Exercises Set 6

Exercise 1 (Abstract state construction and addition)

The aim of this exercise is (i) to contrast the behavioral definition of a (state space) system with earlier definitions, (ii) to illustrate the abstract state construction by specializing it to input/output maps, and (iii) to show a concrete example where the abstract state construction ideas gives very useful results.

1. In the classical theory of state space systems (see, for example, section 2.1 of the book "Topics in Mathematical System Theory" by R.E. Kalman, P.L. Falb, and M.A. Arbib) a 'classical' state space system is defined by

(i) A *state-transition map*

$$\phi : (\mathbb{T}^2)_+ \times \mathbb{X} \times \mathfrak{U} \to \mathbb{X},$$

where $(\mathbb{T}^2)_+ := \{(t_1, t_2) \in \mathbb{T}^2 \mid t_2 \geq t_1\}$ and $\mathfrak{U} \subset \mathbb{U}^{\mathbb{T}}$.

(ii) A *read-out map*

$$r : \mathbb{X} \times \mathbb{U} \times \mathbb{T} \to \mathbb{Y}.$$

Intuitively, $\phi(t_1, t_0, \mathrm{x}, u)$ is *'the state reached at time t_1 starting from state* x *at time t_0 by applying input u'*, and $r(\mathrm{x}, \mathrm{u}, t)$ is *'the value of the output when the system is in state* x*, the value of the input is $u(t) = \mathrm{u}$, and the time is t'.*

These satisfy a number of axioms:

1. $\mathfrak{U} \subset \mathbb{U}^{\mathbb{T}}$ is memoryless
2. $\forall t \in \mathbb{T}, \mathrm{x} \in \mathbb{X}, u \in \mathfrak{U} : \phi(t, t, \mathrm{x}, u) = \mathrm{x}$
3. $$[(t_1, t_0) \in (\mathbb{T}^2)_+, \mathrm{x} \in \mathbb{X}, u_1, u_2 \in \mathfrak{U}, \text{ and } u_1(t) = u_2(t) \text{ for } t_0 \leq t < t_1] \Rightarrow [\phi(t_1, t_0, \mathrm{x}, u_1) = \phi(t_1, t_0, \mathrm{x}, u_2)]$$
4. $[(t_2, t_1), (t_1, t_0) \in (\mathbb{T}^2)_+, \mathrm{x} \in \mathbb{X}, u \in \mathfrak{U}] \Rightarrow [\phi(t_2, t_0, \mathrm{x}, u) = \phi(t_2, t_1, \phi(t_1, t_0, \mathrm{x}, u), u)]$.

Note how much more complicated all this is, compared to our behavioral definition of state system.

1.1 Explain what the corresponding $\mathfrak{B}_{\text{full}}$ is. Prove that it satisfies the state space axiom.

1.2 Consider

$$\overset{\bullet}{\mathrm{x}} = f(\mathrm{x}, \mathrm{u}, t), \quad y = h(\mathrm{x}, \mathrm{u}, t).$$

Explain how this defines a classical state space system, under reasonable conditions on f and $\mathfrak{U}$. You need to specify a domain and co-domain of f, h, choose an appropriate $\mathfrak{U}$, and specify ϕ, r.

2. Let $\Sigma = (\mathbb{T}, \mathbb{U} \times \mathbb{Y}, \mathfrak{B})$ be a dynamical system (in the behavioral sense). Call Σ an *input/output map* if $\mathfrak{B}$ is the graph of a map $F : \mathfrak{U} \subset \mathbb{U}^{\mathbb{T}} \to \mathbb{Y}^{\mathbb{T}}$, i.e., if

$$\mathfrak{B} = \{(u, y) \mid u \in \mathfrak{U} \text{ and } y = F(u)\}.$$

Call it a *non-anticipating input/output map* if the map F is non-anticipating , i.e., if

$$[u_1, u_2 \in \mathfrak{U} \wedge t \in \mathbb{T} \wedge u_1(t') = u_2(t') \text{ for } t' \leq t] \Rightarrow [F(u_1)(t) = F(u_2)(t)].$$

2.1 Under what conditions on $\mathfrak{U}$ and F is an input/output map time-invariant?

Assume henceforth that $\mathfrak{U}$ is memoryless.

In order to avoid (smoothness) problems that are not germane to our aims, we assume in the remainder of this exercise that the time-axis is $\mathbb{R}$ or $\mathbb{Z}$ and that the systems under consideration are time-invariant.

2.2 Specialize the equivalence relation R_- encountered in the past-canonical state construction as discussed in the lectures in the context of behaviors, to the case of an input/output map. (Actually, it is this equivalence relation on $\mathfrak{U}$ or on $\mathfrak{U}_{|\mathbb{T}\cap(-\infty,0)}$ that originally was called *Nerode equivalence*).

2.3 Consider this past-canonical state representation. Prove that it is a 'classical' dynamical system, by specifying the corresponding maps ϕ and r. Is this state space system observable (as a latent variable system)?

