Extracting the memory of a system:

STATE CONSTRUCTION

Chaire Francqui, Lecture III, May 12, 2004

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STATE SPACE SYSTEMS

THEME

How do we formalize the memory of a dynamical system?
When is a variable a state variable?
How do state equations look like?

How are state equations constructed, algorithmically, starting from other representations ?

To what extent are state representations of a system unique?

The state is a latent variable with special properties ...

... ***What's a latent variable system ?***

LATENT VARIABLES

‘First principles’ models contain auxiliary vaiables:

- **interconnection variables in modeling**
- **state variables is systems theory**
- **the nodes in trellis diagrams in coding**
- **the labels of the vertices in automata**
- **non-terminal symbols in formal languages**
- **Ω is probability theory**

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We now first formalize dynamical systems which contain such variables, called latent variables,

$\leftrightarrow$ manifest variables (whose behavior the model aims at).

LATENT VARIABLES

A dynamical system with latent variables =

$$\Sigma_{\mathbb{L}} = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\mathrm{full}})$$

$\mathbb{T} \subset \mathbb{R}$, **the *time-axis* (= the set of relevant time instances)**

$\mathbb{W}$, **the *signal space* (= space of variables the model aims at)**

$\mathbb{L}$, **the *latent variable space* (= space of auxiliary model variables)**

$$\mathfrak{B}_{\mathrm{full}} \subset (\mathbb{W} \times \mathbb{L})^{\mathbb{T}} : \quad \textit{the full behavior}$$

(= the pairs $(w, \ell) : \mathbb{T} \to \mathbb{W} \times \mathbb{L}$ the model declares possible)

THE MANIFEST BEHAVIOR

Call the elements of $\mathbb{W}$ ***'manifest' variables*** **,**
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The latent variable system $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ **induces the** ***manifest system*** $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$**, with** ***manifest behavior***

$$\mathfrak{B} = \{w : \mathbb{T} \to \mathbb{W} \mid \exists\ \ell : \mathbb{T} \to \mathbb{L} \text{ such that } (w, \ell) \in \mathfrak{B}_{\text{full}}\}$$

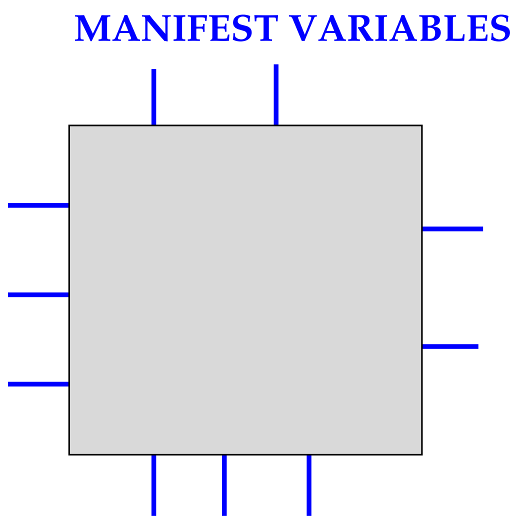
In equations for $\mathfrak{B}$**, the latent variables are** ***'eliminated'*****.**

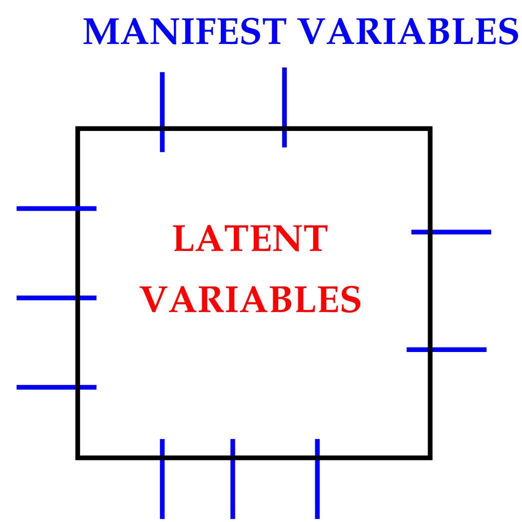
Note the clear notion of equivalent models that emerges!

