European Embedded Control Institute

Graduate School on Control — Spring 2010

The Behavioral Approach to Modeling and Control

Lecture IX

DISSIPATIVE SYSTEMS

Theme

- **Energy exchange with the external world.**
- **Energy storage in a system.**
- **State and storage.**

Outline

- **Dissipative systems;**
- **Spectral factorization;**
- **Storage functions.**

Dissipative Systems

Dissipation inequality

Physical examples:

- **Resistive electrical circuits;**
- **Mechanical systems with friction;**
- **...**

Dissipation inequality

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- **Resistive electrical circuits;**
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Energy supplied to system $\rightsquigarrow$ **supply rate variable** F_Σ

Dissipation inequality

Physical examples:

- **Resistive electrical circuits;**
- **Mechanical systems with friction;**
- **...**

Energy supplied to system $\rightsquigarrow$ **supply rate variable** F_Σ

- **Electrical circuits:** $V^\top I$ **with** V **(resp.** I**) vector of voltages (resp. currents)**
- **Mechanical systems:** $F^\top \frac{d}{dt}x$ **with** F **(resp.** x**) vector of forces (resp. displacements)**

Dissipation inequality

Energy supplied to system $\rightsquigarrow$ **supply rate variable** F_Σ

Energy stored in system $\rightsquigarrow$ **storage variable** F_S

Dissipation inequality

Energy supplied to system $\rightsquigarrow$ **supply rate variable** F_Σ

Energy stored in system $\rightsquigarrow$ **storage variable** F_S

- **Electrical circuits:** $\frac{1}{2}C \cdot V^2$ **for capacitor,** $\frac{1}{2}L \cdot I^2$ **for inductor**
- **Mechanical systems:** $\frac{1}{2}K \cdot x^2$ **for spring**

Dissipation inequality

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In a dissipative system,
energy cannot be stored faster than it is supplied

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Dissipation equality

Lossless systems: $F_\Sigma = \frac{d}{dt}F_S$

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Dissipation equality

Lossless systems: $F_\Sigma = \frac{d}{dt} F_S$

Now, linear time-invariant finite-dimensional systems, with quadratic supply rates

Technical notes

QDFs

$$Q_\Phi(w) := \begin{bmatrix} w^\top & \frac{dw}{dt}^\top & \dots \end{bmatrix} \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots \\ \vdots & \vdots & \cdots \\ \Phi_{k,0} & \Phi_{k,1} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} w \\ \frac{dw}{dt} \\ \vdots \end{bmatrix}$$

$$= \sum_{k,\ell=0}^{L} \left(\frac{d^k w}{dt^k}\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell w}{dt^\ell}\right)$$

QDFs

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$$= \sum_{k,\ell=0}^{L} \left(\frac{d^k w}{dt^k}\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell w}{dt^\ell}\right)$$

$$\sum_{k,\ell=0}^{L} \zeta^k \Phi_{k,\ell} \eta^\ell$$

Setting the stage

LTI systems $\rightsquigarrow$ **supply, dissipation, storage are quadratic functionals of the system variables and their derivatives**

Dissipation equality:

$$Q_\Phi(w) = Q_\Delta(w) + \frac{d}{dt} Q_\Psi(w)$$

where $w \in \mathscr{B}$

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where $w \in \mathscr{B}$

...equalities along $\mathscr{B}$ are cumbersome to work with...

Setting the stage

Controllable system

$$w = M(\tfrac{d}{dt})\ell \rightsquigarrow M(\xi)$$

Power ('supply rate')

$$Q_\Phi \rightsquigarrow \Phi(\zeta, \eta)$$

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Controllable system

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$$Q_\Phi \rightsquigarrow \Phi(\zeta,\eta)$$

$$Q_\Phi(w) = Q_\Phi(M(\tfrac{d}{dt})\ell)$$

$$\Phi'(\zeta,\eta) := M(\zeta)^\top \Phi(\zeta,\eta) M(\eta)$$

$Q_{\Phi'}$ acts on free variable ℓ, i.e. $\mathscr{C}^\infty$

Example

Mass-spring-damper system $m\frac{d^2q}{dt^2} + c\frac{d}{dt}q + kq - F = 0$**; then**

$$R(\xi) = \begin{bmatrix} m\xi^2 + c\xi + k & -1 \end{bmatrix}, \; M(\xi) = \begin{bmatrix} 1 \\ m\xi^2 + c\xi + k \end{bmatrix}$$

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Power $F\frac{d}{dt}q$ **can be written as** $Q_\Sigma(w)$**, with**

$$\Sigma(\zeta,\eta) = \frac{1}{2}\begin{bmatrix} 0 & \zeta \\ \eta & 0 \end{bmatrix}$$