Note: Input/output maps cannot cope with initial states, a basic ingredient of any serious theory of dynamical systems. Systems, even input/output systems, are simply not maps. The output depends not only on the input, but also on the initial state. This situation occurs in physical systems (electrical circuits, mechanical devices, etc.) as well as in man-made systems (computers, etc.). Granted, most introductory systems courses firmly engrain the concept of a system as an input/output map. There is simply no excuse for this. The framework collapses in the first example. As such, i/o maps cannot deal with uncontrollable systems, let alone autonomous ones.

In section 2.1 of the book "Topics in Mathematical System Theory" by R.E. Kalman, P.L. Falb, and M.A. Arbib, there is also a definition of an input/output system that attempts to include initial states by letting the input/output map depend on 'parameters', intended to capture the 'initial state'. But, what is the state depends on system, and it is impossible to construct the state space before one has a model to begin with. But if all one knows is input/output systems, this leads to a vicious circle. It is this type of contorted definitions that show that behaviors provide all around a much better approach to dynamics (of open and closed systems alike).

3. Notwithstanding these limitations of i/o maps, there are some nice examples of i/o maps and their state representation. We now discuss one of them.

Consider a real number with a finite decimal expansion, $x = \sum_{t\in\mathbb{Z}} \mathrm{x}_t 10^t$, written in decimal notation as

$$\cdots \mathrm{x}_\mathrm{n}\mathrm{x}_{\mathrm{n}-1} \cdots \mathrm{x}_1\mathrm{x}_0, \mathrm{x}_{-1}\mathrm{x}_{-2} \cdots .$$

Associate with this number the map $x : \mathbb{Z} \to \mathbb{D} := \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, defined by $x(t) := \mathrm{x}_{-t}$. Call $\mathfrak{D}$ the collection of all such maps. Note that

$$\mathfrak{D} = \{x : \mathbb{Z} \to \mathbb{D} \mid \ \exists t_0, t_1 : x(t) = 0 \text{ for } t \notin [t_0, t_1]\}.$$

Actually, it is customary to leave the zeros that precede this t_0 and follow this t_1 blank.

3.1 Let $F : \mathfrak{D}^2 \to \mathfrak{D}$ be the map that corresponds to the usual addition of real numbers with a finite decimal expansion: $F(a, b) := a + b$. This map is not even linear: there is no justice in this world. Prove that the input space is memoryless and that F is a shift-invariant non-anticipating i/o map.

3.2 Construct very concretely

(i) the past-canonical state space

(ii) the corresponding state transition map

(iii) the corresponding read-out map.

Hint: $\mathbb{X} = \{0, 1\}$.

3.3 Note that there exists $u^\star \in \mathfrak{D}^2$ such that $\forall u \in \mathfrak{D}^2 \;\; \exists t : u(t') = u^\star(t')$ for $t' \leq t$. This input $u^\star$ is independent of time. What is the past-canonical state induced by $u^\star$? Let $u \in \mathfrak{D}^2$ be given. Until what time is the past-canonical state equal to the one induced by $u^\star$? How do you recognize this time from $u = (a, b)$?

This yields a very concrete algorithm for adding two real numbers given in decimal expansion. From a, b recognize the 'initial time', before which the state and output are 0, and from then on proceed to compute the output by iterating ϕ and r. Explain.

The algorithm is often attributed (although this is disputed) to Simon Stevin (1548-1620), a Flemish mathematician and engineer who was also born in Bruges and spent most of his professional life in service of the Dutch government (in casu Prince Maurits of Orange). Stevin is well-known in the Low Countries, but not so much abroad. He is one of the few mathematicians for whom a statue has been erected (on a square in the center of Bruges). He was very influential in popularizing decimal numbers, among other things by formalizing the algorithms for addition, multiplication, division, etc. Actually, the notation for decimal numbers which we use nowadays is 'almost' due to Stevin. Napier (the inventor of logarithms) made an essential improvement in Stevin's notation for indicating what is before and after the comma, clinching the notation that is still in use in our computer age.

Exercise 2 (The Hankel operator in continuous time)

The aim of this exercise is to generalize some of the results on the Hankel matrix to continuous-time systems.

Algorithmically, discrete-time systems are much more convenient to deal with than their continuous-time counterparts. This is brought very prominently into evidence by the role of the Hankel matrix, its rank, its sub-matrices, its SVD, in obtaining state space representations of convolutions. In this exercise, we examine some aspects of the continuous-time counterpart.

The basic problem is to represent the convolution

$$y(t) = \int_0^t H(t')u(t-t')\ dt', \quad t \geq 0, \tag{1}$$

by the finite-dimensional state space system

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx, \quad x(0) = 0. \tag{2}$$

The central question is when and how $H : \mathbb{R}_+ \to \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$ can be expressed as $H(t) = Ce^{At}B$ for $t \geq 0$ (verify this!).

