OPEN DYNAMICAL SYSTEMS:

Basic concepts and examples

Chaire Francqui, Lecture I, May 5, 2004

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THEMES

- Mathematical models, the behavior
- Dynamical systems
- Examples
- Linear differential systems $\leadsto$ 3 basic theorems:

1. Characterization as submodules and Structure of kernel representations
2. Elimination theorem
3. Input/output representability

Generalities

What is a model? As a mathematical concept.

Generalities

What is a model?

What is a dynamical system? As a mathematical concept.

Generalities

What is a model?

What is a dynamical system?

What is the role of differential equations in thinking about dynamical models?

Generalities

Intuition

We have a ‘phenomenon’ that produces ‘outcomes’ (‘events’).
We wish to model the outcomes that can occur.

Before we model the phenomenon:
the outcomes are in a set, which we call the *universum*.

After we model the phenomenon:
the outcomes are declared (thought, believed)
to belong to the *behavior* of the model,
a subset of the universum.

This subset is what we call the mathematical model.

Generalities

This way we arrive at the

Definition

A *mathematical model* is a subset $\mathfrak{B}$ of a universum $\mathfrak{U}$ of outcomes,

$$\mathfrak{B} \subseteq \mathfrak{U}.$$

$\mathfrak{B}$ is called the *behavior* of the model.

Features

- Generality, applicability
- shows the role of model equations
- $\rightsquigarrow$ notion of equivalent models
- $\rightsquigarrow$ notion of more powerful model
- Structure, symmetries
- ...

In these lectures, we will only consider deterministic models.

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Stochastic models: there is a map P (the **'probability'**)

$$\boxed{P : \mathcal{A} \to [0, 1]}$$

with $\mathcal{A}$ a 'σ-algebra' of subsets of $\mathfrak{U}$.

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with $\mathcal{A}$ a 'σ-algebra' of subsets of $\mathfrak{U}$.

$P(\mathfrak{B}) =$ 'the degree of certainty (belief, plausibility, relative frequency) that outcomes are in $\mathfrak{B}$';

$\cong$ the degree of validity of $\mathfrak{B}$ as a model.

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Fuzzy models: there is a map μ (the 'membership function')

$$\boxed{\mu : \mathfrak{U} \to [0,1]}$$

$\mu(x) =$ 'the extent to which $x \in \mathfrak{U}$ belongs to the model'.

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Stochastic models: there is a map P (the 'probability')

$$\boxed{P : \mathcal{A} \to [0, 1]}$$

with $\mathcal{A}$ a 'σ-algebra' of subsets of $\mathfrak{U}$.

Determinism: $\mathcal{A} = \{, \mathfrak{B}, \mathfrak{B}^{\text{complement}}, \mathfrak{U}\}$.

Fuzzy models: there is a map μ (the 'membership function')

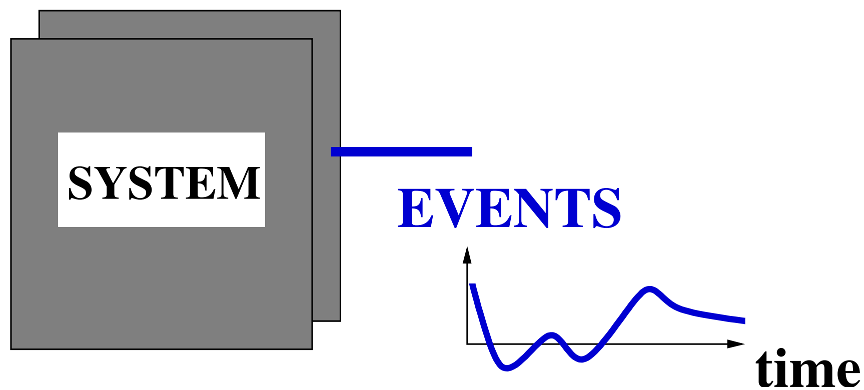
$$\boxed{\mu : \mathfrak{U} \to [0, 1]}$$

Determinism: μ is 'crisp':

$$\text{image}(\mu) = \{0, 1\}, \;\; \mathfrak{B} = \mu^{-1}(\{1\}) := \{x \in \mathfrak{U} \mid \mu(x) = 1\}$$

Dynamical systems

In dynamics, the outcomes are functions of time $\leadsto$



Which event trajectories are possible?

Dynamical systems

Definition

A dynamical system **=** $\Sigma := (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

with $\mathbb{T} \subseteq \mathbb{R}$**, the *time-axis* (= the relevant time instances),**
$\mathbb{W}$**, the *signal space***
(= where the variables take on their values),
$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ ***the behavior*** **(= the admissible trajectories).**

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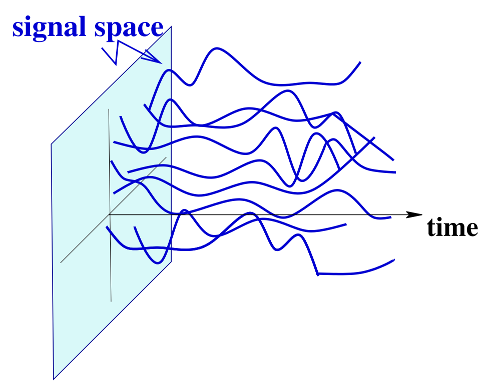

Totality of 'legal' trajectories =: the behavior

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For a trajectory ('an event') $w : \mathbb{T} \to \mathbb{W}$, we thus have:

$w \in \mathfrak{B}$: the model allows the trajectory w,
$w \notin \mathfrak{B}$: the model forbids the trajectory w.

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Usually,
$\mathbb{T} = \mathbb{R}$, or $[0, \infty)$, etc. (in continuous-time systems),
or $\mathbb{Z}$, or $\mathbb{N}$, etc. (in discrete-time systems).

Dynamical systems

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Usually,
$\mathbb{W} \subseteq \mathbb{R}^{\mathtt{w}}$ (in lumped systems),
a function space
(in distributed systems, time a distinguished variable),
a finite set (in DES)' etc.

Dynamical systems

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Emphasis:

$\mathbb{T} = \mathbb{R}$,
$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,
$\mathfrak{B}$ = solutions of system of linear constant coefficient
ODE's, or difference eqn's, or PDE's.

More structure

$$\boldsymbol{\Sigma} = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

is said to be

More structure

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

is said to be **linear**

if $\mathbb{W}$ is a vector space, and $\mathfrak{B}$ a linear subspace of $\mathbb{W}^{\mathbb{T}}$.

More structure

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

is said to be time-invariant

if $\mathbb{T} = \mathbb{R}, \mathbb{R}_+, \mathbb{Z},$ **or** $\mathbb{Z}_+$ **and if** $\mathfrak{B}$ **satisfies**

$$\sigma^t \mathfrak{B} \subseteq \mathfrak{B} \text{ for all } t \in \mathbb{T}.$$

σ^t **denotes the shift,** $\sigma^t f(t') := f(t' + t)$**.**

More structure

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

is said to be **differential**

if $\mathbb{T} = \mathbb{R}$, or $\mathbb{R}_+$, etc., and if $\mathfrak{B}$ is the solution set of a (system of) ODE's.

a **difference system** if, etc.

More structure

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

is said to be **symmetric**

w.r.t. the transformation group $\{T_g, g \in \mathfrak{G}\}$ on $\mathbb{W}^{\mathbb{T}}$

if $\mathbb{T}_g \mathfrak{B} = \mathfrak{B}$ for all $g \in \mathfrak{G}$.

