CONTROLLABILITY and OBSERVABILITY in a NEW PERSPECTIVE

Chaire Francqui, Lecture II, May 12, 2004

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THEME

Central notions in all of system and control theory:

controllability and observability

in the setting and language of behavioral models.

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Central notions in all of system and control theory:

controllability and observability

in the setting and language of behavioral models.

- **Formal definitions**
- **Tests for controllability and observability**
- **Image representations**
- **Stabilizability**
- **PDE's**

CONTROLLABILITY

The time-invariant system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be

controllable

if for all $w_1, w_2 \in \mathfrak{B}$ there exists $w \in \mathfrak{B}$ and $T \geq 0$ such that

$$w(t) = \begin{cases} w_1(t) & t < 0 \\ w_2(t - T) & t \geq T \end{cases}$$

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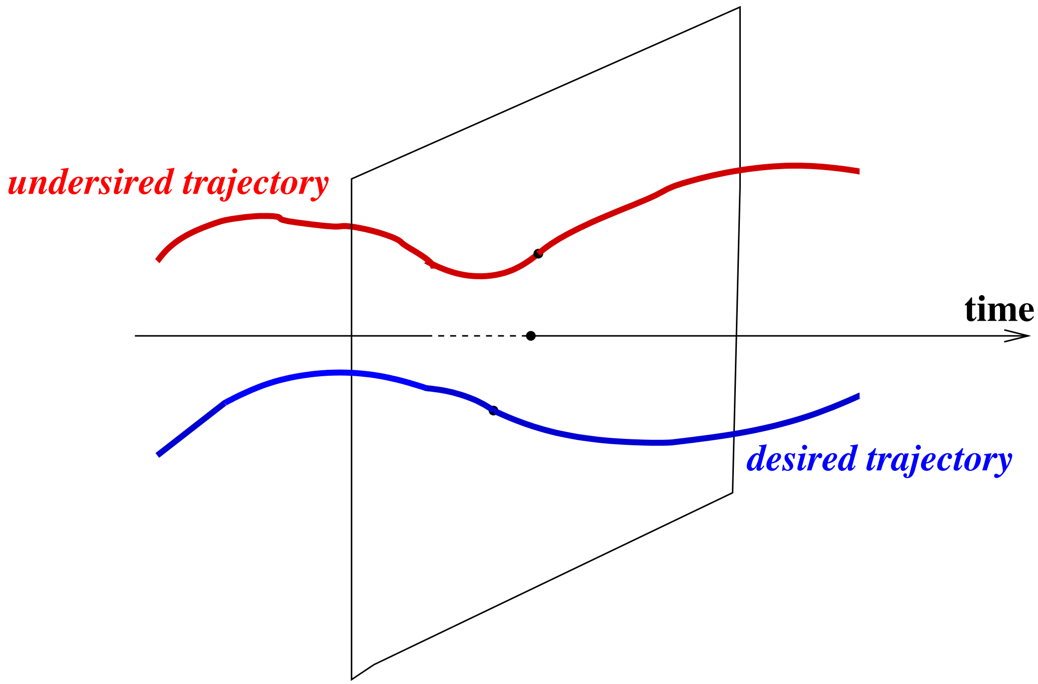
Controllability :$\Leftrightarrow$

legal trajectories must be ‘patch-able’, ‘concatenable’.