THE MANIFEST BEHAVIOR

Call the elements of $\mathbb{W}$ ***'manifest' variables*** **,**

those of $\mathbb{L}$ ***'latent' variables*** **.**





THE NOTION OF STATE

A state system :=

A latent variable system in which the latent variable has a special 'splitting' property.

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The latent variable system

$$\Sigma_X = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$$

is said to be a ***state system*** if

$$(w_1, x_1), (w_2, x_2) \in \mathfrak{B}_{\text{full}}, t_0 \in \mathbb{T}, \quad \text{and} \quad x_1(t_0) = x_2(t_0)$$

imply

$$(w_1, x_1) \underset{t_0}{\wedge} (w_2, x_2) \in \mathfrak{B}_{\text{full}}.$$

$\underset{t_0}{\wedge}$ **denotes *concatenation* at t_0, defined as**

$$f_1 \underset{t_0}{\wedge} f_2(t) := \begin{cases} f_1(t) \text{ for } t < t_0 \\ f_2(t) \text{ for } t \geq t_0 \end{cases}$$

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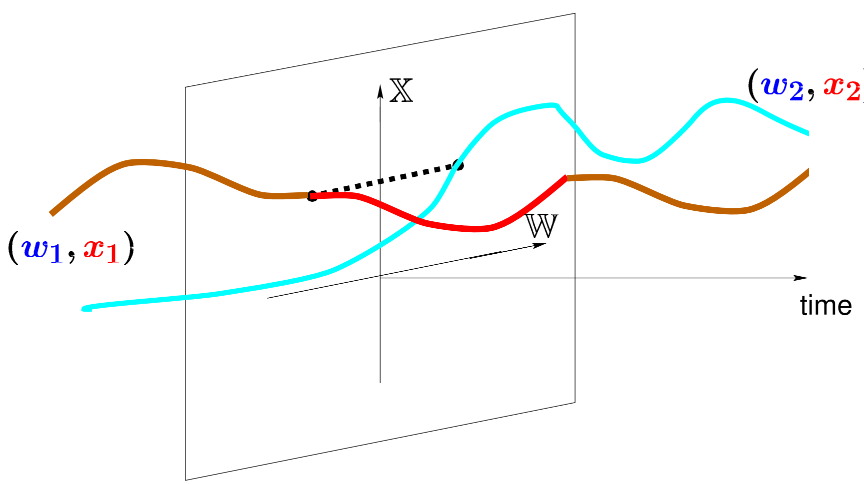
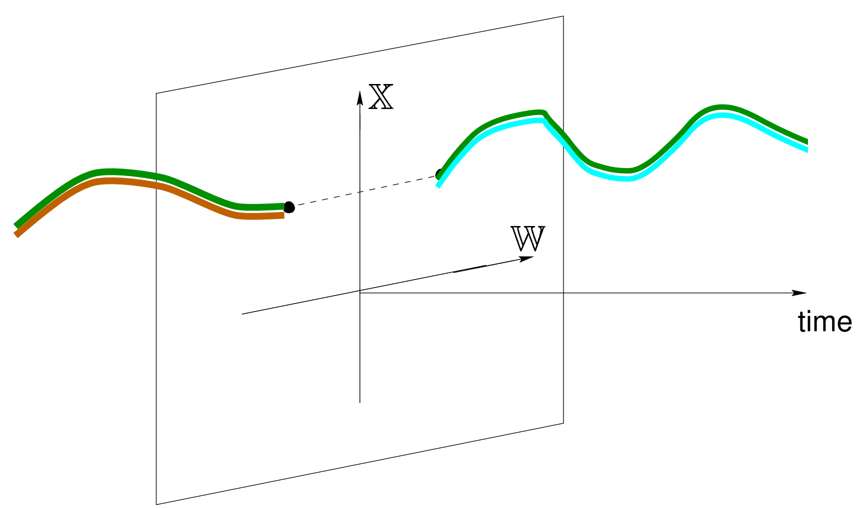
In pictures:





This definition is the implementation of the idea:

The state at time t, $x(t)$, contains all the information (about (w, x)!) that is relevant for the future behavior.

The state = the memory.

