Exercises Set 1

Exercise 1 (Linear static models)

The aim of this exercise is (i) to let you think of the nature of behavioral equations (in the context of systems without dynamics) and (ii) to show some parallels between behavioral equation representations of static linear systems and those for linear differential systems studied later in the course.

Let $(\mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ be a linear mathematical model (this is *newspeak*, it means nothing else than that $\mathfrak{B}$ is a linear subset of $\mathbb{R}^{\mathtt{w}}$).

1. (a) Prove that $\mathfrak{B}$ admits a behavioral equation representation

$$Rw = 0 \tag{1}$$

with $R \in \mathbb{R}^{\bullet \times \mathtt{w}}$.

Call (1) a *kernel representation* of $\mathfrak{B}$, and a *minimal kernel representation* of $\mathfrak{B}$ if, among all such kernel representations of $\mathfrak{B}$, rowdim(R) is as small as possible.

   (b) What is the relation between rank(R) and dim$(\mathfrak{B})$?

   (c) What is the relation between rowdim(R) and dim$(\mathfrak{B})$ if (1) is a minimal kernel representation?

   (d) Prove that (1) is a minimal kernel representation iff the matrix R has full row rank. *full row rank* := rank = rowdim.

   (e) Is it true that if (1) is not minimal, then you can simply cancel equations from (1), i.e. delete rows from R, without changing $\mathfrak{B}$? Reflect on the analogy with differential systems.

   (f) Prove that if (1) is a minimal kernel representation of $\mathfrak{B}$, then $R'w =$ is another kernel representation of the same behavior iff $R' = UR$ with $U \in \mathbb{R}^{\bullet \times \bullet}$ non-singular.

2. (a) Prove that $\mathfrak{B}$ admits a behavioral equation representation

$$w = M\ell \tag{2}$$

with $M \in \mathbb{R}^{\mathtt{w} \times \bullet}$ and ℓ a latent variable.

Call (2) an *image representation* of $\mathfrak{B}$, and a *minimal image representation* of $\mathfrak{B}$ if, among all image representations of $\mathfrak{B}$, dim(ℓ) = coldim(M) is as small as possible.

   (b) Formulate the analogs of the above results obtained for kernel representations, for image representations.

3. Prove that the manifest behavior of the latent variable behavioral equation representation

$$Rw = M\ell \tag{3}$$

with ℓ a latent variable, $R \in \mathbb{R}^{\bullet \times \mathtt{w}}, M \in \mathbb{R}^{\mathtt{w} \times \bullet},$ and rowdim(R) = rowdim(M) is linear.

(3) is the natural outcome (emphasized throughout the course, and illustrated, e.g., by exercise 2) of a first principles modeling procedure, and has obviously both kernel and image representations as special cases.

Exercise 2 (Resistive circuits)

The aim of this exercise is (i) to convince you that first principles models invariably lead to behavioral equations containing latent variables, and (ii) to present a formal mathematical setting for obtaining models of resistive circuits.

A *digraph* is a triple $(\mathfrak{N}, \mathfrak{E}, \pi)$, with $\mathfrak{N}$ a (finite) set, the set of *nodes*, $\mathfrak{E}$ a (finite) set, the set of *edges*, and $\pi : \mathfrak{E} \to \mathfrak{N}^2$ the *incidence map.* If $\pi(e) = (n_1, n_2)$, then we call n_1 the *source* and n_2 the *sink* of e.

1. Draw a not-too-trivial digraph, and specify the associated $\mathfrak{N}, \mathfrak{E},$ and π.

The notion of a digraph is standard, although often the sloppy definition in which $\mathfrak{E}$ is viewed as a subset of $\mathfrak{N}^2$ is preferred above the above accurate one. This follows the time-honored pedagogical principle that confusion is good. Naturally, 'good' means less work and less thinking for professors...

Less standard is the notion of a *digraph with leaves.* It is a quintuple $(\mathfrak{N}, \mathfrak{E}, \mathfrak{L}, \pi, \nu)$, with $(\mathfrak{N}, \mathfrak{E}, \pi)$ a digraph, $\mathfrak{L}$ a (finite) set, the set of *leaves*, and $\nu : \mathfrak{L} \to \mathfrak{N}$ the *leave incidence map.* A graph with leaves is thus a graph in which some 'edges' (called leaves) are connected to only one node. Note that there is no point in giving a direction to the leaves.

2. Draw a not-too-trivial digraph with leaves, and specify the associated $\mathfrak{N}, \mathfrak{E}, \mathfrak{L}, \pi,$ and ν.

We view of a resistive circuit as a black box with a number of wires, external terminals, sticking out of it. Inside the black box there are resistors, connected to each other and to the external terminals. We think of this interconnection pattern as an 'architecture'. The aim of the modeling is to describe the behavior of the voltages (potentials) and currents at the external terminals. We will assume that the currents are positive when they flow into the black box. This avoids having to define directions at the external terminals. For simplicity, we consider only linear resistors, although many of the ideas are just as well valid for the nonlinear case.