CONTROLLABILITY



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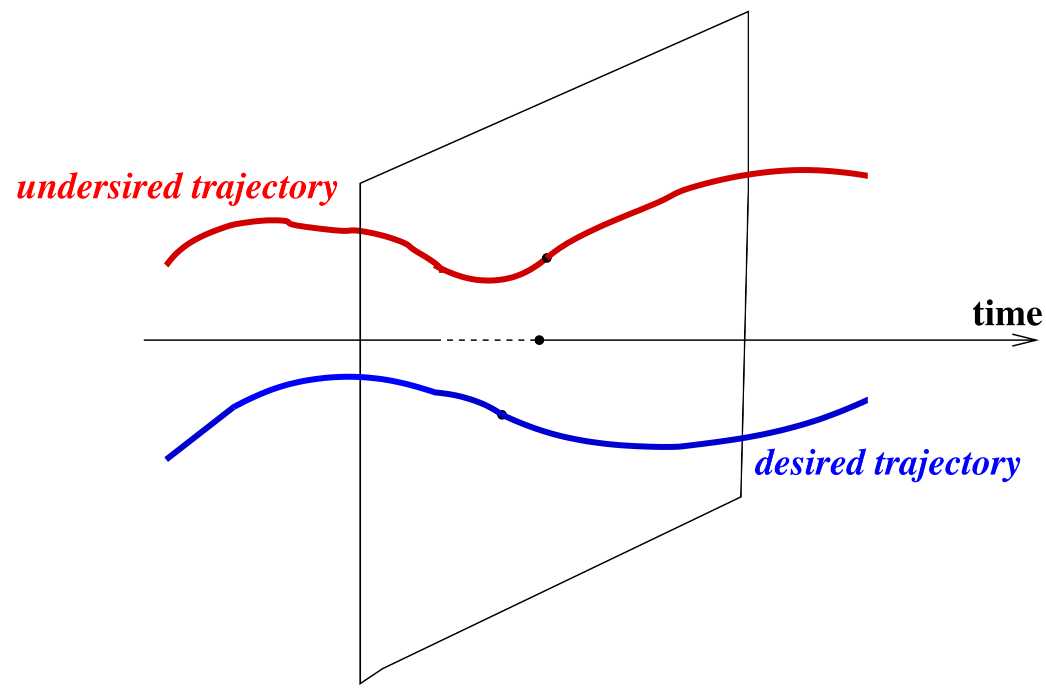
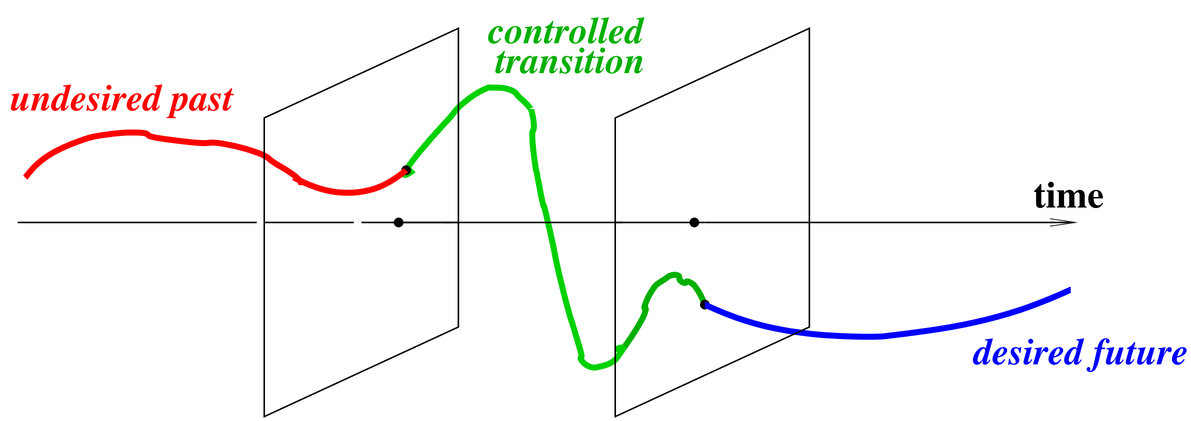
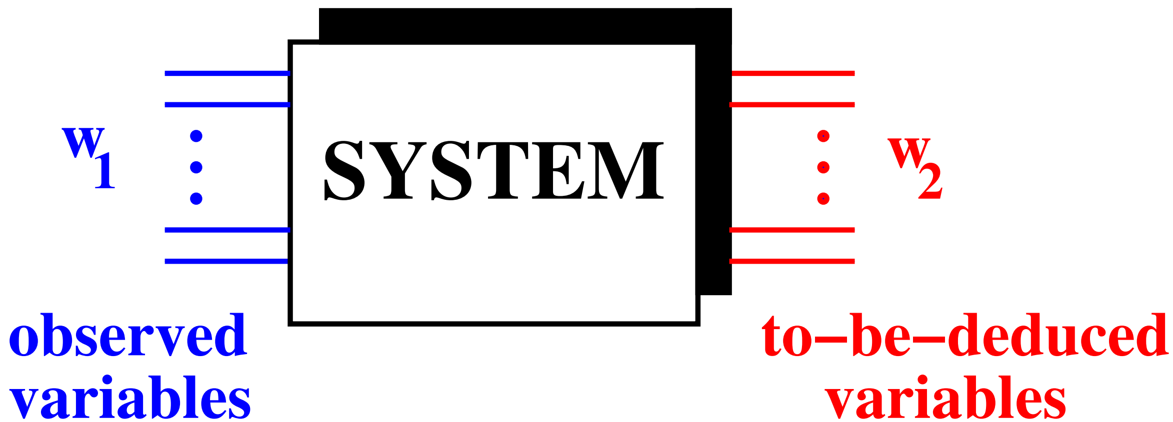




OBSERVABILITY



¿ **Is it possible to deduce** w_2 **from** w_1 ***and the system model*** **?**

OBSERVABILITY

Consider the system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$. Each element of $\mathfrak{B}$ hence consists of a pair of trajectories (w_1, w_2):

w_1 **: observed;**

w_2 **: to-be-deduced.**

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Definition: w_2 is said to be

observable from w_1

if $((w_1, w_2') \in \mathfrak{B}$, and $(w_1, w_2'') \in \mathfrak{B}) \Rightarrow (w_2' = w_2'')$,
i.e., if on $\mathfrak{B}$, there exists a map $w_1 \mapsto w_2$.

