Solutions Exercises Set 1

Exercise 1 (Linear static models)

The aim of this exercise is (i) to let you think of the nature of behavioral equations (in the context of systems without dynamics) and (ii) to show some parallels between behavioral equation representations of static linear systems and those for linear differential systems studied later in the course.

Pre-amble. Equip $\mathbb{R}^{\mathtt{n}}$ with the standard Euclidean inner product. Let $R \in \mathbb{R}^{\bullet \times n}$. Then $\ker(R) = \mathfrak{B}$ iff the transposes of the rows of R form a set of generators of $\mathfrak{B}^{\perp}$. Let $M \in \mathbb{R}^{m \times \bullet}$. Then $\mathrm{im}(M) = \mathfrak{B}$ iff the columns of M form a set of generators of $\mathfrak{B}$.

Let $(\mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ be a linear mathematical model (this is *newspeak*, it means nothing else than that $\mathfrak{B}$ is a linear subset of $\mathbb{R}^{\mathtt{w}}$).

1. (a) Prove that $\mathfrak{B}$ admits a behavioral equation representation

$$Rw = 0 \tag{1}$$

with $R \in \mathbb{R}^{\bullet \times \mathtt{w}}$.

Let $\{ r_1, r_2, \ldots, r_k \}$ be a set of generators of $\mathfrak{B}^{\perp}$ and take

$$R = \mathrm{col}(r_1^{\top}, r_2^{\top}, \ldots, r_k^{\top}).$$

Call (1) a *kernel representation* of $\mathfrak{B}$, and a *minimal kernel representation* of $\mathfrak{B}$ if, among all such kernel representations of $\mathfrak{B}$, $\mathrm{rowdim}(R)$ is as small as possible.

(b) What is the relation between $\mathrm{rank}(R)$ and $\dim(\mathfrak{B})$?

$$\mathrm{rank}(R) = \dim(\mathfrak{B}^{\perp}) = n - \dim(\mathfrak{B})$$

(c) What is the relation between $\mathrm{rowdim}(R)$ and $\dim(\mathfrak{B})$ if (1) is a minimal kernel representation?

(1) is a minimal kernel representation of $\mathfrak{B}$ iff the transposes of its columns form a basis for $\mathfrak{B}^{\perp}$. Hence in this case

$$\mathrm{rowdim}(R) = \mathrm{rank}(R) = n - \dim(\mathfrak{B}).$$

(d) Prove that (1) is a minimal kernel representation iff the matrix R has full row rank.
full row rank := rank = rowdim.

The transposes of the columns of R form a basis for $\mathfrak{B}^{\perp}$ iff the columns are linearly independent. Hence (1) is a minimal kernel representation iff R is of full row rank.

(e) Is it true that if (1) is not minimal, then you can simply cancel equations from (1), i.e., delete rows from R, without changing $\mathfrak{B}$? Reflect on the analogy with differential systems.

Let $Rw = 0$ be a kernel representation of $\mathfrak{B}$, $R = \mathrm{col}(r_1^\top, r_2^\top, \dots, r_{\mathrm{rowdim}(R)}^\top)$. Then $\{r_1^\top, r_2^\top, \dots, r_{\mathrm{rowdim}(R)}^\top\}$ forms a set of generators of $\mathfrak{B}^\perp$. There is then a subset $\{r'_1, r'_2, \dots, r'_k\}$ that forms a basis for $\mathfrak{B}^\perp$. Define

$$R' := \mathrm{col}\big((r'_1)^\top, (r'_2)^\top, \dots, (r'_k)^\top\big).$$

Then $R'w = 0$ is a minimal kernel representation of $\mathfrak{B}$ that has been obtained by deleting rows from R.

Comment: For linear differential systems, it is, in contrast, in general not possible to obtain a minimal kernel representation by simply cancelling equations in a non-minimal kernel representation.

(f) Prove that if (1) is a minimal kernel representation of $\mathfrak{B}$, then $R'w =$ is another kernel representation of the same behavior iff $R' = UR$ with $U \in \mathbb{R}^{\bullet\times\bullet}$ non-singular.

Follows from the observation that $\{r_1, r_2, \dots, r_{\dim(\mathfrak{B}^\perp)}\}$ is a basis for $\mathfrak{B}^\perp$, then $\{r'_1, r'_2, \dots, r'_{\dim(\mathfrak{B}^\perp)}\}$ is another basis iff there exists a nonsingular matrix $V \in \mathbb{R}^{\dim(\mathfrak{B}^\perp)\times\dim(\mathfrak{B}^\perp)}$, such that $[r_1\ r_2 \cdots\ r_{\dim(\mathfrak{B}^\perp)}] = [r'_1\ r'_2 \cdots\ r'_{\dim(\mathfrak{B}^\perp)}]V$.

