Exercises Set 3

Exercise 1 (Controllability and interconnection)

The aim of this exercise is (i) to illustrate the behavioral concept of controllability and (ii) to show its fragility under system operations.

1. Let $\mathfrak{B}_1 \in \mathfrak{L}^\bullet$ be described by
$$R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2, \tag{1}$$
and $\mathfrak{B}_2 \in \mathfrak{L}^\bullet$ be described by
$$R_3(\frac{d}{dt})w_3 = R_4(\frac{d}{dt})w_4. \tag{2}$$
Define their *series (or cascade) interconnection* by the full behavioral equations (1,2) combined with
$$w_2 = w_3.$$
Of course, we assume that the dimensions are such that this equation makes sense. In the manifest behavior, consider (w_1, w_4) as the manifest variables and (w_2, w_3) as latent variables (i.e., in behavioral equations for the manifest behavior of the series connection, w_2 and w_3 are eliminated).
Consider the system with transfer function $\frac{1}{s}$, i.e.,
$$\frac{d}{dt}y_1 = u_1$$
and the system with transfer function s, i.e.,
$$y_2 = \frac{d}{dt}u_2.$$
Are these systems controllable? Compute behavioral equations for the manifest behavior of the series connection defined by $u_2 = y_1$. Is this system controllable? What is its transfer function? Now consider the series connection in opposite order, i.e., defined by $u_1 = y_2$. Compute behavioral equations for the manifest behavior of this series connection. Is this system controllable? What is its transfer function? Are the two series connections the same? If not, give a signal that belongs to the manifest behavior of one, but not the other. Does series connection of single-input/single-output connections 'commute'?
2. Define, in the above spirit of series connection, *parallel connection*.
3. Decide, by means of a proof or a counterexample, which of the above operations preserve controllability. Of course, we assume that we deal with systems in $\mathfrak{L}^\bullet$, and that the dimensions are appropriate:
   (a) series connection
   (b) parallel connection
   (c) addition, i.e., does $\mathfrak{B}_1, \mathfrak{B}_2$ controllable imply $\mathfrak{B}_1 + \mathfrak{B}_2$ controllable?
   (d) intersection
   (e) elimination
   (f) action of a linear differential operator i.e., does $\mathfrak{B}$ controllable and $F \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ imply $F(\frac{d}{dt})\mathfrak{B}$ controllable?
   (g) the inverse action of a linear differential i.e., does $\mathfrak{B}$ controllable and $F \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ imply that $\{w \mid F(\frac{d}{dt})w \in \mathfrak{B}\}$ (this is a system in $\mathfrak{L}^\bullet$!) is controllable?

Exercise 2 (Moving average)

The aim of this exercise is to illustrate that the notion of controllability can even shed some light on some very common algorithms.

Throughout this exercise, the time-axis is $\mathbb{Z}$.

1. (Recall first exercise 2 of set 2). Let σ denote, as usual, the *shift*: $\sigma(f)(t) := f(t+1)$. Let $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi,\xi^{-1}]$ and consider the system of difference equations
$$R(\sigma,\sigma^{-1})w = 0.$$
This defines the dynamical system $\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$. Prove that this system is controllable iff
$$\text{rank}(R(\lambda,\lambda^{-1})) = \text{ constant over } 0 \neq \lambda \in \mathbb{C}.$$
Prove by an example that you cannot dispense of 'puncturing' 0 from $\mathbb{C}$ in this test.

2. Consider the system defined by
$$w_2(t) = \frac{1}{T}\ \Sigma_{t'=1,2,\ldots,T}\, w_1(t-t'). \tag{3}$$
This algorithm is called a *moving average (MA)* smoothing. $T \in \mathbb{N}$ is called the *averaging window*. It is very frequently used in order to filter out noise, detecting trends, etc. When T is large, it is tempting to replace this algorithm by
$$w_2(t) = w_2(t-1) + \frac{1}{T}(w_1(t-1) - w_1(t-T-1)). \tag{4}$$
   (a) Do (3) and (4) have the same transfer function?
   (b) Compare, by counting the number of additions and multiplications required per time-step, (3) and (4) from the computational complexity point of view.
   (c) Do (3) and (4) define the same system (of course, in the behavioral sense, the one and only way...)?
   (d) Is (3) controllable? Is (4) controllable?
   (e) Find a controllable $\oplus$ autonomous decomposition of (4).
   (f) Would you call (3) stable (we use 'stable' as meaning $[w_1(t) = 0, (w_1,w_2) \in \mathfrak{B}] \Rightarrow [w_2(t) \to t \to \infty]$)? (4)? Does this conclusion make (4) useless as an algorithm?

   Conclude that stability is definitely not solely an issue about transfer functions, notwithstanding what is written in an endless number of papers in the IEEE Transactions on Automatic Control.

3. A very close relative of (3) is
$$w_2(t) = \frac{1}{2T+1}\ \Sigma_{t'=-T,\ldots,-1,0,1,\ldots,T}\, w_1(t-t'). \tag{5}$$
We have seen during the lectures that every system in $\mathfrak{L}^\bullet$ admits an componentwise input/output partition with a proper transfer function. This result generalizes to the discrete time case (you may want to prove this): every system admits a componentwise i/o partition in which the input does not anticipate the output (sometimes this is called *'causality'*). Consider the system defined by
$$p(\sigma,\sigma^{-1})y = q(\sigma,\sigma^{-1})u,$$
with $0 \neq p, q \in \mathbb{R}[\xi,\xi^{-1}]$. Determine in terms of the transfer function when the input does not anticipate the output. What is this non-anticipating input/output partition for (3)? Do you have a choice? Same question for (5). In the first case, the answer is what every sensible person would expect. In the second case the answer is a bit absurd. What is the explanation of this annoying aspect of (5)? The problem, it appears, is with the notions of input and output. They are much more context dependent that we have been led to believe.

4. Another close relative of (3) is *exponential weighting:*

$$w_2(t) = \frac{1}{\rho^{-1} - 1} \; \Sigma_{t' \in \mathbb{N}} \rho^{t'} w_1(t - t') \tag{6}$$

with $\rho \in (0, 1)$ the *weighting parameter*. Convolutions as (6) or the continuous time analogs are of course very much related to our linear difference or differential systems. For the case at hand the associated difference equation is

$$w_2(t) = \rho w_2(t-1) + \frac{\rho}{\rho^{-1} - 1} \; w_1(t-1) \tag{7}$$

(a) (6) has the drawback that it hard to give a very concrete characterization of the behavior, since it unclear for which $w_1 : \mathbb{Z} \to \mathbb{R}$ the infinite sum is finite. Prove that the infinite sum is finite when w_1 is bounded. Prove that (6) combined with w_1 bounded, and (7) combined with w_1, w_2 bounded have the same behavior.

(b) Compare the computational complexity of (3), (6), and (7).

(c) Is (7) controllable?

Obviously, these are reasons enough to prefer exponential weighting implemented by (7) over Moving Average for data smoothing.