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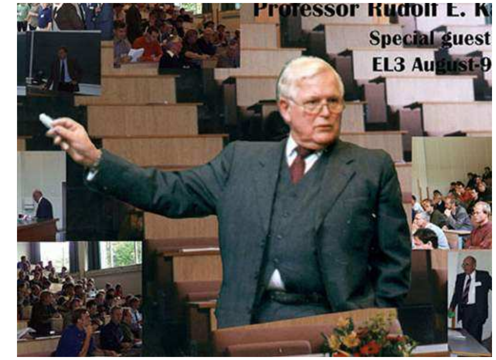
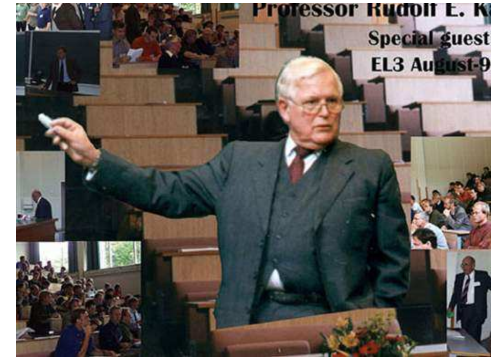
i.e., if on $\mathfrak{B}$, there exists a map $w_1 \mapsto w_2$.

Very often manifest = observed, latent (auxiliary variables introduced in the modeling process) = to-be-deduced.
We then speak of an observable (latent variable) system.



Special case: classical Kalman definitions for

$$\frac{d}{dt}x = f(x,u), \;\; y = h(x,u).$$



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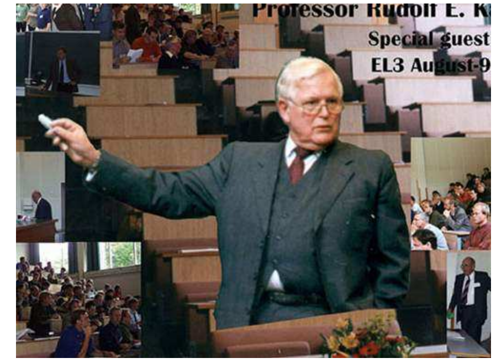
controllability: **variables = (input, state)**

If a system is not (state) controllable, why is it?

Insufficient influence of the control?

Or bad choice of the state?



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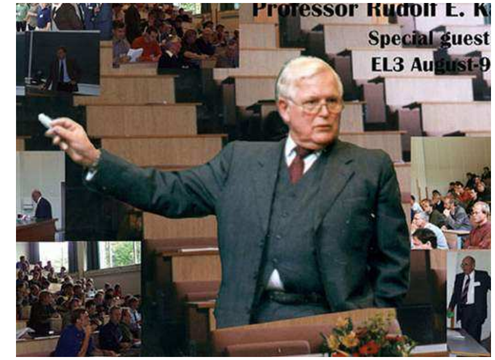
to-be-deduced = state.

Why is it so interesting to try to deduce the state, of all things? The state is a derived notion, not a 'physical' one.



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Why is it so interesting to try to deduce the state, of all things? The state is a derived notion, not a 'physical' one.

Kalman definitions address rather special situations.

TESTS for CONTROLLABILITY

Given a system representation, derive algorithms in terms of the parameters for controllability.

Consider the system $\mathfrak{B} \in \mathfrak{L}^\bullet$ defined by

$$R(\frac{d}{dt})w = 0.$$

Under what conditions on $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ does it define a controllable system?

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Theorem: $R(\frac{d}{dt})w = 0$ **defines a controllable system**

$$\Leftrightarrow$$

$\mathrm{rank}(R(\lambda)) =$ **constant over** $\lambda \in \mathbb{C}$.

TESTS for CONTROLLABILITY

Notes:

- If $R(\frac{d}{dt})w = 0$ is minimal ($\Leftrightarrow$ R of f.r.r.), then

  controllability $\Leftrightarrow$ $R(\lambda)$ is of full row rank $\forall \ \lambda \in \mathbb{C}$.

  Equivalently, R is left-invertible as a polynomial matrix ($\Leftrightarrow$ 'left prime').