Examples:

1. time-invariance, time-reversibility
2. permutation symmetry, rotation symmetry, translation symmetry, Euclidean symmetry,
3. etc., etc.

A series of examples

Planetary motion

Let's put Kepler and Newton in this setting.

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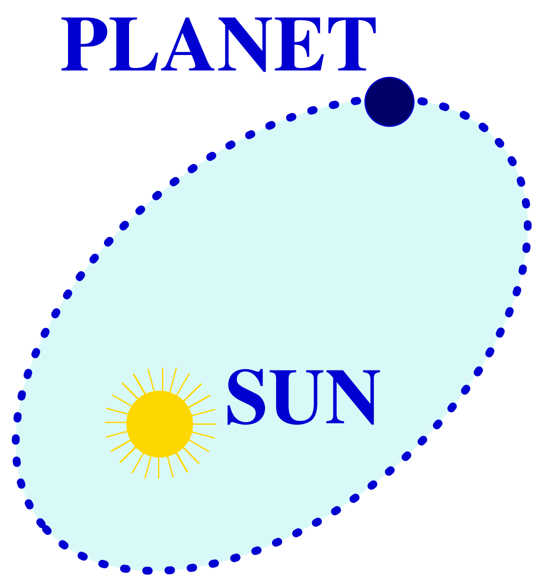
How does it move?

Kepler's laws

Johannes Kepler (1571-1630)



Kepler's first law:

Ellipse, sun in focus

Kepler's laws

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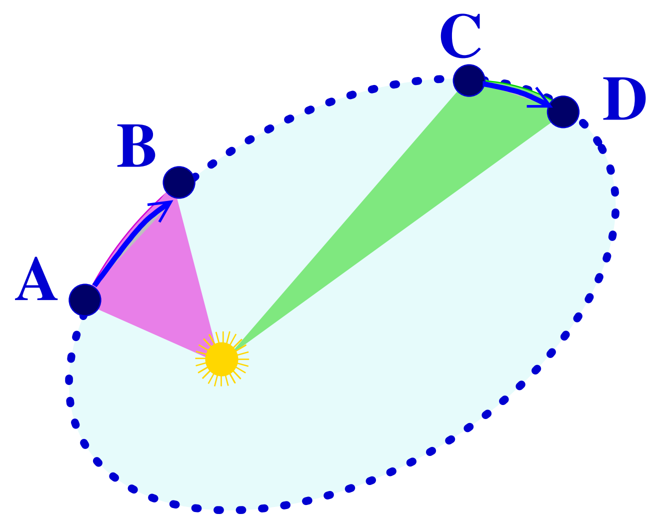

Kepler's second law:

= areas in = times

Kepler's laws

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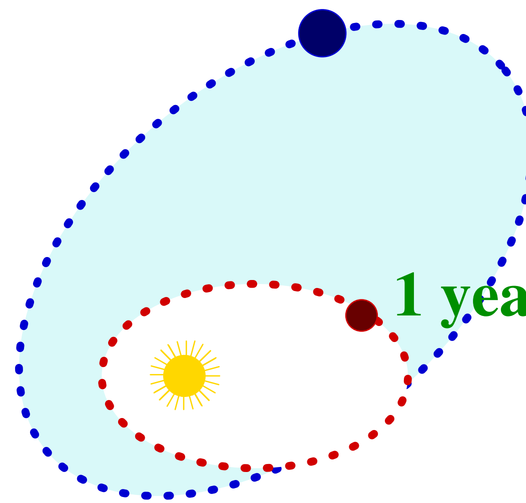

Kepler's third law:

$$(\text{period})^2 = (\text{diameter})^3$$

Kepler's laws

This obviously defines a dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$\mathbb{T} = \mathbb{R}$,
$\mathbb{W} = \mathbb{R}^3$,
$\mathfrak{B} =$ **all** $w : \mathbb{R} \to \mathbb{R}^3$ **that satisfy Kepler's 3 laws.**

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Nice example of a dynamical model 'without equations'.

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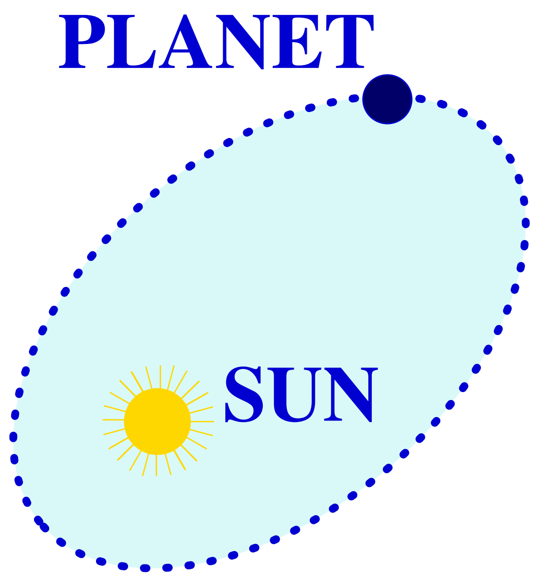
Is it a differential system?

This question turned out to be of revolutionary importance...

The state of the planet

What determines the orbit uniquely?



The state of the planet

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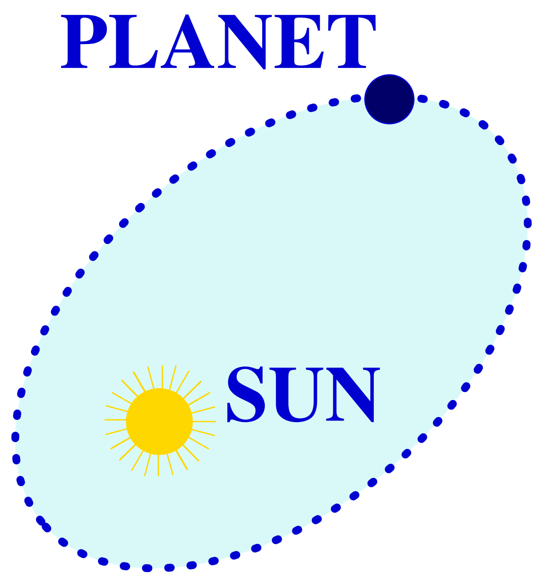

The position?