The past and the future are 'independent', conditioned on (given) the present state.

$\cong$ **Markovianity! Splitting!**

Examples of state systems:

1. Discrete-time systems.

A latent variable system described by a difference equation that is

first order in the latent variable x, and

zero-th order in the manifest variable w:

$$F(x(t+1), x(t), w(t), t) = 0.$$

Examples of state systems:

1. **Discrete-time systems.**

2. **Continuous-time systems.**

A latent variable system described by a differential equation that is

first order **in the** **latent** **variable** x**, and**

zero-th order **in the** **manifest** **variable** w**:**

$$F(\frac{d}{dt}x(t), x(t), w(t), t) = 0.$$

In particular, the ubiquitous

$$\frac{d}{dt}x(t) = f(x(t), u(t)), y(t) = h(x(t), u(t)); w(t) = (u(t), y(t)).$$

Examples of state systems:

1. Discrete-time systems.

2. Continuous-time systems.

3. Automata.

4. Trellis diagrams.

5. QM: $\frac{d}{dt}\psi = i\hbar H(\psi)\ , \quad p = |\psi|^2;$

$\psi =$ the 'wave function';
$p(x,t) =$ the 'probability' density of the particle's position.
wave function = latent, state, observables = manifest ??

For discrete time state systems $\rightsquigarrow$

Theorem: The latent variable system

$$\Sigma_X = (\mathbb{Z}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$$

is a state system

if **(and *only if*,**

provided the behavior is 'complete') $\mathfrak{B}_{\text{full}}$ **admits a representation as a difference equation that is**

first order **in the latent variable** x**, and**

zero-th order **in the manifest variable** w**:**

$$F(x(t+1), x(t), w(t), t) = 0.$$

STATE FOR DIFFERENTIAL SYSTEMS

Here we meet the notorious $\mathfrak{C}^\infty$-difficulty:

concatenation and $\mathfrak{C}^\infty$ don't mix.

We hence modify the state axiom to:

The latent variable system $\Sigma_X = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathfrak{B}_{\text{full}}) \in \mathfrak{L}^{\mathtt{w}+\mathtt{n}}$ is said to be a *state system* if

$$(w_1, x_1), (w_2, x_2) \in \mathfrak{B}_{\text{full}}, t_0 \in \mathbb{T}, \quad \text{and} \quad x_1(t_0) = x_2(t_0)$$

$$\text{imply} \quad (w_1, x_1) \underset{t_0}{\wedge} (w_2, x_2) \in \mathfrak{B}_{\text{full}}^{\text{closure}}.$$

'Closure' w.r.t., the $\mathfrak{L}^{\text{loc}}$-topology.

$\Leftrightarrow$: if $(w_1, x_1) \underset{t_0}{\wedge} (w_2, x_2)$ is a weak sol'n of corr. ODE.

DESCRIPTOR SYSTEMS

<u>Theorem</u>: The latent variable system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathfrak{B}_{\text{full}}) \in \mathfrak{L}^{\mathtt{w}+\mathtt{n}}$ is a state system

if and only if

$\mathfrak{B}_{\text{full}}$ admits a kernel representation that is
first order in the latent variable x, and
zero-th order in the manifest variable w.

In other words, iff there exist $E, F, G \in \mathbb{R}^{\bullet \times \bullet}$ such that this kernel representation takes the form of a *descriptor system:*

$$E\frac{d}{dt}x + Fx + Gw = 0.$$

MINIMALITY

We can consider two types of minimality of state representations:

1. Minimality of the number of *equations*
2. Minimality of the number of *state variables*

We discuss mainly the second one.

Definition: The state system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathfrak{B}_{\text{full}}) \in \mathfrak{L}^{\mathtt{w}+\mathtt{n}}$ is said to be ***state-minimal*** if, whenever $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}'}, \mathfrak{B}'_{\text{full}}) \in \mathfrak{L}^{\mathtt{w}+\mathtt{n}'}$ is another state system with the same manifest behavior, there holds

$$\mathtt{n} \leq \mathtt{n}'.$$

TRIMNESS

One more definition...