  $P \in \mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_2}[\xi]$ is *leftt-invertible*

  $:\Leftrightarrow \exists \ Q \in \mathbb{R}^{\mathtt{n}_2 \times \mathtt{n}_1}[\xi]$ such that $PQ = I_{\mathtt{n}_1}$

TESTS for CONTROLLABILITY

Notes:

- $\frac{d}{dt}x = Ax + Bu, w = (x, u)$ **is controllable iff**

$$\operatorname{rank}([A - \lambda I \;\; B]) = \dim(x) \;\; \forall \, \lambda \in \mathbb{C}.$$

Popov-Belevich-Hautus test **for controllability.**

Of course,

$$\Leftrightarrow \operatorname{rank}([B \;\; AB \;\; \cdots \;\; A^{\dim(x)-1}B]) = \dim(x).$$

TESTS for CONTROLLABILITY

Notes:

- When is

$$p(\frac{d}{dt})w_1 = q(\frac{d}{dt})w_2$$

  controllable? $p, q \in \mathbb{R}[\xi]$, not both zero.

  Iff p and q are co-prime. No common factors!

  Testable via Sylvester matrix, etc.

  Generalizable.

TESTS for CONTROLLABILITY

Notes:

- **Example:** Our electrical circuit is controllable unless

$$CR_C = \frac{L}{R_L} \text{ and } R_C = R_L.$$

  Reasonable physical systems can be uncontrollable.

TESTS for CONTROLLABILITY

Notes:

- When is

$$R(\frac{d^2}{dt^2})w = 0$$

  controllable?

  same conditions on R...

- $\exists$ nonlinear, time-varying generalizations.
- 'Real' algorithms: use image representation.
- If $\mathfrak{B} \in \mathfrak{L}^\bullet$ is controllable, transfer with $T > 0$ arbitrarily small.

TESTS for OBSERVABILITY

Given a system representation, derive algorithms in terms of the parameters for observability.
Consider the system defined by

$$R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2.$$

Under which conditions on $R_1, R_2 \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ is w_2 observable from w_1?

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Theorem: In the system $R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2$, w_2 is observable from w_1

$$\Leftrightarrow$$

$$\mathrm{rank}(R_2(\lambda)) = \dim(w_2) \text{ for all } \lambda \in \mathbb{C}.$$

TESTS for OBSERVABILITY

Notes:

- In $R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2$, w_2 is observable from $w_1 \Leftrightarrow$ $R_2(\lambda)$ is of full column rank $\forall\ \lambda \in \mathbb{C}$.

  Equivalently, iff R_2 is *right-invertible* as a polynomial matrix ($\Leftrightarrow$ right-prime).

  $P \in \mathbb{R}^{n_1 \times n_2}[\xi]$ is *right-invertible*

  $:\Leftrightarrow\ \exists\ Q \in \mathbb{R}^{n_2 \times n_1}[\xi]$ such that $QP = I_{n_2}$.

- Equivalently, iff $\exists$ a representation

$$R_1(\frac{d}{dt})w_1 = 0, \quad w_2 = R_2(\frac{d}{dt})w_1$$

  This representation puts observability into evidence.

TESTS for OBSERVABILITY

Notes:

- In $\frac{d}{dt}x = Ax + Bu, y = Cx, w_1 = (u, y), w_2 = x$ the state x is observable from the input/output (u, y) iff

$$\operatorname{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = \dim(x) \ \forall \lambda \in \mathbb{C}.$$

Popov-Belevich-Hautus test for observability.

Of course, $\Leftrightarrow \operatorname{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\dim(x)-1} \end{bmatrix}\right) = \dim(x).$

TESTS for OBSERVABILITY

Notes:

- When is in

$$p(\frac{d}{dt})w_1 = q(\frac{d}{dt})w_2$$

w_2 observable from w_1? $p, q \in \mathbb{R}[\xi]$.

Iff q is a non-zero constant. No zeros!

TESTS for OBSERVABILITY

Notes:

- In the behavioral language, we can speak of 'a controllable system' but not of 'an observable system'! But we will call the latent variable system

$$R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$$

observable (as a system!) if the latent variable ℓ is observable from the manifest variable w.
Conditions, e.g. $\exists$ equivalent representation

$$R(\frac{d}{dt})w = 0 \qquad \ell = R'(\frac{d}{dt})w$$

$R(\frac{d}{dt})w = 0$ hence specifies the manifest behavior.
We can therefore speak of a controllable & observable latent variable, and hence state system.

TESTS for OBSERVABILITY

Notes:

- The RLC circuit is observable iff $CR_C \neq \frac{L}{R_L}$
  Reasonable physical systems can be unobservable.
- When is in
  $$R_1(\frac{d^2}{dt^2})w_1 = R_2(\frac{d^2}{dt^2})w_2$$
  w_2 observable from w_1 ? Same conditions on R_2.
- $\exists$ nonlinear, time-varying generalizations.
- ‘Real’ algorithms: use computer algebra.
- If observable, deduction on $[0, T], T > 0$ arbitrarily small.