The state of the planet

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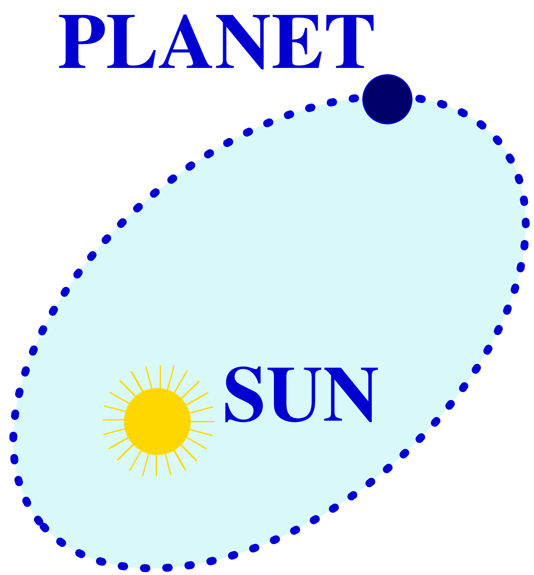
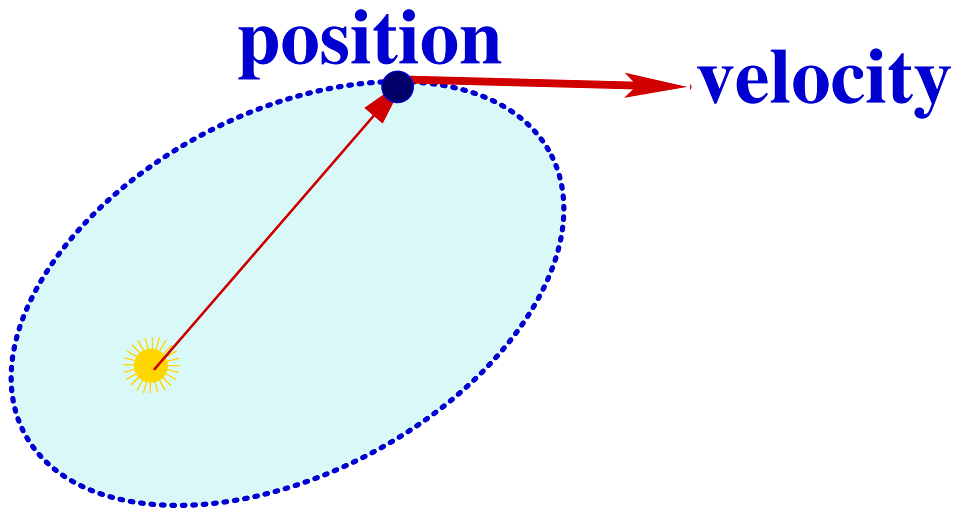


The position and the direction of motion?

The state of the planet



The state = position and velocity

The equation of the planet

Consequence:

acceleration = function of position and velocity

$$\frac{d^2}{dt^2}w(t) = A(w(t), \frac{d}{dt}w(t))$$

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$$\frac{d^2}{dt^2}w(t) = A\left(w(t), \frac{d}{dt}w(t)\right)$$

$\rightsquigarrow$ **via calculus and calculation**

$$\frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|w(t)|^2} = 0$$

The equation of the planet

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The equation of the planet

$\leadsto$ via calculus and calculation

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Is it really equivalent to K.1, K.2, K.3 ?

Kepler’s laws

K.1, K.2, & K.3

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K.1, K.2, & K.3

$$\Updownarrow$$

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Hypotheses non fingo

Kepler's laws

K.1, K.2, & K.3

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Hypotheses non fingo

$\downdownarrows$

$$F = m\frac{d^2}{dt^2}w$$

More examples

Flows

$$\frac{d}{dt}x(t) = f(x(t)),$$

$\mathfrak{B}$ = all state trajectories.

More examples

Observed flows

$$\frac{d}{dt}x(t) = f(x(t)); \quad y(t) = h(x(t)),$$

$\mathfrak{B} =$ **all possible output trajectories.**

Note:

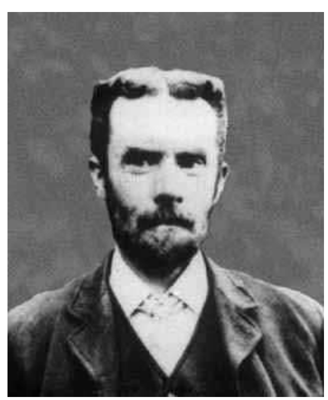
1. **It may be impossible to express $\mathfrak{B}$ as the solutions of a differential equation involving only y.**
2. **The auxiliary (latent variable) nature of x.**

More examples

Input / output systems

$$f_1\left(y(t), \frac{d}{dt}y(t), \frac{d^2}{dt^2}y(t), \ldots, t\right)$$
$$= f_2\left(u(t), \frac{d}{dt}u(t), \frac{d^2}{dt^2}u(t), \ldots, t\right)$$

$\mathbb{T} = \mathbb{R}$ **(time),**
$\mathbb{W} = \mathbb{U} \times \mathbb{Y}$ **(input $\times$ output signal spaces),**
$\mathfrak{B} =$ **all input / output pairs.**



More examples

Input / output systems

$$f_1\left(y(t), \frac{d}{dt}y(t), \frac{d^2}{dt^2}y(t), \ldots, t\right) = f_2\left(u(t), \frac{d}{dt}u(t), \frac{d^2}{dt^2}u(t), \ldots, t\right)$$

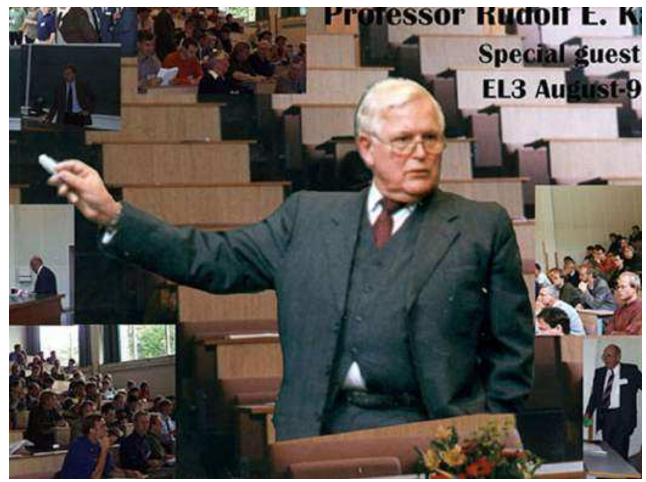
$\mathbb{T} = \mathbb{R}$ **(time),**
$\mathbb{W} = \mathbb{U} \times \mathbb{Y}$ **(input $\times$ output signal spaces),**
$\mathfrak{B} =$ **all input / output pairs.**
Of course, more is required to justify calling
u 'input' and y 'output'.

Not a good starting point!! Why not?

More examples

Input / state / output systems

$$\frac{d}{dt}x(t) = f(x(t), u(t), t), \;\; y(t) = h(x(t), u(t), t)$$

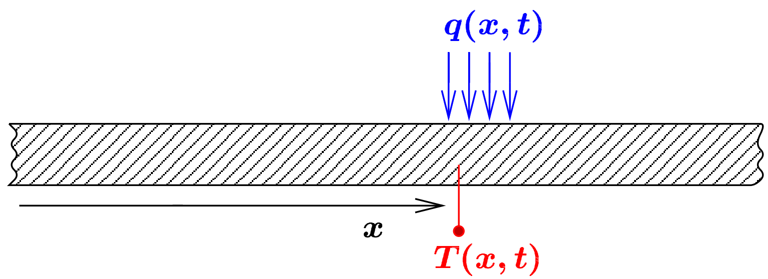


What do we want to call the behavior?
the (u, y, x)'s, or the (u, y)'s?

Is the (u, y) behavior described by a differential eq'n?

More examples

Heat diffusion



Diffusion **describes the evolution of the temperature $T(x,t)$ ($x \in \mathbb{R}$ position, $t \in \mathbb{R}$ time) along a uniform bar (infinitely long), and the heat $q(x,T)$ supplied to the bar. $\rightsquigarrow$ the PDE**

$$\frac{\partial}{\partial t}T = \frac{\partial^2}{\partial x^2}T + q$$

$\mathbb{T} = \mathbb{R}$ **(time),**

$\mathbb{W} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^2)$ **all (temperature, heat) distributions,**

$\mathfrak{B} =$ **all $T(\cdot,t), q(\cdot,t)$-pairs that satisfy the PDE.**

Note: We view t as a distinguished variable.

