$$

Actually, there holds that $\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ is described by a system of difference equations (1) for some $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi, \xi^{-1}]$ *if and only if* it is linear, time-invariant, and complete. This allows to discuss difference equations without introducing difference equations, if you know what I mean. A similar characterization for linear time-invariant differential systems does not exist, even though some fine minds have broken their head over it.

Comment: A (wrong) similar characterization for $\mathbb{T} = \mathbb{R}$ could have been

$$\boxed{\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}} \text{ iff } \mathfrak{B} \text{ is linear, time-invariant, and locally specified.}}$$

($\mathfrak{B}$ is *locally specified* $:\Leftrightarrow$ $w \in \mathfrak{B}$ iff there is $\varepsilon > 0$ such that $w|_{[t,t+\varepsilon]} \in \mathfrak{B}|_{[t,t+\varepsilon]}$ for all $t \in \mathbb{R}$.)

It is easy to see that $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ implies that $\mathfrak{B}$ is locally specified. However, the converse is unfortunately not correct. Hence differential equations define locally specified behaviors: a trajectory is legal iff all its germs (the behavior on an arbitrarily small sliding interval) are legal. But there are locally specified behaviors that do not come from a differential equation! See the footnote in the paper: J.C. Willems, *Paradigms and puzzles in the theory of dynamical systems,* IEEE Transactions on Automatic Control, Volume 36, pages 259-294, 1991.

4. State the analog for kernel representations (1) of the structure theorem for kernel representations of linear differential systems. Define carefully all the terms. You need not prove the result. As Fermat demonstrated, with the help of Wiles: *If you can state it correctly, you need not prove it!*

1. **Let $U \in \mathbb{R}^{\bullet \times \bullet}[\xi, \xi^{-1}]$ be unimodular. Then $R(\sigma, \sigma^{-1})w = 0$ and $U(\sigma, \sigma^{-1})R(\sigma, \sigma^{-1})w = 0$ have the same behavior.**
2. **Let $R_1(\sigma, \sigma^{-1})w_1 = 0$ have behavior $\mathfrak{B}_1$, and $R_2(\sigma, \sigma^{-1})w_2 = 0$ have behavior $\mathfrak{B}_2$. Then $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ iff there is $F \in \mathbb{R}^{\bullet \times \bullet}[\xi, \xi^{-1}]$ such that $R_2 = F R_1$.**
3. **$R(\sigma, \sigma^{-1})w = 0$ is minimal iff R is of full row rank.**
4. **Let $R(\sigma, \sigma^{-1})w = 0$. All minimal kernel representations with the same behavior are obtained by pre-multiplying R by an arbitrary unimodular polynomial matrix of suitable dimension.**

5. Does all this imply that any linear time-invariant complete system $(\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ admits a (minimal) representation of the form

$$R(\sigma)w = 0$$

for some $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ and one of the form

$$R(\sigma^{-1})w = 0$$

for some $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$?

Note: in a hairsplitting sense, this notation is a bit sloppy. Better would have been to say: Prove that there is a representation involving only forward difference, and one involving only backward differences (ordinary mortals call these *lags*).

Let $R(\sigma, \sigma^{-1}) = R_\ell \xi^\ell + R_{\ell+1}\xi^{\ell+1} + \cdots + R_L \xi^L$. Define R_+ and R_- by

$$R_+(\sigma, \sigma^{-1}) := \xi^{-\ell} R(\sigma, \sigma^{-1}), \qquad R_-(\sigma, \sigma^{-1}) := \xi^{-L} R(\sigma, \sigma^{-1}).$$

Obviously then $R(\sigma, \sigma^{-1})w = 0$, $R_+(\sigma, \sigma^{-1})w = 0$, and $R_-(\sigma, \sigma^{-1})w = 0$ define the same system but $R_+(\sigma, \sigma^{-1})$ contains only forward differences (in a sense $R_+ \in \mathbb{R}^{\bullet \times \mathtt{w}}(\xi)$) and $R_- \in \mathbb{R}^{\bullet \times \mathtt{w}}(\sigma, \sigma^{-1})$ contains only backward differences (in a sense $R_- \in \mathbb{R}[\xi^{-1}]$).

6. Is the appropriate setting for linear difference equations $(\mathbb{Z}_+, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ (time-axis $\mathbb{Z}_+$, or $\mathbb{N}$ if you like) ordinary (matrix) polynomials or Laurent (matrix) polynomials?

If the time-axis is $\mathbb{T} = \mathbb{Z}_+$, then $(\sigma^{-1}w)(0)$ is not defined (it is unwarranted, of course, to set $(\sigma^{-1}w)(0) = 0$)! Hence for $\mathbb{T} = \mathbb{Z}_+$, $\mathbb{R}[\xi]$ and difference equations with forward differences only are called for. So, no Laurent when $\mathbb{T} = \mathbb{Z}_+$.