IMAGE REPRESENTATIONS

Representations of $\mathfrak{L}^\bullet$:

$$R(\tfrac{d}{dt})w = 0$$

called a *'kernel' representation*. Sol'n set $\in \mathfrak{L}^\bullet$, by definition.

$$R(\tfrac{d}{dt})w = M(\tfrac{d}{dt})\ell$$

called a *'latent variable' representation* of

$\mathfrak{B} = (R(\tfrac{d}{dt}))^{-1} M(\tfrac{d}{dt})\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\ell)$**. El. th'm $\Rightarrow \in \mathfrak{L}^\bullet$.**

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Missing link:

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¿¿ Which kernels are also images ?? Controllability!

IMAGE REPRESENTATIONS

Theorem: (Controllability and image representations):

The following are equivalent for $\mathfrak{B} \in \mathfrak{L}^\bullet$:

1. **$\mathfrak{B}$ is controllable,**

2. **$\mathfrak{B}$ admits an image representation,**

IMAGE REPRESENTATIONS

Theorem: (Controllability and image representations):

The following are equivalent for $\mathfrak{B} \in \mathfrak{L}^\bullet$:

1. $\mathfrak{B}$ is controllable,
2. $\mathfrak{B}$ admits an image representation,
3. for any $a \in \mathbb{R}^{\mathtt{w}}[\xi]$,
   $$a^\top[\tfrac{d}{dt}]\mathfrak{B} \text{ equals } 0 \text{ or all of } \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}),$$
4. $\mathbb{R}^{\mathtt{w}}[\xi]/\mathfrak{N}_{\mathfrak{B}}$ is torsion free,
5. etc.

NUMERICAL TEST for CONTROLLABILITY

- **Image representation leads to an effective numerical test!**

NUMERICAL TEST for CONTROLLABILITY

Start with $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, in kernel representation, with $R \in \mathbb{R}^{\mathtt{w}}[\xi]$, $\rightsquigarrow$ submodule $\mathfrak{R} = \mathfrak{N}_{\mathfrak{B}}$ of $\mathbb{R}^{\mathtt{w}}[\xi]$, generated by transposes of the rows $r_1, \cdots, r_{\mathrm{g}}$ of R.

Compute a set of generators $m_1, \cdots, m_{\mathrm{g}'}$, of the right syzygy of $\mathfrak{R}$: the submodule

$$\mathfrak{M} = \{m \in \mathbb{R}^{\mathtt{w}}[\xi] \mid Rm = 0\}.$$

NUMERICAL TEST for CONTROLLABILITY

Compute a set of generators $m_1, \cdots, m_{g'}$, of the **right syzygy** of $\mathfrak{R}$: the submodule

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Compute a set of generators, $r'_1, \cdots, r'_{g''}$, of the **left syzygy** of $\mathfrak{M}$: the submodule

$$\mathfrak{R}' = \{r' \in \mathbb{R}^{\mathtt{w}}[\xi] \mid r'^{\top}\mathfrak{M} = 0\}$$

Controllability $\Leftrightarrow r_k^{\top} \in \mathfrak{R}' \ \forall k$

i.e., $\mathfrak{R}' = \mathfrak{R}$ (inclusion $\supseteq$ obvious).

$\Rightarrow$ Numerical test for controllability on coefficients of R.

OBSERVABLE IMAGE REPRESENTATION

- $\exists$ an observable image representation $\cong$ 'flatness':

Theorem (Contr. and observable image repr'ns):
The following are equivalent for $\mathfrak{B} \in \mathfrak{L}^\bullet$:

1. $\mathfrak{B}$ is controllable,
2. $\mathfrak{B}$ admits an image representation,
3. $\mathfrak{B}$ admits an observable image representation:

$$w = M(\frac{d}{dt})\ell$$

in which ℓ is observable from w.

- $\exists$ similar results for time-varying systems.
- $\exists$ partial results for nonlinear systems.

STABILIZABILITY

The system $\Sigma = (\mathbb{T}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ is said to be **stabilizable** if, for all $w \in \mathfrak{B}$, there exists $w' \in \mathfrak{B}$ such that