More examples

Codes

$\mathbb{A}$ = the code alphabet, say, $\mathbb{A} = \mathbb{F}^{\mathtt{w}}$, $\mathbb{F}$ a finite field,
$\mathbb{I}$ = an index set, say,

$\mathbb{I} = (1, \cdots, \mathtt{n})$ in block codes,
$\mathbb{I} = \mathbb{N}$ or $\mathbb{Z}$ in convolutional codes,

$\mathfrak{C} \subseteq \mathbb{A}^{\mathbb{I}}$ = the code; yields the system $\Sigma = (\mathbb{I}, \mathbb{A}, \mathfrak{C})$.

Redundancy structure, error correction possibilities, etc., are visible in the code behavior $\mathfrak{C}$.

It is the central object of study.

The encoder and decoder can be put (temporarily) into the background.

More examples

Example:

The following error detecting code:
$\mathbb{I} = \mathbb{Z}, \mathbb{A} = \mathbb{F} = \{0, 1\}$,
$\mathfrak{B}$ = **all compact support sequences** $w : \mathbb{Z} \to \mathbb{F}$ **such that**

$$w(t) = p_0 \ell(t) + p_1 \ell(t-1) + \cdots + p_{\mathrm{n}} \ell(t-\mathrm{n})$$

for some $\ell : \mathbb{Z} \to \mathbb{F}$, **with** $p_0, p_1, \ldots, p_{\mathrm{n}} \in \mathbb{F}$ **design parameters.**

More examples

Formal languages

$\mathbb{A}$ $=$ a (finite) alphabet,

$\mathfrak{L} \subseteq \mathbb{A}^*$ $=$ the language = all 'legal' 'words' $a_1 a_2 \cdots a_k \cdots$

$\mathbb{A}^*$ $=$ all finite strings with symbols from $\mathbb{A}$.

yields the system $\Sigma = (\mathbb{N}, \mathbb{A}, \mathfrak{L})$.

Examples: All words appearing in the *van Dale*

All LaTeX documents.

More examples

Thermodynamics

Thermodynamics is the only theory of a general nature of which I am convinced that it will never be overthrown.

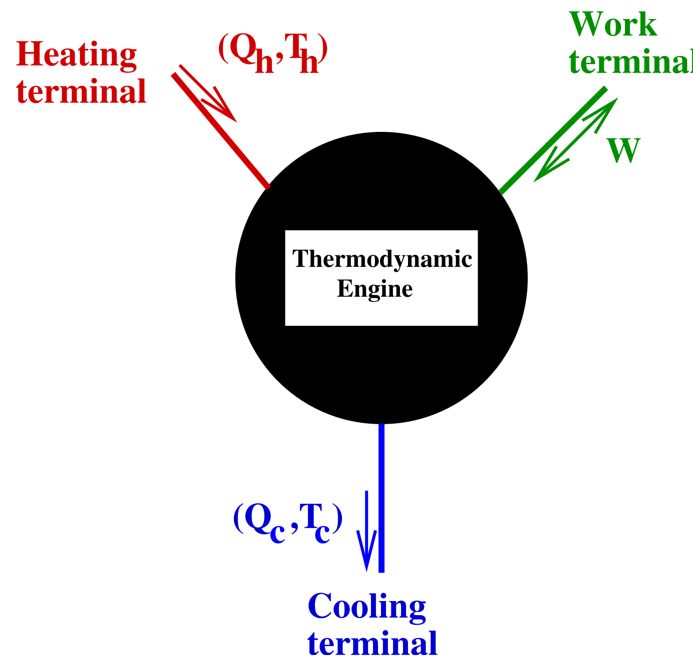
Albert Einstein

The law that entropy always increases – the second law of thermodynamics – holds, I think, the supreme position among the laws of nature.

Arthur Eddington

More examples

Thermodynamics



time-axis: $\mathbb{R}$

Q: Variables of interest? **A:** Q_h, T_h, Q_c, T_c, W

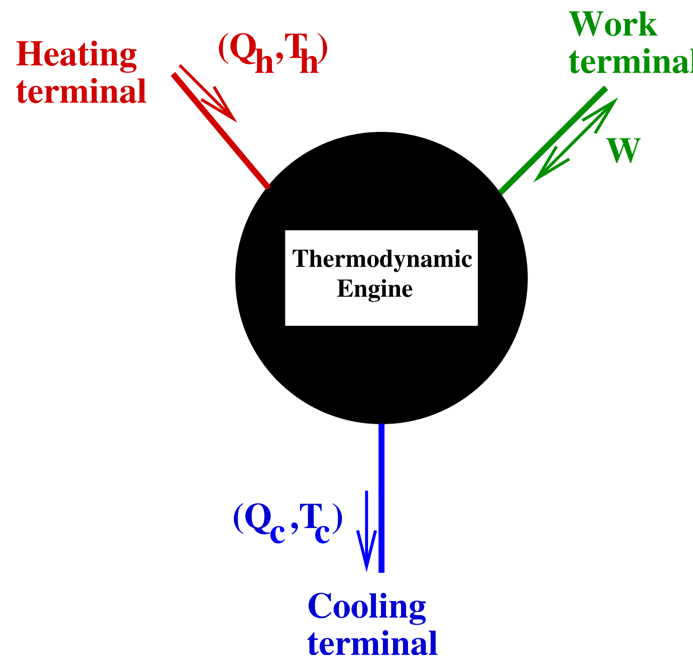
$\rightsquigarrow$ **signal space:** $\mathbb{W} = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$

Behavior $\mathfrak{B}$: a suitable family of trajectories.

But, there are some universal laws that restrict the $\mathfrak{B}$'s that are 'thermodynamic'.

More examples

Thermodynamics



First and second law:

$$\oint (Q_h - Q_c - W)\,dt = 0; \quad \oint \left(\frac{Q_h}{T_h} - \frac{Q_c}{T_c}\right) dt \leq 0.$$

These laws deal with ‘open’ systems.

But not with input/output systems!

Conclusion

The framework

$$\boldsymbol{\Sigma} = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

with $\mathfrak{B}$ = the 'behavior'

is a very flexible, universal, pedagogical approach to the theory of models in general and dynamical systems in particular!

Linear Differential Systems

Polynomials

$\mathbb{R}[\xi]$:= the polynomials with real coefficients in the 'indeterminate' ξ.

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Addition, multiplication defined in the obvious way.

This makes $\mathbb{R}[\xi]$ **into a 'ring'.**

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This makes $\mathbb{R}^{\mathrm{n}}[\xi]$ into a 'module' over $\mathbb{R}[\xi]$.

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Addition, multiplication by polynomials ('scalar multiplication') defined in the obvious way.

This makes $\mathbb{R}^{\mathrm{n}}[\xi]$ **into a 'module' over** $\mathbb{R}[\xi]$**.**

A *submodule* of $\mathbb{R}^{\mathrm{n}}[\xi]$ **is a subset of** $\mathbb{R}^{\mathrm{n}}[\xi]$ **that is closed under addition and scalar multiplication.**

Polynomials

$\mathbb{R}[\xi]$:= the polynomials with real coefficients in the 'indeterminate' ξ.

A *polynomial vector* is a vector of polynomials, $\rightsquigarrow \mathbb{R}^{\mathrm{n}}[\xi]$

Example of a submodule: Let $g_1, g_2, \ldots, g_{\mathrm{k}} \in \mathbb{R}^{\mathrm{n}}[\xi]$ and consider all elements of the form

$$p_1 g_1 + p_2 g_2 + \cdots + p_{\mathrm{k}} g_{\mathrm{k}}$$

with $p_1, p_2, \ldots, p_{\mathrm{k}} \in \mathbb{R}[\xi]$. This is obviously a submodule of $\mathbb{R}^{\mathrm{n}}[\xi]$, called the submodule 'generated by' $g_1, g_2, \ldots, g_{\mathrm{k}}$ ('the generators').