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be ***trim*** if, $\forall$ $\mathrm{w}_0 \in \mathbb{R}^{\mathtt{w}}$, $\exists w \in \mathfrak{B}$ such that $w(0) = \mathrm{w}_0$. The state system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathfrak{B}_{\mathrm{full}}) \in \mathfrak{L}^{\mathtt{w}+\mathtt{n}}$ is said to be ***state-trim*** if, $\forall$ $\mathrm{x}_0 \in \mathbb{R}^{\mathtt{n}}$, $\exists (w, x) \in \mathfrak{B}_{\mathrm{full}}$ such that $x(0) = \mathrm{x}_0$.

Theorem: (Minimality of state representations):
The state system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathfrak{B}_{\mathrm{full}}) \in \mathfrak{L}^{\mathtt{w}+\mathtt{n}}$ is *state-minimal* iff it is *state trim* and the state x is *observable* from w.

Observability $:\Leftrightarrow$ x can be deduced from w.
I.e., $\exists$ $X \in \mathbb{R}^{\mathtt{n} \times \mathtt{w}}[\xi]$ such that

$$(w, x) \in \mathfrak{B}_{\mathrm{full}} \Rightarrow x = X\left(\frac{d}{dt}\right)w.$$

FURTHER RESULTS

1. **State isomorphism theorem.**

Assume $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathfrak{B}_{\text{full}})$ $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}}, \mathfrak{B}'_{\text{full}}) \in \mathfrak{L}^{\mathtt{w}+\mathtt{n}}$, both state-minimal, same manifest behavior

$\Rightarrow$ there exists a nonsingular $S \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that

$$[(w, x) \in \mathfrak{B}_{\text{full}} \text{ and } (w, x') \in \mathfrak{B}'_{\text{full}}] \Leftrightarrow [x' = Sx].$$

The minimal state representation is unique up to a choice of the basis in the state space.

FURTHER RESULTS

1. **State isomorphism theorem.**

2. **Controllability.**

The manifest behavior is controllable iff there exists a state representation of it whose state behavior is controllable.

If the manifest behavior is controllable then any state-minimal state representation of is state-controllable.

FURTHER RESULTS

1. State isomorphism theorem.

2. Controllability.

3. Descriptor systems.

$\exists$ algorithms acting on E, F, G in a descriptor repr'ion to verify its state-minimality, its eq'n minimality, both.

$$E\frac{d}{dt}x + Fx + Gw = 0 \quad \text{and} \quad E'\frac{d}{dt}x' + F'x' + G'w = 0$$

are two minimal (state- & eq'ion-minimal) repr'ions of the same man. b'ior iff $\exists$ nonsingular $T, S \in \mathbb{R}^{\bullet \times \bullet}$ such that

$$E' = TES, F' = TES, G' = TG.$$

Uniqueness up to choice of basis in state and equation space.

FURTHER RESULTS

1. State isomorphism theorem.

2. Controllability.

3. Descriptor systems.

4. Notation: Introduce the 'invariant' (the 'state cardinality')

$$\mathtt{n} : \mathfrak{L}^{\bullet} \to \mathbb{N}$$

$\mathtt{n}(\mathfrak{B})$:= dimension of the minimal state associated with $\mathfrak{B}$.

All ‘classical’ results remain valid, except, (fortunately!) the celebrated (non-)equivalence:
state-minimality $\Leftrightarrow$ state-observability + state-controllability.

Non-controllable systems are very ‘real’ and they allow state-minimal (non-controllable) state representation.

I/S/O SYSTEMS

Finally...

It is possible to combine the input/output partition and the state representation, leading to the ubiquitous:

$$\frac{d}{dt}x = Ax + Bu, \;\; y = Cx + Du, \;\; w = (u, y).$$

u **is input := free,**
y **is output := bound by** u**,**
x **is state := ‘splitting’ = memory.**

Notation:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Theorem: Let $\mathfrak{B} \in \mathfrak{L}^\bullet$.