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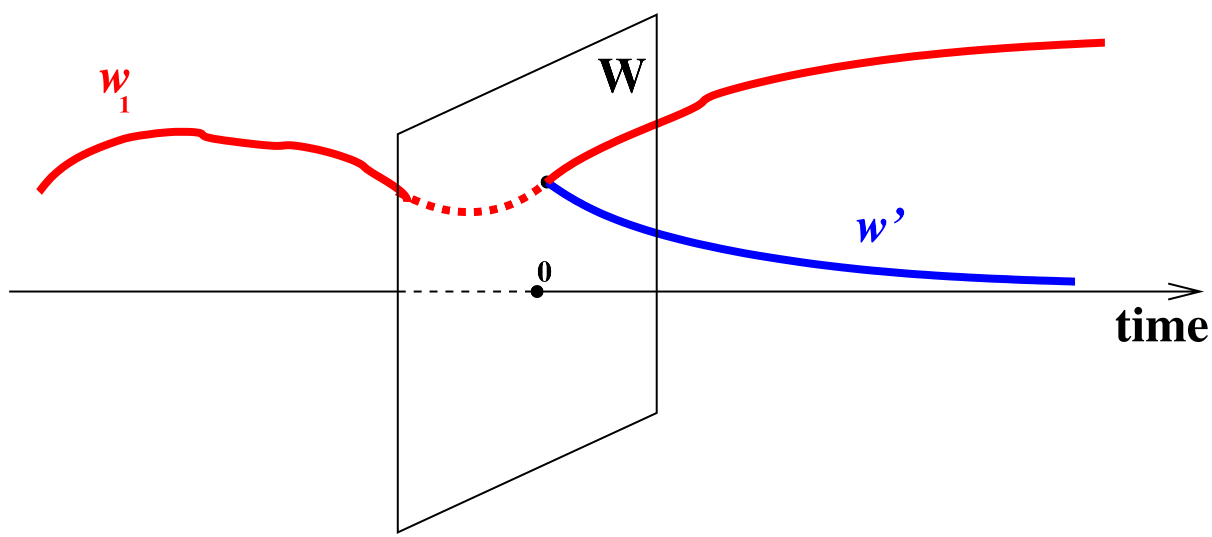
Stabilizability :$\Leftrightarrow$

legal trajectories can be steered to a desired point.



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$\mathrm{rank}(R(\lambda)) =$ **constant over** $\{\lambda \in \mathbb{C} \mid \mathrm{Real}(\lambda) \geq 0\}$.

CONTROLLABLE PART

Every $\mathfrak{B} \in \mathfrak{L}^\bullet$ admits a decomposition

$$\mathfrak{B} = \mathfrak{B}_{\text{controllable}} \oplus \mathfrak{B}_{\text{autonomous}}$$

with $\mathfrak{B}_{\text{controllable}} \in \mathfrak{L}^\bullet$ the 'controllable part' of $\mathfrak{B}$ def. (e.g.) by

$$\mathfrak{B}_{\text{controllable}} := \{w \in \mathfrak{B} \mid \forall\, t_0, t_1 \in \mathbb{R},\ \exists\, w' \in \mathfrak{B}$$

of compact support such that $w(t) = w'(t)$ for $t \in [t_0, t_1]\}$

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Exercises:

1. **Define the charactersitic pol. of an autonomous system $\in \mathfrak{L}^\bullet$.**
2. **Determine stabilizability in terms of the above decomposition.**

RECAP

- **Controllability** := trajectories in behavior are patchable
- **Observability** := to-be-deduced variables reconstructible from observed signal and system behavior
- Controllability in $\mathfrak{L}^\bullet$ $\Leftrightarrow \exists$ an (observable) image representation
- There are effective numerical tests for verifying controllability and observability
- **Stabilizability** := all sol'ns can be steered to 0
- These central concepts in systems and control take on a much more intrinsic meaning for behavioral systems

PDE's

What of this generalizes to PDE's?

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, **the set of independent variables, often** $\mathtt{n} = 4$**,**
$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, **the set of dependent variables,**
$\mathfrak{B} =$ **sol'ns of a linear constant coefficient system of PDE's.**

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Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, **and consider**

$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0. \quad (*)$$

Define the associated behavior

$$\mathfrak{B} = \{ w \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds} \}.$$

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<u>Notation</u> for $\mathtt{n}$**-D linear differential systems:**

$$(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}, \quad \text{or } \mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}.$$

Examples: Maxwell's eq'ns, diffusion eq'n, wave eq'n, . . .

$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}$$

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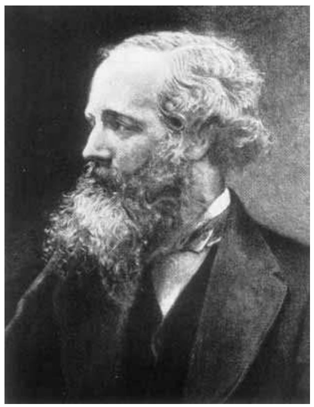

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\end{aligned}
$$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ **(time and space)** $\mathtt{n} = 4$,

$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

(electric field, magnetic field, current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$, $\mathtt{w} = 10$,

$\mathfrak{B}$ = **set of solutions to these PDE's.**

Note: 10 variables, 8 equations! $\Rightarrow \exists$ free variables.

SUBMODULE THEOREM

$R \in \mathbb{R}^{\bullet \times \bullet}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ **defines** $\mathfrak{B} = \ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}))$**,**
but not vice-versa.

¿¿ $\exists$ 'intrinsic' characterization of $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$??