Example

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Since $F=m\frac{d^2q}{dt^2}+c\frac{d}{dt}q+kq$**, can rewrite** $Q_\Sigma(w)$ **as** $Q_{\Sigma'}(q)$**, with**

$$\Sigma(\zeta,\eta)=\begin{bmatrix}1 & m\zeta^2+c\zeta+k\end{bmatrix}\frac{1}{2}\begin{bmatrix}0 & \zeta\\ \eta & 0\end{bmatrix}\begin{bmatrix}1\\ m\eta^2+c\eta+k\end{bmatrix}$$

Frequency-domain characterization of dissipativity

When is a system dissipative?

Dissipation equality:

$$Q_{\Phi} = \frac{d}{dt} Q_{\Psi} + Q_{\Delta}$$

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for all compact-support trajectories $w \in \mathscr{B}$

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If $w = M(\frac{d}{dt})\ell$**, equivalent to**

$$Q_{\Phi'}(\ell) \geq 0 \text{ **for all** } \ell \in \mathscr{C}^\infty$$

with $\Phi'(\zeta,\eta) = M(\zeta)^\top \Phi(\zeta,\eta) M(\eta)$

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Fourier transformation leads to

$$\Phi'(-i\omega, i\omega) = M(-i\omega)^\top \Phi(-i\omega, i\omega) M(i\omega) \geq 0$$

for all $\omega \in \mathbb{R}$

¡A frequency-domain inequality!

When is a system dissipative?

We just proved:

Theorem: `image` $M(\frac{d}{dt})$ **is** Φ**-dissipative if and only if** $M(-i\omega)^\top \Phi(-i\omega, i\omega) M(i\omega) \geq 0$ **for all** $\omega \in \mathbb{R}$

Characterization of dissipativity

Characterizations of dissipativity

Theorem: **The following conditions are equivalent:**

- $\int_{-\infty}^{+\infty} Q_\Phi(\ell)dt \geq 0$ **for all** $\mathscr{C}^\infty$ **compact-support** ℓ**;**
- Q_Φ **admits a storage function;**
- Q_Φ **admits a dissipation rate**

Given Q_Φ**, storage and dissipation are one-one:**

$$\begin{aligned} \frac{d}{dt}Q_\Psi &= Q_\Phi - Q_\Delta \\ (\zeta+\eta)\Psi(\zeta,\eta) &= \Phi(\zeta,\eta) - \Delta(\zeta,\eta) \end{aligned}$$

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¿Given Φ, how to find dissipation/storage functions?

Spectral factorization

Dissipation in an algebraic setting: spectral factorization

$$(\zeta+\eta)\Psi(\zeta,\eta)+\Delta(\zeta,\eta)=\Phi(\zeta,\eta)$$

¿How to compute Δ and Ψ?

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Let $\zeta=-\xi$, $\eta=\xi$; then $\Delta(-\xi,\xi)=\Phi(-\xi,\xi)$

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Let $\zeta=-\xi$**,** $\eta=\xi$**; then** $\Delta(-\xi,\xi)=\Phi(-\xi,\xi)$

$Q_\Delta(\ell)\geq 0\ \forall\ \ell\in\mathscr{C}^\infty(\mathbb{R},\mathbb{R}^\bullet)\implies\exists$ **square** $D\in\mathbb{R}^{\bullet\times\bullet}[\xi]$ **such that**

$$\Delta(\zeta,\eta)=D(\zeta)^\top D(\eta)$$

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Spectral factorization: given $\Phi(-\xi,\xi)$**, find square** D **s.t.**

$$\Phi(-\xi,\xi)=D(-\xi)^\top D(\xi)$$

Example

$$\Phi(\zeta,\eta) = 4 + 6\eta + 2\eta^2 + 6\zeta + 9\zeta\eta + 4\zeta\eta^2 + 2\zeta^2 + 4\zeta^2\eta + \eta^2\zeta^2$$

Example

$$\Phi(\zeta,\eta)=4+6\eta+2\eta^2+6\zeta+9\zeta\eta+4\zeta\eta^2+2\zeta^2+4\zeta^2\eta+\eta^2\zeta^2$$

Check if $\Phi(-i\omega,i\omega)\geq 0$ **for all** $\omega\in\mathbb{R}$**:**

$$\Phi(-i\omega,i\omega)=4+5\omega^2+\omega^4$$

a sum of squares, always nonnegative.