Notation:

$$< g_1, g_2, \ldots, g_{\mathrm{k}} > \quad \text{or} \quad < G >$$

if we organize the g's into the matrix $G = \begin{bmatrix} g_1 & g_2 & \cdots & g_{\mathrm{k}} \end{bmatrix}$

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Every submodule of $\mathbb{R}^{\mathrm{n}}[\xi]$ is actually of this form: 'finitely generated'; max. # of generators required is, in fact, n.
Exercise: Prove this.

Polynomials

$\mathbb{R}[\xi]$:= the polynomials with real coefficients in the 'indeterminate' ξ.

A ***polynomial vector*** is a vector of polynomials, $\rightsquigarrow \mathbb{R}^{\mathtt{n}}[\xi]$

A ***polynomial matrix*** is a matrix of polynomials.

Notation: $\mathbb{R}^{\mathtt{n}}[\xi], \mathbb{R}^{\bullet}[\xi], \mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_2}[\xi], \mathbb{R}^{\bullet \times \mathtt{n}}[\xi], \mathbb{R}^{\mathtt{n} \times \bullet}[\xi], \mathbb{R}^{\bullet \times \bullet}[\xi]$.

Add and multiply as if they were ordinary matrices.

Polynomials

In $\mathbb{R}^{n \times n}[\xi]$ we can add and multiply elements:

it is a 'non-commutative' ring.

So, elements could be invertible (in $\mathbb{R}^{n \times n}[\xi]$!).

These are an important type of square polynomial matrices:

Definition: $P \in \mathbb{R}^{n \times n}[\xi]$ **is said to be** ***unimodular*** **if** $\exists$

$Q \in \mathbb{R}^{n \times n}[\xi]$ **such that** $QP = I_{n \times n}$.

This Q **is denoted as** P^{-1}. **Easy:** $PP^{-1} = P^{-1}P = I_{n \times n}$

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Proposition: $P \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}[\xi]$ ***is unimodular if and only if***
$\det(P) = \alpha$***, with*** $0 \neq \alpha \in \mathbb{R}$.

Polynomials

The following result is very useful in proofs. It shows that, by pre- and postmultiplying by unimodular matrices, polynomial matrices can be brought in *Smith form,* a simple, diagonal like form (reminiscent of the Jordan form).

Polynomials

Let $P \in \mathbb{R}^{n_1 \times n_2}[\xi]$. There exist unimodular polynomial matrices $U \in \mathbb{R}^{n_1 \times n_1}[\xi]$ and $V \in \mathbb{R}^{n_2 \times n_2}[\xi]$ such that

$$UPV = \begin{bmatrix} \operatorname{diag}(p_1, p_2, \ldots, p_r) & 0_{r \times (n_2 - r)} \\ 0_{(n_1 - r) \times r} & 0_{(n_1 - r) \times (n_2 - r)} \end{bmatrix}$$

where $r = \operatorname{rank}(P)$, $p_1, p_2, \ldots, p_r \in \mathbb{R}[\xi]$, and p_{k+1} is a factor of p_k for $k = 1, 2, \ldots k - 1$.

$p_1, p_2, \ldots, p_r$ are called the ***invariant factors*** of P.

Polynomials

Exercises:

1. Prove the Smith form. **Hint: Consider a non-zero element of P of smallest degree. Assume that in the same row or column as this element there is a second non-zero element. Now use an elementary row or column operation (meaning: replace this second row or column by the sum of this row or column and a polynomial multiple of the first row or column) to create either a new zero element, or a non-zero element of lower dgeree. Show that this process ends in a finite number of steps, and that it ends if, up to a permutation of rows and columns, the matrix is in Smith form.**

2. Use the Smith form to prove that every submodule of $\mathbb{R}^{\mathtt{n}}[\xi]$ is finitely generated, and that the matrix formed by the generators as columns can be taken to have full row rank, and hence that the number of generators is at most $\mathtt{n}$.

Linear time-invariant differential systems

We discuss the fundamentals of the theory of dynamical systems

$$\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$$

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1. ***linear*** **meaning ('superposition')**

$$((w_1, w_2 \in \mathfrak{B}) \wedge (\alpha, \beta \in \mathbb{R})) \Rightarrow (\alpha w_1 + \beta w_2 \in \mathfrak{B});$$

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2. ***time-invariant*** meaning

$$((w \in \mathfrak{B}) \wedge (t \in \mathbb{R})) \Rightarrow (\sigma^t w \in \mathfrak{B}))$$

3. ***differential,*** meaning
$\mathfrak{B}$ consists of the sol'ns of a system of differential eq'ns.

Linear time-invariant differential systems

Variables: $w_1, w_2, \ldots w_{\mathtt{w}}$, **up to** $\mathtt{n}$**-times differentiated,** $\mathtt{g}$ **equations.** $\rightsquigarrow$

$$\begin{array}{rcl}
\Sigma_{j=1}^{\mathtt{w}} R_{1,j}^{0} w_j + \Sigma_{j=1}^{\mathtt{w}} R_{1,j}^{1} \frac{d}{dt} w_j + \cdots + \Sigma_{j=1}^{\mathtt{w}} R_{1,j}^{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w_j & = & 0 \\
\Sigma_{j=1}^{\mathtt{w}} R_{2,j}^{0} w_j + \Sigma_{j=1}^{\mathtt{w}} R_{2,j}^{1} \frac{d}{dt} w_j + \cdots + \Sigma_{j=1}^{\mathtt{w}} R_{2,j}^{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w_j & = & 0 \\
\vdots \qquad \vdots & & \vdots \\
\Sigma_{j=1}^{\mathtt{w}} R_{\mathtt{g},j}^{0} w_j + \Sigma_{j=1}^{\mathtt{w}} R_{\mathtt{g},j}^{1} \frac{d}{dt} w_j + \cdots + \Sigma_{j=1}^{\mathtt{w}} R_{\mathtt{g},j}^{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w_j & = & 0
\end{array}$$

Coefficients $R_{\imath,j}^{k}$**: 3 indices!**

- $\imath = 1, \ldots, \mathtt{g}$ **: for the** $\imath$**-th differential equation,**
- $j = 1, \ldots, \mathtt{w}$ **: for the variable** w_j **involved,**
- $k = 1, \ldots, \mathtt{n}$ **: for the order** $\frac{d^k}{dt^k}$ **of differentiation.**

Linear time-invariant differential systems

In vector/matrix notation:

$$w = \begin{bmatrix} w_1 \\ w_2, \\ \vdots \\ w_{\mathtt{w}} \end{bmatrix}, \quad R_{\mathtt{k}} = \begin{bmatrix} R^{\mathtt{k}}_{1,1} & R^{\mathtt{k}}_{1,2} & \cdots & R^{\mathtt{k}}_{1,\mathtt{w}} \\ R^{\mathtt{k}}_{2,1} & R^{\mathtt{k}}_{2,2} & \cdots & R^{\mathtt{k}}_{2,\mathtt{w}} \\ \vdots & \vdots & \cdots & \vdots \\ R^{\mathtt{k}}_{\mathtt{g},1} & R^{\mathtt{k}}_{\mathtt{g},2} & \cdots & R^{\mathtt{k}}_{\mathtt{g},\mathtt{w}} \end{bmatrix}.$$

Linear time-invariant differential systems

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Yields

$$\boxed{R_0 w + R_1 \frac{d}{dt} w + \cdots + R_n \frac{d^n}{dt^n} w = 0,}$$

with $R_0, R_1, \cdots, R_n \in \mathbb{R}^{g \times \mathtt{w}}$.