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¿¿ $\exists$ 'intrinsic' characterization of $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$??

Is there a mathematical 'object' that characterizes a $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$?

Define the *annihilators* of $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ by

$$\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}] \mid n^\top(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{B} = 0\}.$$

Proposition: $\mathfrak{N}_{\mathfrak{B}}$ **is a** $\mathbb{R}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ **sub-module of** $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$.

SUBMODULE THEOREM

$$\mathfrak{N}_{\mathfrak{B}} := \{ n \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathrm{n}}] \mid n^\top (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}) \mathfrak{B} = 0 \}.$$

Let $< R^\top >$ denote the submodule of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathrm{n}}]$ spanned by the transposes of the rows of R. Obviously $< R^\top > \subseteq \mathfrak{N}_{\ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}))}$. But, indeed:

$$\mathfrak{N}_{\ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}))} = < R^\top >$$

SUBMODULE THEOREM

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$$\mathfrak{N}_{\ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}))} = < R^\top >$$

Associate with the submodule $\mathfrak{M}$ of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ the system

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid n^\top(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0 \; \forall \, n \in \mathfrak{M}\}$$

Again, every sub-module of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ is finitely generated (but number of generators may be $> \mathtt{w}$), $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$.

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Theorem 1: $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}} \overset{1:1}{\longleftrightarrow}$ **submodules of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$**

ELIMINATION THEOREM

The fundamental principle, and hence the elimination theorem generalize to PDE's!

Which PDE's describe $(\rho, \vec{E}, \vec{j})$ in Maxwell's equations ?

Eliminate $\vec{B}$ from Maxwell's equations $\rightsquigarrow$

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\ \varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{j} &= 0. \end{aligned}$$

$$\boxed{R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0}$$

is called a kernel representation of the associated $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$.

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Another representation: image representation

$$\boxed{w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\ell.}$$

'Elimination' thm $\quad \Rightarrow \quad \mathrm{im}(M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})) \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ **!**

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Which linear diff. systems admit an image representation???

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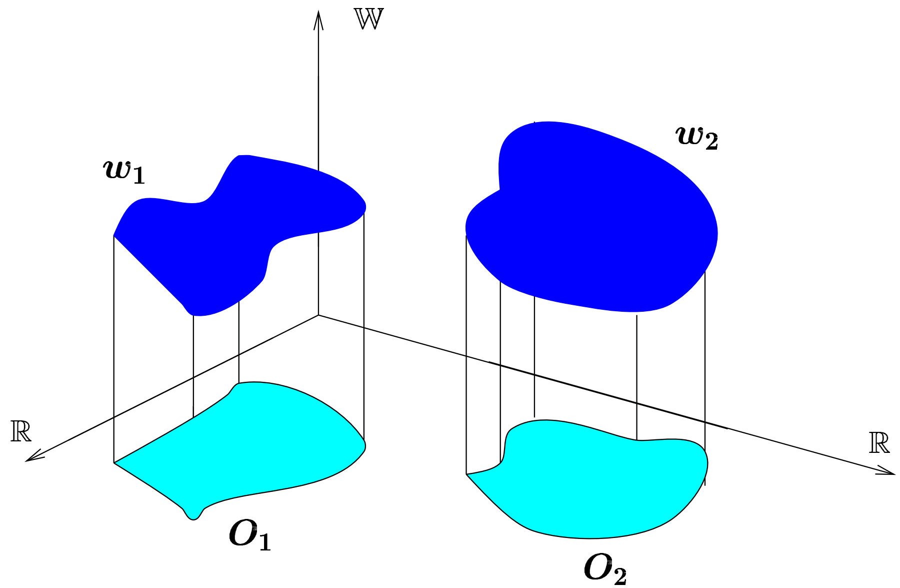
Which linear diff. systems admit an image representation???

$\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ admits an image representation iff it is 'controllable'.