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Note that

$$\Phi(-\xi,\xi) = 4 - 5\xi^2 + \xi^4 = (\xi - 2)(\xi - 1)(\xi + 1)(\xi + 2)$$

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We can choose

$$\Delta(\zeta,\eta)=(\zeta+1)(\zeta-2)(\eta+1)(\eta-2)$$

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Note that

$$\Phi(-\xi,\xi) = 4 - 5\xi^2 + \xi^4 = (\xi - 2)(\xi - 1)(\xi + 1)(\xi + 2)$$

We can also choose

$$\Delta'(\zeta,\eta) = (\zeta + 1)(\zeta + 2)(\eta + 1)(\eta + 2)$$

and so forth...

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Solvable if and only if $\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$.

¡Frequency domain condition for dissipativity!

Spectral factorization

Spectral factorization: given $\Phi(-\xi,\xi)$, find square matrix D s.t.

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Spectral factorize $\Phi(-\xi,\xi) = D(-\xi)^\top D(\xi)$, define

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$\Phi(-\xi,\xi) = \Delta(-\xi,\xi) \Longrightarrow$ **there exists $\Psi(\zeta,\eta)$ s.t.**

$$\Phi(\zeta,\eta) - \Delta(\zeta,\eta) = (\zeta+\eta)\Psi(\zeta,\eta)$$

Then storage function is

$$\Psi(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - \Delta(\zeta,\eta)}{\zeta+\eta}$$

Remarks

- **Many ways of spectral factorizing the same matrix**
  $\leadsto$ **many dissipation functions**
  $\leadsto$ **many storage functions.**

- **Set of storage functions is convex:**

  Q_{Ψ_1}, Q_{Ψ_2} **storage functions and** $\alpha \in [0,1]$

  $\Longrightarrow \alpha Q_{\Psi_1} + (1-\alpha) Q_{\Psi_2}$ **is storage function**

Example

$$\Phi(\zeta, \eta) = 4 + 6\eta + 2\eta^2 + 6\zeta + 9\zeta\eta + 4\zeta\eta^2 + 2\zeta^2 + 4\zeta^2\eta + \eta^2\zeta^2$$

Example

$$\Phi(\zeta,\eta)=4+6\eta+2\eta^2+6\zeta+9\zeta\eta+4\zeta\eta^2+2\zeta^2+4\zeta^2\eta+\eta^2\zeta^2$$

Since

$$\Phi(-\xi,\xi)=4-5\xi^2+\xi^4=(\xi-2)(\xi-1)(\xi+1)(\xi+2)$$

if we choose the dissipation function

$$\Delta(\zeta,\eta)=(\zeta+1)(\zeta-2)(\eta+1)(\eta-2)$$

we obtain the storage function

$$\Psi(\zeta,\eta)=\frac{\Phi(\zeta,\eta)-\Delta(\zeta,\eta)}{\zeta+\eta}=4+4\eta+4\zeta+5\zeta\eta$$

Example

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Since also

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if we choose the dissipation function

$$\Delta'(\zeta,\eta)=(\zeta+1)(\zeta+2)(\eta+1)(\eta+2)$$

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$$\Psi'(\zeta,\eta)=\frac{\Phi(\zeta,\eta)-\Delta'(\zeta,\eta)}{\zeta+\eta}=\zeta\eta$$

Storage functions

Maximal and minimal storage functions

Theorem: **Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ be controllable and Φ-dissipative. There exist storage functions Q_{Ψ_-} and Q_{Ψ_+} such that for any storage function Q_Ψ it holds**

$$Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}$$

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Q_{Ψ_-} is minimal-, Q_{Ψ_+} is maximal storage function

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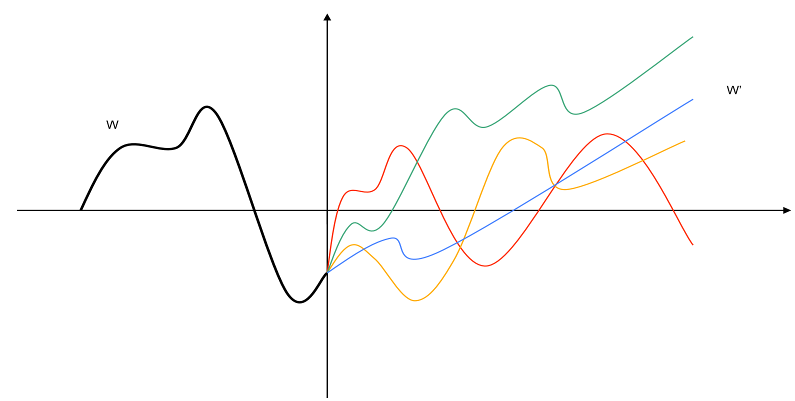
Q_{Ψ_-} is **available storage**:

$$Q_{\Psi_-}(w)(0) = \sup_{\substack{w' \text{ s.t.} \\ w \wedge w' \in \mathscr{B}}} \left(-\int_0^\infty Q_\Phi(w')dt \right)$$



Maximum amount of energy extractable from system.