There exists a componentwise partition $w = (u, y)$, with $\dim(u) = \mathtt{m}(\mathfrak{B}), \dim(y) = \mathtt{p}(\mathfrak{B})$, and matrices

$$A \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}, B \in^{\mathtt{n}(\mathfrak{B})\times\mathtt{m}(\mathfrak{B})}, C \in \mathbb{R}^{\mathtt{p}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}, D \in \mathbb{R}^{\mathtt{p}(\mathfrak{B})\times\mathtt{m}(\mathfrak{B})}$$

such that

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du,$$

is a minimal (eq'n- and state-minimal) state repr'ion of $\mathfrak{B}$.

Note important 'invariants' ('cardinalities):

$$\mathtt{w}(\mathfrak{B}), \mathtt{m}(\mathfrak{B}), \mathtt{p}(\mathfrak{B}), \mathtt{n}(\mathfrak{B})$$

associated with $\mathfrak{B} \in \mathfrak{L}^\bullet$.

$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **is minimal (state + eq'n minimal)**

$\Leftrightarrow$ **it is state-minimal**

$\Leftrightarrow$ **it is state-observable**

$$\Leftrightarrow \mathrm{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\dim(A)-1} \end{bmatrix}\right) = \dim(A).$$

$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **is state controllable (usual Kalman def'n)**

$$\Leftrightarrow \mathrm{rank}(\begin{bmatrix} B & AB & \dots & A^{\dim(A)-1}B \end{bmatrix}) = \dim(A).$$

$\Rightarrow$ **the manifest behavior is controllable.**

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If $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **is minimal (i.e., observable) then**

state controllable iff manifest behavior controllable.

Watch out:

minimality of $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ $\Leftarrow$ **but** $\not\Rightarrow$ **controllable & observable.**

STATE CONSTRUCTION

!! Given a dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$
find a state representation $\Sigma_X = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$
for it !!

This problem is a jewel that has emerged in systems theory (and in computer science) in the 1960's. It has ramifications in the theory of stochastic processes (where they hide under the name hidden Markov models - HMM's), in formal language theory, (more recently) model simplification, etc.

See my webpage for the general set theoretic construction, using 'Nerode equivalence'.

We only consider linear time-invariant differential (c.q. difference) systems.

STATE CONSTRUCTION in DIFFERENTIAL SYSTEMS

!! Given a representation of the manifest behavior

$$\mathfrak{B} \in \mathfrak{L}^\bullet,$$

find a (state-minimal) state representation for it !!

Problem:

Given a 'numerical' specification of the (manifest) behavior, end up with a 'numerical' specification of a state model.

ALGORITHMS

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$\rightsquigarrow$ **the theory around the Hankel matrix.**

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- **Given a kernel, image, or latent variable representation, construct a (minimal) state model** (E, F, G) **or** $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$.

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- Given data (e.g. $u(\cdot), u(\cdot)$), construct a state model $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$.
- Make sure $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is in a special (e.g., balanced) form

STATE MAPS

Let $X(\xi) \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$. The map $X(\frac{d}{dt})$ is called a **state map** for $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ if the full behavior

$$\mathfrak{B}_{\text{full}} = \{(w, x) \mid w \in \mathfrak{B} \text{ and } x = X(\frac{d}{dt})w\}$$

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In a state-minimal repr'ion, x is determined by a state map (because of observability), whence (minimal) state maps exist.

State representation problem: Find a minimal state map.

Most logical : latent variable repr'on $\rightsquigarrow$ state repr'on.
However, we only discuss kernel and image repr'ons.

Define the *'shift-and-cut'* operator σ on $\mathbb{R}[\xi]$ as follows:

$$\sigma : p_0 + p_1\xi + \cdots + p_{\mathrm{n}-1}\xi^{\mathrm{n}-1} + p_{\mathrm{n}}\xi^{\mathrm{n}}$$
$$\mapsto \quad p_1 + p_2\xi + \cdots + p_{\mathrm{n}-1}\xi^{\mathrm{n}-2} + p_{\mathrm{n}}\xi^{\mathrm{n}-1}$$

Extend-able in the obvious term-by-term way to $\mathbb{R}^{\bullet\times\bullet}[\xi]$.

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Extend-able in the obvious term-by-term way to $\mathbb{R}^{\bullet\times\bullet}[\xi]$.