Linear time-invariant differential systems

Yields

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0,$$

with $R_0, R_1, \cdots, R_{\mathtt{n}} \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}$.
Combined with the polynomial matrix

$$R(\xi) = R_0 + R_1 \xi + \cdots + R_{\mathtt{n}} \xi^{\mathtt{n}},$$

we obtain the mercifully short notation

$$R(\frac{d}{dt}) w = 0.$$

Other similar situations:

Difference equations on $\mathbb{Z}_+ \rightsquigarrow$

$$R(\sigma)w = 0;$$

Difference equations on $\mathbb{Z} \rightsquigarrow$

$$R(\sigma, \sigma^{-1})w = 0;$$

PDE's on $\mathbb{R}^{\mathrm{n}} \rightsquigarrow$

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w = 0;$$

Differential delay systems on $\mathbb{R} \rightsquigarrow$

$$R(\frac{d}{dt}, \sigma, \sigma^{-1})w = 0;$$

etc.

<u>Execise</u>: State and prove the appropriate version of the 3 basic theorems for discrete-time systems, with $\mathbb{T} = \mathbb{Z}$ **and** $\mathbb{Z}_+$**.**

The behavior of $R(\frac{d}{dt})w = 0$

What do we mean by the behavior of this system of differential equations?

When do we want to call $w : \mathbb{R} \rightarrow \mathbb{R}^{\mathtt{w}}$ a solution?

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When do we want to call $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ a solution?

Possibilities:

Strong solutions?
Weak solutions?
$\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ (infinitely differentiable) solutions?
Distributional solutions?
Compact support solutions, etc.

The behavior of $R(\frac{d}{dt})w = 0$

$\mathfrak{C}^\infty$-solutions:

$w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ is a $\mathfrak{C}^\infty$-solution of $R(\frac{d}{dt})w = 0$ if

1. w is infinitely differentiable (=: $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$), and
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Weak solutions:

$w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ is a weak solution of $R(\frac{d}{dt})w = 0$ if

1. $\int_{t_0}^{t_1} ||w(t)||\, dt < \infty$ for all $t_0, t_1 \in \mathbb{R}$, and
2. $\int_{-\infty}^{+\infty} (R^\top(-\frac{d}{dt})a)^\top(t)w(t)\, dt = 0$ for all $a \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{rowdim}(R)})$ of compact support (i.e., a is zero outside some finite interval).

The behavior of $R(\frac{d}{dt})w = 0$

$\mathfrak{C}^\infty$-solutions:

$w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ is a **$\mathfrak{C}^\infty$-solution** of $R(\frac{d}{dt})w = 0$ if

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'Pragmatic', easy way out: **$\mathfrak{C}^\infty$ soln's!** Transmits main ideas, easier to handle, easy theory, sometimes (too) restrictive (step-response, state property etc.).

The behavior of $R(\frac{d}{dt})w = 0$

Whence, $R(\frac{d}{dt})w = 0$ defines the system $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ with

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{d}{dt})w = 0\}.$$

Proposition: This system is linear and time-invariant.

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NOTATION

$\mathfrak{L}^{\bullet}$: all such systems

(with any - finite - number of variables)

$\mathfrak{L}^{\mathtt{w}}$: with $\mathtt{w}$ variables

$\mathfrak{B} = \ker(R(\frac{d}{dt}))$

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ (no ambiguity regarding $\mathbb{T}$, $\mathbb{W}$)

The behavior of $R(\frac{d}{dt})w = 0$

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NOMENCLATURE

Elements of $\mathfrak{L}^{\bullet}$: *linear differential systems*

$R(\frac{d}{dt})w = 0$: a *kernel representation* of the corresponding $\Sigma \in \mathfrak{L}^{\bullet}$ or $\mathfrak{B} \in \mathfrak{L}^{\bullet}$

$R(\frac{d}{dt})w = 0$ 'has' behavior $\mathfrak{B}$

Σ or $\mathfrak{B}$: the system *induced by* $R \in \mathbb{R}^{\bullet \times \bullet}[\xi]$

The behavior of $R(\frac{d}{dt})w = 0$

Some other properties of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$:

$$(w \in \mathfrak{B}) \Rightarrow (\tfrac{d}{dt} w \in \mathfrak{B});$$

$$(w \in \mathfrak{B} \;\; \text{and} \;\; p \in \mathbb{R}[\xi]) \Rightarrow (p(\tfrac{d}{dt}) w \in \mathfrak{B});$$

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Further niceties:

$$(w \in \mathfrak{B} \ \text{ and } \ f \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R})) \Rightarrow (f * w \in \mathfrak{B}),$$

$*$ denotes convolution

$\mathfrak{C}^{\infty}$-solutions of $R(\frac{d}{dt})w = 0$ are dense in the set of weak (or distributional) solutions.

3 theorems about $\mathfrak{L}^\bullet$

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Theorem 1:

There is a one-to-one relation between $\mathfrak{L}^{\mathtt{w}}$ and the submodules of $\mathbb{R}^{\mathtt{w}}[\xi]$.

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Theorem 2 (Elimination theorem):

$\mathfrak{L}^\bullet$ is closed under projection.

3 theorems about $\mathfrak{L}^\bullet$

Theorem 1:

There is a one-to-one relation between $\mathfrak{L}^{\mathtt{w}}$
and the submodules of $\mathbb{R}^{\mathtt{w}}[\xi]$.

Theorem 2 **(Elimination theorem)**:

$\mathfrak{L}^\bullet$ is closed under projection.

Theorem 3 **(Input / output representation)**:

For each $\mathfrak{B} \in \mathfrak{L}^\bullet$ there is a componenentwise partition
of the variables into inputs and outputs.
This partition is not unique, but the numbers
of input and output variables is invariant.

Submodule theorem

$R \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ **defines** $\mathfrak{B} = \ker(R(\frac{d}{dt}))$**, but not vice-versa!**

Obviously, $R(\frac{d}{dt})w = 0$ **and** $U(\frac{d}{dt})R(\frac{d}{dt})w = 0$ **define the same behavior whenever** U **is unimodular.**

¿¿ $\exists$ 'intrinsic' characterization of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$??

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¿¿ $\exists$ **'intrinsic' characterization of** $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ **??**

Is there a mathematical 'object' that characterizes a $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$**?**

Define the ***annihilators*** **of** $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ **by**

$$\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{\mathtt{w}}[\xi] \mid n^{\top}(\tfrac{d}{dt})\mathfrak{B} = 0\}.$$

Proposition: $\mathfrak{N}_{\mathfrak{B}}$ **is a** $\mathbb{R}[\xi]$ **sub-module of** $\mathbb{R}^{\mathtt{w}}[\xi]$**.**

Submodule theorem

$$\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{\mathtt{w}}[\xi] \mid n^\top(\tfrac{d}{dt})\mathfrak{B} = 0\}.$$

Let $< R^\top >$ denote the submodule of $\mathbb{R}^{\mathtt{w}}[\xi]$ spanned by the transposes of the rows of R. Obviously $< R^\top > \subseteq \mathfrak{N}_{\ker(R(\frac{d}{dt}))}$. And, indeed:

$$\mathfrak{N}_{\ker(R(\frac{d}{dt}))} =< R^\top >$$

Note: Depends on $\mathfrak{C}^\infty$. False for compact support soln's.