Notation: $(A, B, C) \Rightarrow H$.

Associate with (1) the *Hankel operator,* defined by

$$\boxed{y(t) = \int_0^\infty H(t+t')u(t')\ dt'.} \tag{3}$$

The Hankel operator tells us how past inputs are mapped to future outputs (verify this!). We are on purpose a bit vague about the domain of $\mathfrak{H}_H$. Until we say more about H, this must remain vague. If this bothers you, assume $H \in \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{\mathtt{p}\times\mathtt{m}})$, and take $\{u \in \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{m}}) \mid u \text{ has compact support}\}$ as the domain of $\mathfrak{H}_H$.

1. Prove that the following conditions for representability are equivalent:
   (a) $\exists\ (A, B, C)$ such that $(A, B, C) \Rightarrow H$.
   (b) H satisfies a differential equation, i.e., $\exists\ P \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi], \det(P) \neq 0$ such that $P(\frac{d}{dt})H = 0$. Verify that this is equivalent to the existence of a $0 \neq p \in \mathbb{R}[\xi]$ such that $p(\frac{d}{dt})H = 0$.
   (c) H is *factorizable*, i.e., there exist $\mathtt{n} \in \mathbb{N}, F \in \mathfrak{C}^\infty(\mathbb{R}_+, \mathbb{R}^{\mathtt{p}\times\mathtt{n}})$ and $G \in \mathfrak{C}^\infty(\mathbb{R}_+, \mathbb{R}^{\mathtt{n}\times\mathtt{p}})$ such that
   $$H(t' + t'') = F(t')G(t'') \text{ for } t', t'' \in \mathbb{R}_+.$$
   (d) $\mathrm{im}(\mathfrak{H}_H)$ is finite dimensional.

2. Each of these conditions involves an integer: $\dim(x)$, the 'order' of the differential equation (define this, but reflect on the proper definition), the $\mathtt{n}$ in the factorization, the dimension of $\mathrm{im}(\mathfrak{H}_H)$. Under the obvious definitions of minimality, this leads to the question when the state representation, the differential equation, the factorization are minimal, how minimal elements are related, and how the minimal value of these integers are related. State the definitions, the conditions for minimality, the relations between minimal elements, and the relation between the integers. You need not give a proof.

3. Prove that the following stability conditions are equivalent:
   (a) $\exists\ (A, B, C)$ with A Hurwitz, such that $(A, B, C) \Rightarrow H$.
   (b) H satisfies a stable differential equation, i.e., there exist $P \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi], \det(P) \neq 0$, Hurwitz, such that $P(\frac{d}{dt})H = 0$. You may wish to verify that this is equivalent to the existence of a $0 \neq p \in \mathbb{R}[\xi]$, Hurwitz, such that $p(\frac{d}{dt})H = 0$.

(c) H is $\mathcal{L}_2$-*factorizable* i.e., there exist $\mathtt{n} \in \mathbb{N}, F \in \mathcal{L}_2(\mathbb{R}_+, \mathbb{R}^{\mathtt{p}\times\mathtt{k}})$ and $G \in \mathcal{L}_2(\mathbb{R}_+, R^{\mathtt{k}\times\mathtt{p}})$ such that

$$H(t' + t'') = F(t')G(t'') \text{ for } t', t'' \in \mathbb{R}_+.$$

(d) $\int_0^\infty ||H(t)||\, dt < \infty$ (hence $\mathfrak{H}_H$ maps $\mathcal{L}_2(\mathbb{R}_+, \mathbb{R}^{\mathtt{m}})$ into $\mathcal{L}_2(\mathbb{R}_+, \mathbb{R}^{\mathtt{p}})$, and $\mathrm{im}(\mathfrak{H}_H)$ is finite dimensional.

4. It is well-known that $\int_0^\infty ||H(t)||\, dt < \infty$ implies that $\mathfrak{H}_H$ maps $\mathcal{L}_2(\mathbb{R}_+, \mathbb{R}^{\mathtt{m}})$ into $\mathcal{L}_2(\mathbb{R}_+, \mathbb{R}^{\mathtt{p}})$, and that it has an SVD, i.e., a decomposition:

$$\mathfrak{H}_H(\cdot) = \Sigma_{\mathtt{k}\in\mathbb{N}} \sigma_{\mathtt{k}} u_{\mathtt{k}} < v_{\mathtt{k}}, \cdot >_{\mathcal{L}_2(\mathbb{R}_+, \mathbb{R}^{\mathtt{m}})},$$

where the $u_{\mathtt{k}}$'s and the $v_{\mathtt{k}}$'s are orthonormal, and

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\mathtt{k}} \geq \cdots \geq 0$$

are the SV's of $\mathfrak{H}_H$.

Prove the existence of this SVD in the case $\int_0^\infty ||H(t)||\, dt < \infty$ and $\mathrm{im}(\mathfrak{H}_H)$ finite dimensional, from (i) the factorizability and (ii) from the existence of a balanced state representation.