2. (a) Prove that $\mathfrak{B}$ admits a behavioral equation representation

$$w = M\ell \tag{2}$$

with $M \in \mathbb{R}^{\mathtt{w}\times\bullet}$ and ℓ a latent variable.

Let $\{m_1, m_2, \dots, m_k\}$ be a set of generators of $\mathfrak{B}$ and take $M = [m_1\ m_2 \cdots m_k]$. Obviously, then, $\mathfrak{B} = \mathrm{im}(M)$. Hence (2) is an image representation of $\mathfrak{B}$.

Call (2) an *image representation* of $\mathfrak{B}$, and a *minimal image representation* of $\mathfrak{B}$ if, among all image representations of $\mathfrak{B}$, $\dim(\ell) = \mathrm{coldim}(M)$ is as small as possible.

(b) Formulate the analogs of the above results obtained for kernel representations, for image representations.

(b') $\mathrm{rank}(M) = \dim(\mathfrak{B})$.

(c') (2) is a minimal image representation of its image $\mathfrak{B}$ iff $\mathrm{coldim}(M) = \dim(\mathfrak{B})$.

(d') (2) is a minimal image representation iff M is of full column rank.

(e') If (2) is not a minimal image representation of its image $\mathfrak{B}$, then a minimal one is obtained by cancelling suitable columns of M.

(f') If (2) is a minimal image representation of its image $\mathfrak{B}$, then $w = M'\ell'$ is another minimal image representation of $\mathfrak{B}$ iff there exists a non-singular matrix V such that $M' = MV$.

3. Prove that the manifest behavior of the latent variable behavioral equation representation

$$Rw = M\ell \tag{3}$$

with ℓ a latent variable, $R \in \mathbb{R}^{\bullet \times \mathtt{w}}, M \in \mathbb{R}^{\mathtt{w} \times \bullet},$ and $\text{rowdim}(R) = \text{rowdim}(M)$ is linear. (3) is the natural outcome (emphasized throughout the course, and illustrated, e.g., by exercise 2) of a first principles modeling procedure, and has obviously both kernel and image representations as special cases.

Let w_1, w_2 belong to the manifest behavior. Hence there are ℓ_1, ℓ_2, such that $Rw_1 = M\ell_1$, $Rw_2 = M\ell_2$. Let $\alpha, \beta \in \mathbb{R}$. Then $R(\alpha w_1 + \beta w_2) = M(\alpha \ell_1 + \beta \ell_2)$. Hence $\alpha w_1 + \beta w_2$ belongs to the manifest behavior, which is hence a linear subspace of $\mathbb{R}^{\mathtt{n}}$.

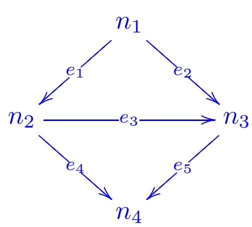
Exercise 2 (Resistive circuits)

The aim of this exercise is (i) to convince you that first principles models invariably lead to behavioral equations containing latent variables, and (ii) to present a formal mathematical setting for obtaining models of resistive circuits.

A *digraph* is a triple $(\mathfrak{N}, \mathfrak{E}, \pi)$, with $\mathfrak{N}$ a (finite) set, the set of *nodes*, $\mathfrak{E}$ a (finite) set, the set of *edges*, and $\pi : \mathfrak{E} \to \mathfrak{N}^2$ the *incidence map*. If $\pi(e) = (n_1, n_2)$, then we call n_1 the *source* and n_2 the *sink* of e.

1. Draw a not-too-trivial digraph, and specify the associated $\mathfrak{N}, \mathfrak{E},$ and π.

The notion of a digraph is standard, although often the sloppy definition in which $\mathfrak{E}$ is viewed as a subset of $\mathfrak{N}^2$ is preferred above the above accurate one. This follows the time-honored pedagogical principle that confusion is good. Naturally, 'good' means less work and less thinking for professors ...