Submodule theorem

$$\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{\mathtt{w}}[\xi] \mid n^\top(\tfrac{d}{dt})\mathfrak{B} = 0\}.$$

$$\mathfrak{N}_{\ker(R(\frac{d}{dt}))} = < R^\top >$$

Denote by $\mathfrak{M}$ the set of submodules of $\mathbb{R}^{\mathtt{n}}[\xi]$. Associate elements of $\mathfrak{L}^{\mathtt{w}}$ with those of $\mathfrak{M}$ as follows:

$$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}} \quad \mapsto \quad \mathfrak{N}_{\mathfrak{B}} \in \mathfrak{M}$$

$$M \in \mathfrak{M}, \quad \mapsto \quad \{w \mid n^\top(der)w = 0 \forall n \in \mathfrak{M}$$

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Theorem 1:

$$\mathfrak{L}^{\mathtt{w}} \overset{1:1}{\longleftrightarrow} \mathfrak{M}$$

Submodule theorem

Consequence (Structure of kernel representations):

1. Let $R_1(\frac{d}{dt})w = 0$ have behavior $\mathfrak{B}_1$, and $R_2(\frac{d}{dt})w = 0$ have behavior $\mathfrak{B}_2$. Then $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ if and only if $\exists \ F \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ such that $R_2 = FR_1$.

2. $R(\frac{d}{dt})w = 0$ is a minimal kernel representation (has a minimal number of rows over all kernel representations of a given behavior) if and only if R is of full row rank.

3. Let U be unimodular. Then $R(\frac{d}{dt})w = 0$ and $U(\frac{d}{dt})R(\frac{d}{dt})w = 0$ have the same behavior.

4. Let $R(\frac{d}{dt})w = 0$ be minimal. All minimal kernel representations with the same behavior are obtained by pre-multiplying R by a unimodular polynomial matrix.

Elimination theorem

Motivation: In many problems, we want to eliminate variables.

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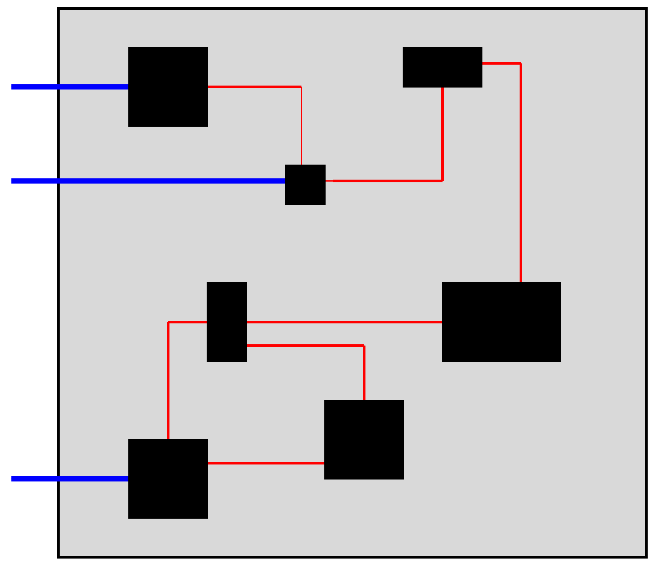

$\leadsto$ model containing both variables the model aims at ('manifest' variables), and auxiliary variables introduced in the modeling process ('latent' variables).

¿ Can these variables be eliminated from the equations ?

Elimination theorem

This leads to the following important question, first in polynomial matrix language. Consider

$$R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2.$$

Obviously, the behavior of the (w_1, w_2)'s is described by a system of differential equation.

¿ But is the behavior of the w_1's alone also ?

Elimination theorem

In the language of behaviors:

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1 + \mathtt{w}_2}$**. Define**

$$\mathfrak{B}_1 = \{ w_1 \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}) \mid \ \exists \ w_2 \text{ such that } (w_1, w_2) \in \mathfrak{B} \}.$$

Does $\mathfrak{B}$ **belong to** $\mathfrak{L}^{\mathtt{w}_1}$ **?**

Elimination theorem

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Does $\mathfrak{B}$ belong to $\mathfrak{L}^{\mathtt{w}_1}$?

Theorem 2: It does!

$\mathfrak{L}^\bullet$ is closed under projection !!

Proof: follows from the 'fundamental principle'.
Exercise 1: Prove the eliminationn th'm, using the fund. pr.
Exercise 2: Prove the fund. pr. for const. coeff. lin. ODE's.

Elimination theorem

Example: ▶ The ubiquitous

$$\frac{d}{dt}x = Ax + Bu; \quad y = Cx + Du, \quad w = (u, y).$$

Which eq'ns describe the (u, y) (input-output) behavior?

Elimination theorem $\Rightarrow$ it is a system of differential eq'ns:

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u$$

with P square and $\det(P) \neq 0$. Why the latter?: soon!

Elimination theorem

Example: ▶ The descriptor system

$$\frac{d}{dt}Ex + Fx + Gw = 0.$$

Which eqn's describe the w behavior?

Elimination theorem $\Rightarrow$ it is a system of differential eq'ns:

$$R(\frac{d}{dt})w = 0$$

! Compute $(E, F, G) \mapsto R$.
Row dimension of minimal kernel representation?

Elimination theorem

Example: Consider a linear RLCTG circuit.

First principles modeling ($\cong$ CE's, KVL, & KCL)

$\leadsto$ **linear constant coefficient differential equations. All of them algebraic, only the L's and C's differential.**

These include as variables both the external port and the internal branch voltages and currents.

Elimination theorem

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First principles modeling ($\cong$ CE's, KVL, & KCL)
$\leadsto$ linear constant coefficient differential equations. All of them algebraic, only the L's and C's differential.
These include as variables both the external port and the internal branch voltages and currents.
Can the port behavior be described by a system of linear constant coefficient differential equations?
YES, because:

1. The CE's, KVL, & KCL are all linear constant coefficient differential equations.
2. The elimination theorem.

Row dimension of minimal kernel representation?
Interesting things can be said, using passivity! ...

Elimination theorem

There exist effective algorithms for

$$(R_1, R_2) \mapsto R$$

incorporating, if desired, minimality of $R(\frac{d}{dt})w = 0$.

$\rightsquigarrow$ **Computer algebra.**

Start from R_1, R_2. We want to compute R.
Find a set of generators $n_1, n_2, \ldots, n_{\mathrm{g}}$ for the 'left syzygy' of the module $< R_2 >$, i.e. for the module

$$\{n \in \mathbb{R}^{\mathrm{coldim}(R_2)}[\xi] \mid n^\top R_2 = 0\}.$$

Define $N = \begin{bmatrix} n_1 & n_2 & \cdots & n_{\mathrm{g}} \end{bmatrix}$. Then $R = N^\top R_1$.

Elimination theorem

It follows from all this that $\mathfrak{L}^\bullet$ has very nice properties. In particular, it is closed under:

- **Intersection:** $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}) \Rightarrow (\mathfrak{B}_1 \cap \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}})$.
- **Addition:** $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}) \Rightarrow (\mathfrak{B}_1 + \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}})$.
- **Projection:** $(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}) \Rightarrow (\Pi_{w_1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1})$.
  Π_{w_1} := projection
- **Action of a linear differential operator:**
  $(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}, P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_1}[\xi]) \Rightarrow (P(\frac{d}{dt})\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2})$.
- **Inverse image of a linear differential operator:**
  $(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2}, P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_1}[\xi]) \Rightarrow (P(\frac{d}{dt}))^{-1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}$.