We formally define a resistive circuit as a digraph with leaves $(\mathfrak{N}, \mathfrak{E}, \mathfrak{L}, \pi, \nu)$, called the *interconnection architecture*, and a map $\rho : \mathfrak{E} \to [0, \infty)$, called the *resistance assignment.*

Intuition:

- $\mathfrak{L}$: the external terminals
- $\mathfrak{E}$: the internal branches with the resistors. Note that since we deal with a digraph (and not simply with a graph) we have somehow chosen a positive direction for the currents in these branches. (I do not like this aspect, since this direction is not present 'in nature'. It is possible to avoid it, but this is somewhat 'heavy' for circuits containing only 2-terminal elements, as is the case studied here).
- $\rho(e)$: tells what the value of the resistance (say in ohms) is in the branch corresponding to edge e
- all the edges and leaves incident to the same node are assumed connected (think 'soldered together').

3. Draw a not-too-trivial resistive circuit, and specify the associated $\mathfrak{N}, \mathfrak{E}, \mathfrak{L}, \pi, \nu$ and ρ.

4. We are now ready to set up behavioral equations. Of course, we take as manifest variables the voltages (potentials) at, and the currents into the external terminals. That is what the model aims at. But we need auxiliary variables to come up with a model. For the latent variables we take the voltages (potentials) at the internal nodes, at the connections, and the currents in the internal branches. Note that the assumption that the potentials of the nodes are well defined comes down to assuming *Kirchhoff's voltage law.*

Let $| \; |$ denote *cardinality*, i.e. $|S|$ denotes the number of elements of the (finite) set S.

In order to set up behavioral equations,

- Number the elements of $\mathfrak{L}$ as $\{1, 2, \ldots, |\mathfrak{L}|\}$, and denote the associated voltage and current vectors at the external terminals as $V = (V_1, V_2, \ldots, V_{|\mathfrak{L}|})$ and $I = (I_1, I_2, \ldots, I_{|\mathfrak{L}|})$. Denote the vector of manifest variables as $w = (V, I)$. Hence $\mathbb{W}$, the space of manifest variables, $= \mathbb{R}^{2|\mathfrak{L}|}$.
- Enumerate the elements of $\mathfrak{N}$ as $\{n_1, n_2, \ldots, n_{|\mathfrak{N}|}\}$, and those of $\mathfrak{E}$ as $\{e_1, e_2, \ldots, n_{|\mathfrak{E}|}\}$. Take for the space of latent variables $\mathbb{L} = \mathbb{R}^{|\mathfrak{N}|+|\mathfrak{L}|}$. The physical meaning of a typical element

$$((V_{n_1}, V_{n_2}, \ldots, V_{n_{|\mathfrak{N}|}}), (I_{e_1}, I_{e_2}, \ldots, I_{e_{|\mathfrak{E}|}}))$$

  of $\mathfrak{L}$ is obvious from the notation.
- The full behavioral equations are given by

  1. Kirchhoff's current laws: For each $n \in \mathfrak{N}$ the following equation holds

$$\Sigma_{\{e \in \mathfrak{E} | n \text{ is the sink of } e\}} I_e + \Sigma_{\{t \in \mathfrak{L} | t \text{ is incident to } t\}} I_t = \Sigma_{\{e \in \mathfrak{E} | n \text{ is the source of } e\}} I_e.$$

  2. Compatibility, or Kirchhoff's voltage laws, if you like: For each $\ell \in \mathfrak{L}$, the following equation holds

$$V_\ell = V_{\nu(\ell)}$$

  3. Constitutive equations: For each $e \in \mathfrak{E}$ with $(n_1, n_2) = \pi(e)$, the following equation holds

$$V_{n_1} + \rho(e) I_e = V_{n_2}.$$

Set up these equations for the circuit which you defined in 3.

In the sequel you may consider either the circuit which you defined in 3, or a completely general linear resistive circuit. This illustrates that the general, if treated with the proper notation, is often easier than the specific.

5. Let $\mathfrak{B}$ be the manifest behavior of the circuit defined by $((\mathfrak{N}, \mathfrak{E}, \mathfrak{L}, \pi, \nu), \rho)$.

Prove that $\mathfrak{B}$ is a linear subspace of $\mathbb{R}^{2|\mathfrak{L}|}$.

Let $\mathbf{1_n}$ denote the $\mathtt{n}$-dimensional column vector with all 1's.

Prove that

$$\begin{array}{lll} (V, I) \in \mathfrak{B} & \Rightarrow & \mathbf{1}_{|\mathfrak{L}|}^\top I = 0 \\ (V, I) \in \mathfrak{B} & \Rightarrow & \forall \, \alpha \in \mathbb{R} : (V + \alpha \mathbf{1}_{|\mathfrak{L}|}, I) \in \mathfrak{B} \end{array}$$

6. Prove *passivity:* $(V, I) \in \mathfrak{B} \Rightarrow V^\top I \geq 0$.

7. Prove $\dim(\mathfrak{B}) \leq |\mathfrak{L}|$. It can be shown that equality holds. Try it!

8. Prove *reciprocity:* $(V_1, I_1), (V_2, I_2) \in \mathfrak{B} \Rightarrow V_1^\top I_2 = V_2^\top I_1$.

9. Assume that $\mathfrak{B}$ admits an *admittance* representation, i.e., e behavioral representation of the form

$$I = AV.$$

Prove, using [6-8], that $A = A^\top \geq 0$ (≥ 0 means that the matrix is nonnegative definite).