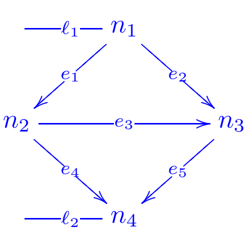


$\mathfrak{N} = \{ n_1, n_2, n_3, n_4 \}$
$\mathfrak{E} = \{ e_1, e_2, e_3, e_4, e_5 \}$

$\pi(e_1) = (n_1, n_2)$
$\pi(e_2) = (n_1, n_3)$
$\pi(e_3) = (n_2, n_3)$
$\pi(e_4) = (n_2, n_4)$
$\pi(e_5) = (n_3, n_4)$

Less standard is the notion of a *digraph with leaves.* It is a quintuple $(\mathfrak{N}, \mathfrak{E}, \mathfrak{L}, \pi, \nu)$, with $(\mathfrak{N}, \mathfrak{E}, \pi)$ a digraph, $\mathfrak{L}$ a (finite) set, the set of *leaves*, and $\nu : \mathfrak{L} \to \mathfrak{N}$ the *leave incidence map.* A graph with leaves is thus a graph in which some 'edges' (called leaves) are connected to only one node. Note that there is no point in giving a direction to the leaves.

2. Draw a not-too-trivial digraph with leaves, and specify the associated $\mathfrak{N}, \mathfrak{E}, \mathfrak{L}, \pi,$ and ν.



$\mathfrak{N} = \{ n_1, n_2, n_3, n_4 \}$
$\mathfrak{E} = \{ e_1, e_2, e_3, e_4, e_5 \}$
$\mathfrak{L} = \{ \ell_1, \ell_2 \}$

$\nu(\ell_1) = n_1$
$\nu(\ell_2) = n_2$

$\pi(e_1) = (n_1, n_2)$
$\pi(e_2) = (n_1, n_3)$
$\pi(e_3) = (n_2, n_3)$
$\pi(e_4) = (n_2, n_4)$
$\pi(e_5) = (n_3, n_4)$

We view a resistive circuit as a black box with a number of wires, external terminals, sticking out of it and through which the circuit can be connected to the environment. Inside the black box there are resistors, connected to each other and to the external terminals. We think of this interconnection pattern as an 'architecture'. The aim of the modeling is to describe the behavior of the voltages (potentials) and currents at the external terminals. We will assume that the currents are positive when they flow into the black box. This avoids having to define directions at the external terminals. For simplicity, we consider only linear resistors, although many of the ideas are just as well valid for the nonlinear case.

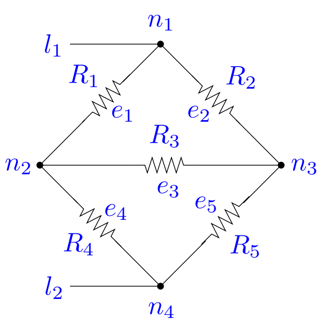
We formally define a resistive circuit as a digraph with leaves $(\mathfrak{N}, \mathfrak{E}, \mathfrak{L}, \pi, \nu)$, called the *interconnection archi-*

tecture, and a map $\rho : \mathfrak{E} \to [0, \infty)$, called the *resistance assignment*.

Intuition:

- $\mathfrak{L}$: the external terminals.
- $\mathfrak{E}$: the internal branches with the resistors. Note that since we deal with a digraph (and not simply with a graph) we have somehow chosen a positive direction for the currents in these branches. (I do not like this aspect, since this direction is not present 'in nature'. It is possible to avoid it, but this is somewhat 'heavy' for circuits containing only 2-terminal elements, as is the case studied here).
- $\rho(e)$: tells what the value of the resistance (say in ohms) is in the branch corresponding to edge e .
- all the edges and leaves incident to the same node are assumed connected (think 'soldered together').

3. Draw a not-too-trivial resistive circuit, and specify the associated $\mathfrak{N}, \mathfrak{E}, \mathfrak{L}, \pi, \nu$ and ρ.



$\mathfrak{N} = \{ n_1, n_2, n_3, n_4 \}$
$\mathfrak{E} = \{ e_1, e_2, e_3, e_4, e_5 \}$
$\mathfrak{L} = \{ \ell_1, \ell_2 \}$

$\nu(l_1) = n_1$
$\nu(\ell_2) = n_2$

$\pi(e_1) = (n_1, n_2)$
$\pi(e_2) = (n_1, n_3)$
$\pi(e_3) = (n_2, n_3)$
$\pi(e_4) = (n_2, n_4)$
$\pi(e_5) = (n_3, n_4)$

$\rho(e_k) = R_k, \quad k = 1, \ldots, 5$

<u>Comment:</u> This circuit is the "Wheatstone bridge". It is used to measure resistances by putting a voltage source across the external terminals, and measuring the current I_3 through edge e_3. This current is zero iff

$$R_5 = R_4 \frac{R_1}{R_2}.$$

Now, by putting a resistor of unknown value in edge e_5 and changing the value of the calibrating resistor R_4 in edge e_4 until $I_3 = 0$, the value of R_5 is obtained. R_1 and R_2 are assumed to be known. They are used to adapt the measuring range.