Repeated use of the cut-and-shift on $P \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ yields the ***'stack' operator*** Σ_P, defined by

$$\Sigma_P := \begin{bmatrix} \sigma(P) \\ \sigma^2(P) \\ \vdots \\ \sigma^{\mathrm{degree}(P)}(P) \end{bmatrix}$$

FROM KERNEL to STATE REPRESENTATION

There is a construction (elegant in its simplicity) of a state map in terms of the cut-and-shift and stack operators!

Theorem: Let

$$R(\frac{d}{dt})w = 0$$

be a kernel representation of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$.
Then $\Sigma_R(\frac{d}{dt})$ is a state map for $\mathfrak{B}$. The resulting

$$R(\frac{d}{dt})w = 0\,; \quad x = \Sigma_R(\frac{d}{dt})w$$

need not be minimal. It is trivially state-observable, but it may not be state-trim. Using Gröbner basis techniques it can be trimmed, leading to a minimal state representation.

FROM IMAGE to STATE REPRESENTATION

Theorem: Let $$w = M(\frac{d}{dt})\ell$$

be an image representation of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. Then $\Sigma_M(\frac{d}{dt})$ (acting on ℓ) yields a state map for $\mathfrak{B}$. The resulting

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We obtain a state map that acts on ℓ. If $w = M(\frac{d}{dt})\ell$ is not observable, then the state may not be observable, whence not state-minimal. But, if the image representation is observable, then so is the state. Hence then there is a(n implicit) state map. State trim iff the rows of Σ_M are linearly independent over $\mathbb{R}$. So, trimming only requires deleting rows.

The image repr. gives a very effective state construction.

SINGLE INPUT - SINGLE OUTPUT SYSTEMS

Apply this to

$$p(\frac{d}{dt})y = q(\frac{d}{dt})u$$

with

$$\begin{aligned} p(\xi) &= p_0 + p_1\xi + \cdots + p_{n-1}\xi^{n-1} + p_n\xi^n, \;\; p_n \neq 0 \\ q(\xi) &= q_0 + q_1\xi + \cdots + q_{n-1}\xi^{n-1} + q_n\xi^n \end{aligned}$$

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There are 4 well-known state constructions:

1. **the *observer canonical form***
2. **the *observability canonical form***
3. **the *controllability canonical form***
4. **the *controller canonical form***

SINGLE INPUT - SINGLE OUTPUT SYSTEMS

The cut-and-shift and stack operators yield the polynomial matrix

$$\Sigma_R(\xi) = \begin{bmatrix} p_1 + \cdots + p_{n-1}\xi^{n-2} + p_n\xi^{n-1} & -q_1 - \cdots - q_{n-1}\xi^{n-2} - q_n\xi^{n-1} \\ p_2 + \cdots + p_{n-1}\xi^{n-3} + p_n\xi^{n-2} & -q_2 - \cdots - q_{n-1}\xi^{n-3} - q_n\xi^{n-2} \\ \vdots & \vdots \\ p_{n-1} + p_n\xi & -q_{n-1} - q_n\xi \\ p_n & -q_n \end{bmatrix}$$

It follows that $x = \Sigma_R(\frac{d}{dt})$ is a state map, in fact, a state minimal one, even if the system is not controllable, i.e., when p and q have a common factor.

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To get more convenient minimal state maps, we can take any basis for span of the rows of X.

SINGLE INPUT - SINGLE OUTPUT SYSTEMS

To get more convenient minimal state maps, we can take any basis for span of the rows of X.

One choice: take the rows of Σ_R in reverse order.

A small calculation shows that this choice of the state variables leads to the so-called ***observer canonical form,*** **the i/s/o representation**

$$A = \begin{bmatrix} -p_{n-1}/p_n & 1 & 0 & \cdots & 0 & 0 \\ -p_{n-2}/p_n & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -p_0/p_n & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} q_{n-1} - p_{n-1} q_n/p_n \\ q_{n-2} - p_{n-2} q_n/p_n \\ \vdots \\ q_0 - p_0 q_n/p_n \end{bmatrix},$$

$$C = \begin{bmatrix} 1/p_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} q_n/p_n \end{bmatrix}.$$

Another immediate choice is to pick the state map

$$X(\xi) = \begin{bmatrix} 1 & \star \\ \xi & \star \\ \vdots & \vdots \\ \xi^{n-2} & \star \\ \xi^{n-1} & \star \end{bmatrix}$$

We need to compute the $\star$'s so that the combinations of the rows of Σ_R that yield the first column of X also give the second column.