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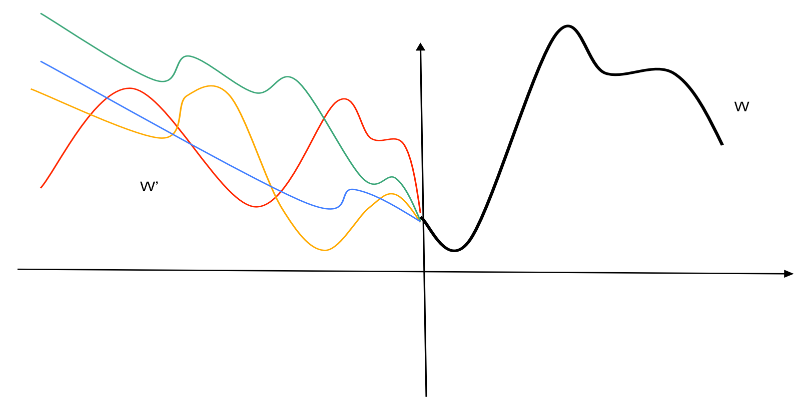
Q_{Ψ_+} **is required supply:**

$$Q_{\Psi_+}(w)(0) = \inf_{\substack{w' \text{ s.t.} \\ w' \wedge w \in \mathscr{B}}} \left(\int_{-\infty}^{0} Q_\Phi(w')dt \right)$$



Minimum energy needed to produce w **from** $t = 0$

Spectral factorization and extremal storage functions

If $\det \Phi(-\xi,\xi) \neq 0$ **and** $\Phi(-i\omega, i\omega) \geq 0$ **for all** $\omega \in \mathbb{R}$**, there exist** H**,** A **s.t.**

$$\Phi(-\xi,\xi) = H(-\xi)^\top H(\xi) = A(-\xi)^\top A(\xi)$$

where

$$\det(H(\lambda)) = 0 \Longrightarrow \lambda \in \mathbb{C}_-^0 \quad \text{(“semi-Hurwitz polynomial”)}$$

$$\det(A(\lambda)) = 0 \Longrightarrow \lambda \in \mathbb{C}_+^0 \quad \text{(“semi-anti-Hurwitz polynomial”)}$$

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$\det(H(\lambda)) = 0 \Longrightarrow \lambda \in \mathbb{C}_-^0$ **("semi-Hurwitz polynomial")**

$\det(A(\lambda)) = 0 \Longrightarrow \lambda \in \mathbb{C}_+^0$ **("semi-anti-Hurwitz polynomial")**

In this case,

$$\Psi_-(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - H(\zeta)^\top H(\eta)}{\zeta+\eta}$$

$$\Psi_+(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - A(\zeta)^\top A(\eta)}{\zeta+\eta}$$

Storage functions and the state

Circuit theory folklore: state variables are associated with energy storing elements (capacitors, inductors)

Physics: potential energy in a field dependent on position (and velocity/acceleration)

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¿Can we give rational foundation to the intuition that "storage" is related with "memory"?

Storage functions and the state

<u>**Theorem**</u>: **Let** $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ **be nonsingular. Assume that** $\mathscr{B} = \mathtt{image}\,(M(\frac{d}{dt}))$ **is** Σ**-dissipative.**

Let $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ **be a storage function, and let** $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ **be a state map for** $\mathscr{B}$**.**

Then $\exists\, K = K^\top \in \mathbb{R}^{\bullet\times\bullet}$**,** $E = E^\top \in \mathbb{R}^{\bullet\times\bullet}$ **such that**

$$\Psi(\zeta,\eta) = X(\zeta)^\top K X(\eta)$$

$$\Delta(\zeta,\eta) = \begin{bmatrix} M(\zeta) \\ X(\zeta) \end{bmatrix}^\top E \begin{bmatrix} M(\eta) \\ X(\eta) \end{bmatrix}$$

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¡The storage function is a quadratic function of the state!

Storage functions and the state

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¡The dissipation function
is a quadratic function of the state and of the input!

Summary

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- **Spectral factorization and storage functions;**

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- **Spectral factorization and storage functions;**
- **Extremal storage functions;**

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- **Dissipative systems: storage and dissipation;**
- **Spectral factorization and storage functions;**
- **Extremal storage functions;**
- **Storage function is a function of the state.**

End of Lecture IX