4. We are now ready to set up behavioral equations. Of course, we take as manifest variables the voltages (potentials) at, and the currents into, the external terminals. That is what the model aims at. But we need auxiliary variables to come up with a model. For the latent variables we take the voltages (potentials) at the internal nodes, at the connections, and the currents in the internal branches. Note that the assumption that the potentials of the nodes are well defined comes down to assuming *Kirchhoff's voltage law*.

Let $|\ |$ denote *cardinality*, i.e. $|S|$ denotes the number of elements of the (finite) set S.

In order to set up behavioral equations,

- Number the elements of $\mathfrak{L}$ as $\{1, 2, \ldots, |\mathfrak{L}|\}$, and denote the associated voltage and current vectors at the external terminals as $V = (V_1, V_2, \ldots, V_{|\mathfrak{L}|})$ and $I = (I_1, I_2, \ldots, I_{|\mathfrak{L}|})$. Denote the vector of manifest variables as $w = (V, I)$. Hence $\mathbb{W}$, the space of manifest variables, is $\mathbb{R}^{2|\mathfrak{L}|}$.
- Enumerate the elements of $\mathfrak{N}$ as $\{n_1, n_2, \ldots, n_{|\mathfrak{N}|}\}$, and those of $\mathfrak{E}$ as $\{e_1, e_2, \ldots, e_{|\mathfrak{E}|}\}$. Take for the space of latent variables $\mathbb{L} = \mathbb{R}^{|\mathfrak{N}|+|\mathfrak{L}|}$. The physical meaning of a typical element

  $$((V_{n_1}, V_{n_2}, \ldots, V_{n_{|\mathfrak{N}|}}), (I_{e_1}, I_{e_2}, \ldots, I_{e_{|\mathfrak{E}|}}))$$

  of $\mathfrak{L}$ is obvious from the notation.
- The full behavioral equations are given by
  1. *Kirchhoff's current laws*: For each $n \in \mathfrak{N}$ the following equation holds

     $$\sum_{\{e \in \mathfrak{E} | n \text{ is the sink of } e\}} I_e + \sum_{\{t \in \mathfrak{L} | t \text{ is incident to } n\}} I_t = \sum_{\{e \in \mathfrak{E} | n \text{ is the source of } e\}} I_e.$$

  2. *Compatibility*, or *Kirchhoff's voltage laws*, if you like: For each $\ell \in \mathfrak{L}$, the following equation holds

     $$V_\ell = V_{\nu(\ell)}$$

  3. *Constitutive equations*: For each $e \in \mathfrak{E}$ with $(n_1, n_2) = \pi(e)$, the following equation holds

     $$V_{n_1} + \rho(e) I_e = V_{n_2}.$$

Set up these equations for the circuit which you defined in 3.

In the sequel you may consider either the circuit which you defined in 3, or a completely general linear resistive circuit. This illustrates that the general, if treated with the proper notation, is often easier than the specific.

In order to answer questions 4 to 9, it is useful to introduce, for a digraph with leaves, two matrices, E, the *edge incidence matrix,* and L, the *leave incidence matrix.* E is an $|\mathfrak{E}| \times |\mathfrak{N}|$ matrix, defined as follows. Let $\pi = (\pi_1, \pi_2)$, then E_{ij}, the (i, j)-th element of E, is equal to

$$E_{ij} = \begin{cases} -1 & \textbf{if } n_j = \pi_1(e_i) \neq \pi_2(e_i) \\ +1 & \textbf{if } n_j = \pi_2(e_i) \neq \pi_1(e_i) \\ 0 & \textbf{otherwise.} \end{cases}$$

L is an $|\mathfrak{L}| \times |\mathfrak{N}|$ matrix, with L_{ij}, the (i, j)-th element of L, is equal to

$$L_{ij} = \begin{cases} +1 & \textbf{if } n_j = \nu(i) \\ 0 & \textbf{otherwise.} \end{cases}$$

Finally, let $R \in \mathbb{R}^{|\mathfrak{E}| \times |\mathfrak{E}|}$ be defined by

$$R = \operatorname{diag}\big(\rho(e_1), \rho(e_2), \ldots, \rho(e_{|\mathfrak{E}|})\big).$$