The second column can be obtained by long hand division of q by p, i.e., by computing the polynomial $b(\xi) \in \mathbb{R}[\xi]$ defined by the equation

$$p(\xi)b(\xi^{-1}) = q(\xi) \quad (\text{ modulo } \xi^{-1}\mathbb{R}[\xi^{-1}]).$$

Then $$X(\xi) = \begin{bmatrix} 1 & b_0 \\ \xi & b_1 + b_0\xi \\ \vdots & \vdots \\ \xi^{\mathrm{n}-2} & b_{\mathrm{n}-2} + b_{\mathrm{n}-3}\xi + \cdots + b_0\xi^{\mathrm{n}-2} \\ \xi^{\mathrm{n}-1} & b_{\mathrm{n}-1} + b_{\mathrm{n}-2}\xi + \cdots + b_0\xi^{\mathrm{n}-1} \end{bmatrix}.$$

Then $$X(\xi)=\begin{bmatrix} 1 & b_0 \\ \xi & b_1+b_0\xi \\ \vdots & \vdots \\ \xi^{n-2} & b_{n-2}+b_{n-3}\xi+\cdots+b_0\xi^{n-2} \\ \xi^{n-1} & b_{n-1}+b_{n-2}\xi+\cdots+b_0\xi^{n-1} \end{bmatrix}.$$

This leads to the *observable canonical form,* the i/s/o representation

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{p_0}{p_n} & -\frac{p_1}{p_n} & -\frac{p_2}{p_n} & \cdots & -\frac{p_{n-1}}{p_n} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix},$$

$$C = \begin{bmatrix} 1 \; 0 \cdots 0 \; 0 \end{bmatrix}, \quad D = \begin{bmatrix} b_0 \end{bmatrix}.$$

SINGLE INPUT - SINGLE OUTPUT SYSTEMS

When the SISO system is controllable, and given in image representation by

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} p(\frac{d}{dt}) \\ q(\frac{d}{dt}) \end{bmatrix} \ell$$

with

$$\begin{aligned} p(\xi) &= p_0 + p_1\xi + \cdots + p_{\mathrm{n}-1}\xi^{\mathrm{n}-1} + p_{\mathrm{n}}\xi^{\mathrm{n}}, \;\; p_{\mathrm{n}} \neq 0, \\ q(\xi) &= q_0 + q_1\xi + \cdots + q_{\mathrm{n}-1}\xi^{\mathrm{n}-1} + q_{\mathrm{n}}\xi^{\mathrm{n}}. \end{aligned}$$

SINGLE INPUT - SINGLE OUTPUT SYSTEMS

The cut-and-shift and stack operators yield

$$X(\xi) = \begin{bmatrix} p_1 + \cdots + p_{n-1}\xi^{n-2} + p_n\xi^{n-1} \\ q_1 + \cdots + q_{n-1}\xi^{n-2} + q_n\xi^{n-1} \\ p_2 + \cdots + p_{n-1}\xi^{n-3} + p_n\xi^{n-2} \\ q_2 + \cdots + q_{n-1}\xi^{n-3} + q_n\xi^{n-2} \\ \vdots \\ p_{n-1} + p_n\xi \\ q_{n-1}{}^{n} + q_n\xi \\ q_n \\ p_n \end{bmatrix}.$$

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There are again two ready bases for the linear span of the rows of X:

$$\begin{bmatrix} p_n \\ p_{n-1}+p_n\xi \\ \vdots \\ p_2+\cdots+p_{n-1}\xi^{n-3}+p_n\xi^{n-2} \\ p_1+\cdots+p_{n-1}\xi^{n-2}+p_n\xi^{n-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ \xi \\ \vdots \\ \xi^{n-2} \\ \xi^{n-1} \end{bmatrix}.$$