I/O representation theorem

When is a system variable an input? An output?

I/O representation theorem

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Intuition

Our choice: the input is a free variable which, together with the 'initial conditions', determines the output.

I/O representation theorem

When is a system variable an input? An output?

<u>Intuition</u>

Our choice: the input is a free variable which, together with the 'initial conditions', determines the output.

These concepts (input, output) are strongly domain dependent. We follow the usual Systems & Control setting.

Central is, of course, that the input must in some way *cause* the output.

I/O representation theorem

- In real-time signal processing and control, non-anticipation must be an important feature.
- But not in non-real-time signal processing problems, or when the independent variable is not time.
- In many problems (e.g. computing, signal processing) inputs may have to be structured, in order for machines or algorithms to be able to accept them.
- In control, it is customary to assume that inputs are free, and that outputs are bound (determined by the inputs and the initial conditions). We will follow this tradition.
- Often systems are non-anticipating both forward and backward in time.

I/O representation theorem

$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, is said to be *autonomous* if

$$[w_1, w_2 \in \mathfrak{B}] \wedge [t \in \mathbb{T}] \wedge [w_1(t') = w_2(t') \;\; \forall t' < t] \Rightarrow [w_1 = w_2].$$

Autonomous:= the past implies the future.

'Closed' systems.

I/O representation theorem

Let $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}_1+\mathtt{w}_2}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$. Then w_1 is said to be an *input/output system* with w_1 the *input* and w_2 the *output* if

1. the w_1-behavior $\mathfrak{B}_1 = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1})$, i.e., w_1 is free,
2. for all $w_1 \in \mathfrak{B}_1$, the system $\Sigma_2^{w_1} := (\mathbb{R}, \mathbb{R}^{\mathtt{w}_2}, \mathfrak{B}_2^{w_1})$ is autonomous,

where $\mathfrak{B}_2^{w_1}$ denotes the w_2 behavior with fixed w_1, i.e.,

$$\mathfrak{B}_2^{w_1} := \{w_2 \mid (w_1, w_2) \in \mathfrak{B}\}.$$

input $\cong$ free; output $\cong$ bound (det. by input + initial cond's).

I/O representation theorem

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input $\cong$ free; output $\cong$ bound (det. by input + initial cond's).
In keeping with tradition

$w_1 \to u$; $\mathbb{R}^{\mathtt{w}_1} \to \mathbb{R}^{\mathtt{m}}$ ($\mathtt{m}$ input variables),
$w_2 \to y$; $\mathbb{R}^{\mathtt{w}_2} \to \mathbb{R}^{\mathtt{p}}$ ($\mathtt{p}$ output variables).

I/O representation theorem

Let $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}$, with $\mathbb{R}^{\mathtt{w}} = \mathbb{R}^{\mathtt{m}} \times \mathbb{R}^{\mathtt{p}}$, $\mathtt{w} = \mathtt{m} + \mathtt{p}$.

If $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{m}} \times \mathbb{R}^{\mathtt{p}}, \mathfrak{B})$ is an input/output system,
then we call $w = (u, y)$ an **input/output partition** of w.

I/O representation theorem

Notation for vectors and matrices of rational functions:

$$\mathbb{R}(\xi)^{\mathtt{n}}, \mathbb{R}(\xi)^{\bullet}, \mathbb{R}(\xi)^{\mathtt{n}_1 \times \mathtt{n}_2}, \mathbb{R}(\xi)^{\bullet \times \mathtt{n}}, \mathbb{R}(\xi)^{\mathtt{n} \times \bullet}, \mathbb{R}(\xi)^{\bullet \times \bullet}.$$

A rational function $\frac{a}{b} \in \mathbb{R}(\xi)$ is said to be

proper **if $\mathrm{degree}(a) \leq \mathrm{degree}(b)$,**
strictly proper **if $\mathrm{degree}(a) < \mathrm{degree}(b)$,**
and *bi-proper* if $\mathrm{degree}(a) = \mathrm{degree}(b)$.

$\rightsquigarrow$ **vectors, matrices of (strictly) proper rational functions.**

I/O representation theorem

Proposition: Consider the kernel representation

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u, w = (u, y).$$

Then u is free $\Leftrightarrow$ $\operatorname{rank}([P \;\; Q]) = \operatorname{rank}(P)$,
and y is bound $\Leftrightarrow$ P is of full column rank,
i.e. $\operatorname{rank}(P) = \dim(y)$.

It defines an input/output partition if and only if

$$\operatorname{rank}([P \;\; Q]) = \operatorname{rank}(P) = \dim(y).$$

If it is a minimal kernel representation, then I/O partition if and only if P is square, and $\det(P) \neq 0$.

Call $\boxed{G := P^{-1}Q}$ $\in \mathbb{R}(\xi)^{\mathtt{p} \times \mathtt{m}}$ its transfer function.

I/O representation theorem

Theorem 3:

Every system $\Sigma \in \mathfrak{L}^{\bullet}$ admits an input/output partition.

I/O representation theorem

Theorem 3:

Every system $\Sigma \in \mathfrak{L}^{\bullet}$ admits an input/output partition.

even a componentwise I/O partition

:= some well-chosen components of w are the inputs,

the others are the outputs

$\cong$ up to re-ordering of the variables, $w = (u, y)$,

i.e., $(u, y) = \Pi w$, with Π a permutation matrix.

In fact, $\exists$ a componentwise partition with G proper.

If one can choose the basis, and then the partition, even with G strictly proper.

I/O representation theorem

Notes:

1. For a given $\mathfrak{B} \in \mathfrak{L}^\bullet$, which variables are input variables, and which are input variables, is not fixed.

THIS IS A GOOD THING!

Examples:

An Ohmic resistor $V = RI$ $R \neq 0$ may be viewed as a current controlled or as a voltage controlled device.

Often, a transfer function is bi-proper, and then there is no reason to prefer one input/output partition over another.

etc., etc.

I/O representation theorem

2. The number of input and the number of output variables are fixed by $\mathfrak{B} \in \mathfrak{L}^\bullet$.

I/O representation theorem

Notation: Define the 3 maps $\mathtt{w}, \mathtt{m}, \mathtt{p} : \mathfrak{L}^\bullet \to \mathbb{Z}_+$ by

$$\begin{aligned}
\mathtt{w}(\Sigma) = \mathtt{w}(\mathfrak{B}) &:= \text{ the number of variables} \\
\mathtt{m}(\Sigma) = \mathtt{m}(\mathfrak{B}) &:= \text{ the number of input variables} \\
\mathtt{p}(\Sigma) = \mathtt{p}(\mathfrak{B}) &:= \text{ the number of output variables}
\end{aligned}$$

In terms of the kernel representation $R(\frac{d}{dt})w = 0$, we have

$$\begin{aligned}
\mathtt{w}(\Sigma) &= \mathrm{coldim}(R), \\
\mathtt{m}(\Sigma) &= \mathrm{coldim}(R) - \mathrm{rank}(R), \\
\mathtt{p}(\Sigma) &= \mathrm{rank}(P)
\end{aligned}$$

In particular, $\mathtt{m} + \mathtt{p} = \mathtt{w}$.

End of the Lecture I