In terms of these matrices, the behavioral equations of a resistive electrical circuit may be written as

$$\textbf{KCL}: \quad E^\top I_{\mathfrak{E}} + L^\top I = 0 \qquad \text{(KCL)}$$

$$\textbf{VL}: \quad V = L V_{\mathfrak{N}} \qquad \text{(VL)}$$

$$\textbf{CE}: \quad E V_{\mathfrak{N}} + R I_{\mathfrak{E}} = 0 \qquad \text{(CE)}$$

In (KCL,VL,CE), (V, I) are the manifest variables and $(V_{\mathfrak{N}}, I_{\mathfrak{E}})$ are the latent variables, where $V_{\mathfrak{N}} = (V_{n_1}, V_{n_1}, \ldots, V_{n_{|\mathfrak{N}|}})$ and $V_{\mathfrak{E}} = (V_{e_1}, V_{e_1}, \ldots, V_{e_{|\mathfrak{E}|}})$.

For the Wheatstone bridge we have

$$E = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad R = \mathrm{diag}\big(R_1, R_2, R_2, R_3, R_4, R_5\big),$$

so that (KCL), (VL), and (CE) yield

$$\textbf{KCL:} \quad \begin{array}{rcrcrcl} -I_{e_1} & - & I_{e_2} & + & I_1 & = & 0 \\ I_{e_1} & - & I_{e_3} & - & I_{e_4} & = & 0 \\ I_{e_2} & + & I_{e_3} & - & I_{e_5} & = & 0 \\ I_{e_4} & + & I_{e_5} & + & I_2 & = & 0 \end{array}, \quad \textbf{VL:} \quad \begin{array}{l} V_1 = V_{n_1} \\ V_2 = V_{n_4} \end{array}, \quad \textbf{CE:} \quad \begin{array}{l} R_1 I_{e_1} = -V_{n_1} + V_{n_2} \\ R_2 I_{e_2} = -V_{n_1} + V_{n_3} \\ R_3 I_{e_3} = -V_{n_2} + V_{n_3} \\ R_4 I_{e_4} = -V_{n_2} + V_{n_5} \\ R_5 I_{e_5} = -V_{n_3} + V_{n_4} \end{array}$$

5. Let $\mathfrak{B}$ be the manifest behavior of the circuit defined by $((\mathfrak{N}, \mathfrak{E}, \mathfrak{L}, \pi, \nu), \rho)$. Prove that $\mathfrak{B}$ is a linear subspace of $\mathbb{R}^{2|\mathfrak{L}|}$.

Rather than proving 5–9 for the specific circuit at hand, we prove the claims in their full generality, using equations (KCL,VL,CE).

Since (KCL,VL,CE) are linear equations, linearity of the manifest behavior follows from exercise 1, part 3.

Let $\mathbf{1}_{\mathtt{n}}$ denote the $\mathtt{n}$-dimensional column vector with all 1's. Prove that

$$(V, I) \in \mathfrak{B} \quad \Rightarrow \quad \mathbf{1}_{|\mathfrak{L}|}^{\top} I = 0$$
$$(V, I) \in \mathfrak{B} \quad \Rightarrow \quad \forall\, \alpha \in \mathbb{R} : (V + \alpha \mathbf{1}_{|\mathfrak{L}|}, I) \in \mathfrak{B}$$

Note that $E\mathbf{1}_{|\mathfrak{N}|} = 0$ and $L\mathbf{1}_{|\mathfrak{N}|} = \mathbf{1}_{|\mathfrak{L}|}$. Assume that $(V, I) \in \mathfrak{B}$, i.e., there are $V_{|\mathfrak{N}|}$, $I_{|\mathfrak{E}|}$, such that (KCL,VL,CE) hold. Clearly (KCL) (premultiply (KCL) by $\mathbf{1}_{|\mathfrak{N}|}$) implies $\mathbf{1}_{|\mathfrak{L}|} I = 0$. Moreover, if $(V, I, V_{|\mathfrak{N}|}, I_{|\mathfrak{E}|})$ satisfies (KCL,VL,CE), so does $(V + \alpha\mathbf{1}_{|\mathfrak{L}|}, I, V_{|\mathfrak{N}|} + \alpha\mathbf{1}_{|\mathfrak{N}|}, I_{|\mathfrak{E}|})$. Hence $(V, I) \in \mathfrak{B}$ implies $(V + \alpha\mathbf{1}_{|\mathfrak{L}|}, I) \in \mathfrak{B}$.