The first choice leads to the ***controllable canonical form***

$$A = \begin{bmatrix} -p_{n-1}/p_n & 1 & 0 & \cdots & 0 & 0 \\ -p_{n-2}/p_n & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -p_0/p_n & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} b_1 & b_2 & \cdots & b_{n-1} & b_n \end{bmatrix}, \qquad D = \begin{bmatrix} b_0 \end{bmatrix}.$$

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$$C = \begin{bmatrix} b_1 & b_2 & \cdots & b_{n-1} & b_n \end{bmatrix}, \qquad D = \begin{bmatrix} b_0 \end{bmatrix}.$$

The second choice leads to the **controller canonical form**

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{p_0}{p_n} & -\frac{p_1}{p_n} & -\frac{p_2}{p_n} & \cdots & -\frac{p_{n-1}}{p_n} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \frac{1}{p_n} \end{bmatrix},$$

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Minimality (observability) holds iff p and q are co-prime.

FROM LATENT VARIABLE to STATE REPRESENTATION

Consider the latent variable system $\Sigma_X = (\mathbb{R}, \mathbb{R}^{\mathtt{w}_1+\mathtt{w}_2}, \mathbb{R}^{\mathtt{n}}, \mathfrak{B}_{\text{full}}) \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2+\mathtt{n}}$. Eliminate $w_2 \rightsquigarrow$ $\Sigma'_X = (\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}, \mathbb{R}^{\mathtt{n}}, \mathfrak{B}'_{\text{full}})$. It is easy to deduce directly from the state axiom that Σ'_X is a state system if Σ_X is.

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Construction of a state representation for $\mathfrak{B}$:

1. $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$ **latent variable representation for $\mathfrak{B}$.**
2. **Apply the cut-and-shift and stack operators to $[R \mid -M]$.**
3. **Obtain a state map**

$$x = \Sigma_{[R \mid -M]}(\tfrac{d}{dt})\begin{bmatrix} w \\ \ell \end{bmatrix}.$$

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$\rightsquigarrow$ **a, not nec. minimal, latent var'ble state repr'ion for $\mathfrak{B}$.**

Notes

- **Basic idea of algorithms: from a (e.g. latent variable) representation directly to state model.**

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- **Basic idea of algorithms: from a (e.g. latent variable) representation <u>directly</u> to state model.**
  This complements the existing algorithms
  transfer function $\longrightarrow$ i / s / o representation;
  impulse response $\longrightarrow$ i / s / o representation.

Notes

- **Basic idea of algorithms: from a (e.g. latent variable) representation directly to state model.**
- **$\exists$ Gröbner basis algorithms for state trimming.**
- **Our state construction is easily extended to state / input construction.**
- **Examples of useful special (minimal) state repr'ons:**

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i/s/o representation:

$$\frac{d}{dt}x = Ax + Bu, \ \ y = Cx + Du, w = (u, y),$$

Notes

- **Basic idea of algorithms: from a (e.g. latent variable) representation directly to state model.**
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  output nulling representation:

$$\frac{d}{dt}x = Ax + Bw, \;\; 0 = Cx + Dw,$$

Notes

- Basic idea of algorithms: from a (e.g. latent variable) representation directly to state model.
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- Examples of useful special (minimal) state repr'ons:

  driving variable representation:

$$\frac{d}{dt}x = Ax + Bv, \quad w = Cx + Dv.$$

Notes

- Basic idea of algorithms: from a (e.g. latent variable) representation directly to state model.
- $\exists$ Gröbner basis algorithms for state trimming.
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- Examples of useful special (minimal) state repr'ons:

  Readily deduced from descriptor representation:

$$E\frac{d}{dt}x + Fx + Gw = 0.$$

Notes

- **Basic idea of algorithms: from a (e.g. latent variable) representation directly to state model.**
- **$\exists$ Gröbner basis algorithms for state trimming.**
- **Our state construction is easily extended to state / input construction.**
- **Examples of useful special (minimal) state repr'ons:**
- **Recent advances: from image representation directly to balanced state representation.**

End of the Lecture III