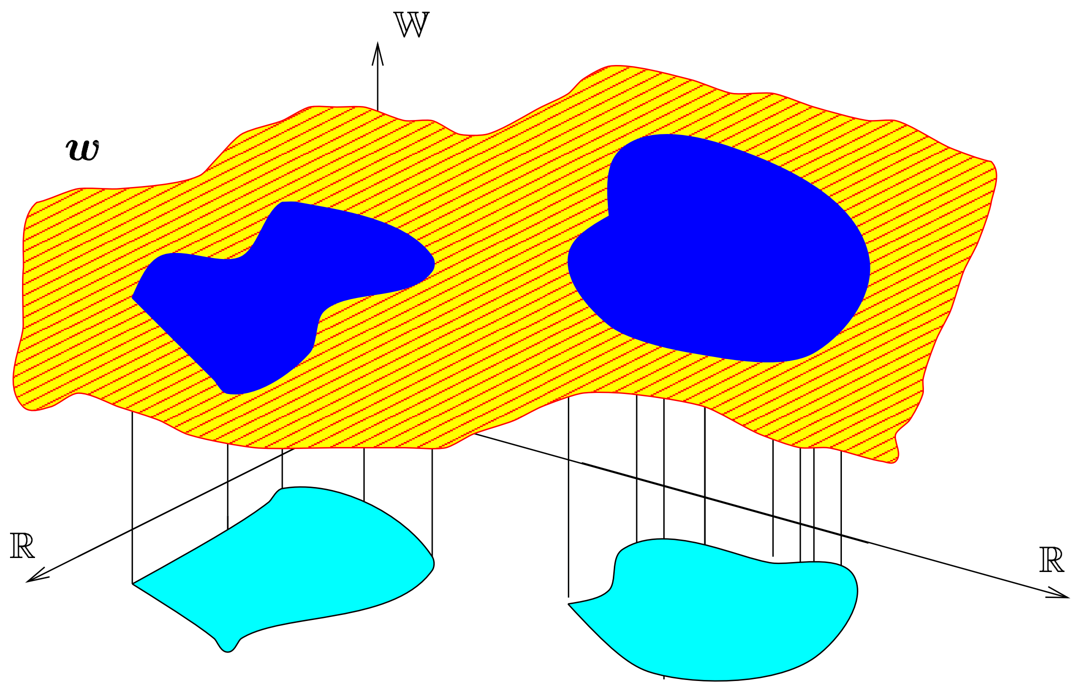
Controllability def'n in pictures:



$$w_1, w_2 \in \mathfrak{B}.$$

CONTROLLABILITY of PDE's

$w \in \mathfrak{B}$ **'patches'** $w_1, w_2 \in \mathfrak{B}$.



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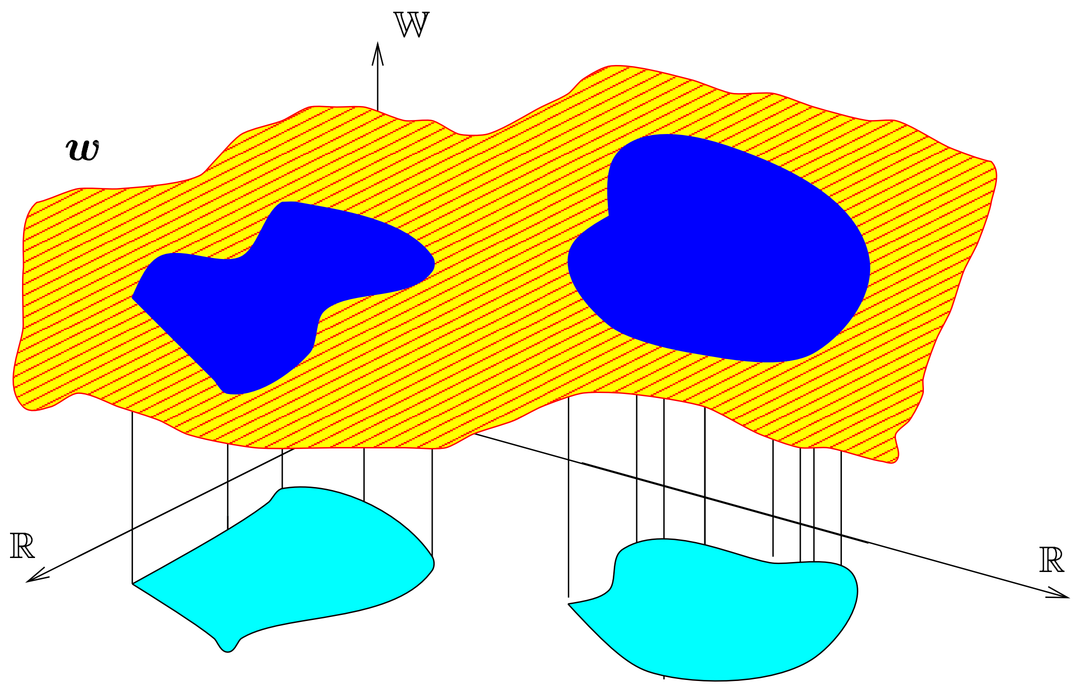

Controllability $:\Leftrightarrow$ 'patch-ability'.

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The following equations in the *scalar potential* $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, generate exactly the solutions to Maxwell's equations:

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
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Proves controllability.

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$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow \exists$ potential!

OBSERVABILITY

Observability of the image representation

$$w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})\ell$$

is defined as: ℓ can be deduced from w,

i.e., $M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})$ should be injective.

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For $\mathrm{n} = 1$, they do. For $\mathrm{n} > 1$, exceptionally so.
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Example: Maxwell's equations do not allow a potential representation that is observable.

End of the Lecture II