6. Prove *passivity:* $(V, I) \in \mathfrak{B} \Rightarrow V^{\top} I \geq 0$.

If $(V, I, V_{\mathfrak{N}}, I_{\mathfrak{E}})$ satisfies (KCL,VL,CE), then

$$V^{\top} I = (L V_{\mathfrak{N}})^{\top} I = V_{\mathfrak{N}}^{\top} L^{\top} I = -V_{\mathfrak{N}}^{\top} E^{\top} I_{\mathfrak{E}} = -(E V_{\mathfrak{N}})^{\top} I_{\mathfrak{E}} = I_{\mathfrak{E}}^{\top} R I_{\mathfrak{E}}.$$

Since, $\rho \geq 0$ this implies $R = R^T \geq 0$, and hence $V^{\top} I \geq 0$.

7. Prove the $\dim(\mathfrak{B}) \leq |\mathfrak{L}|$. It can be shown that equality holds. Try it!

It is easy to see that if L_1, L_2 are linear subspaces of $\mathbb{R}^{\mathtt{n}}$, and $\dim(L_1) + \dim(L_2) > n$, then $L_1 \cap L_2 \neq \{0\}$. Now consider the subspace $\mathfrak{A}$ of $\mathbb{R}^{2|\mathfrak{L}|}$ consisting of all vectors of the form $(V, -V)$, $V \in \mathbb{R}^{|\mathfrak{L}|}$. Obviously $\dim(\mathfrak{A}) = |\mathfrak{L}|$, so if $\dim(\mathfrak{B}) > |\mathfrak{L}|$, there exists an $0 \neq a \in \mathfrak{A} \cap \mathfrak{B}$. Obviously $a = (V, I)$ has $I = -V$ hence $V^T I = -\|V\|^2 < 0$. This is in contradiction with 6.

Comment: The proof that for any passive linear circuit actually $\dim(\mathfrak{B}) = |\mathfrak{L}|$ (the *input cardinality property*) needs to be added.

8. Prove *reciprocity:* $(V_1, I_1), (V_2, I_2) \in \mathfrak{B} \Rightarrow V_1^\top I_2 = V_2^\top I_1$.

Assume that $(V_k, I_k) \in \mathfrak{B}$, $k = 1, 2$. Hence there exist $V_{\mathfrak{N},k}$, $I_{\mathfrak{N},k}$, $k = 1, 2$, such that $(V_k, I_k, V_{\mathfrak{N},k}, I_{\mathfrak{N},k})$ satisfy (KCL,VL,CE). Then

$$V_1^\top I_2 = (LV_{\mathfrak{N},1})^\top I_2 = V_{\mathfrak{N},1}^\top L^\top I_2 = -V_{\mathfrak{N},1}^\top E^\top I_{\mathfrak{E},2} = -(EV_{\mathfrak{N},1})^\top I_{\mathfrak{E},2} = (RI_{\mathfrak{E},1})^\top I_{\mathfrak{E},2} = I_{\mathfrak{E},1}^\top RI_{\mathfrak{E},2}$$

Hence $V_1^\top I_2 = I_{\mathfrak{E},1}^\top RI_{\mathfrak{E},2}$. By the same argument $V_2^\top I_1 = I_{\mathfrak{E},2}^\top RI_{\mathfrak{E},1}$. Since $R = R^T$, reciprocity follows.

9. Assume that $\mathfrak{B}$ admits an *admittance* representation, i.e., a behavioral representation of the form

$$I = AV.$$

Prove, using [6-8], that $A = A^\top \geq 0$ (≥ 0 means that the matrix is nonnegative definite).

If $I = AV$, 8 implies $V_1^\top AV_2 = V_2^\top AV_1$ for all $V_1, V_2 \in \mathbb{R}^{|L|}$. Hence $V_1^\top (A - A^\top)V_2 = 0$ for all $V_1, V_2 \in \mathbb{R}^{|L|}$. This obviously implies $A = A^\top$. In turn 6 implies $V^\top AV \geq 0$. Hence $A = A^\top \geq 0$.

This exercise shows that in order to model a resistive electrical circuit, input/output thinking is awkward, inconvenient, unnatural, and certainly not needed. Also, the derivation of important properties as passivity and reciprocity proceeds very smoothly using the behavioral framework. The behavioral framework is the only reasonable framework already in the case of simple linear memoryless systems, as resistive electrical circuits. There is consequently no reason why this should not be the case for more complex systems. Complexity does not give more structure, to the contrary!