European Embedded Control Institute

Graduate School on Control — Spring 2010

The Behavioral Approach to Modeling and Control

Lecture VII

QUADRATIC AND BILINEAR

DIFFERENTIAL FORMS

Theme

- **Bilinear and quadratic functionals of system variables arise in many applications.**
- **These functionals can be represented by two-variable polynomial matrices.**
- **A calculus can be developed in which operations and properties of the functionals are reflected in those of their representations.**

Outline

- Motivation;
- Bilinear- and quadratic differential forms;
- Two-variable polynomial representation;
- Calculus of B/QDFs;
- Lyapunov theory for higher-order systems.

Motivation

Dynamics and functionals in systems and control

Instances: Lyapunov theory, performance criteria, etc.

Linear case $\Longrightarrow$ *quadratic* and *bilinear* functionals.

Dynamics and functionals in systems and control

Instances: Lyapunov theory, performance criteria, etc.

Linear case $\Longrightarrow$ *quadratic* and *bilinear* functionals.

Usually: state-space equations, constant functionals.

But tearing and zooming do not yield state space equations!

Dynamics and functionals in systems and control

Instances: Lyapunov theory, performance criteria, etc.

Linear case $\Longrightarrow$ *quadratic* and *bilinear* functionals.

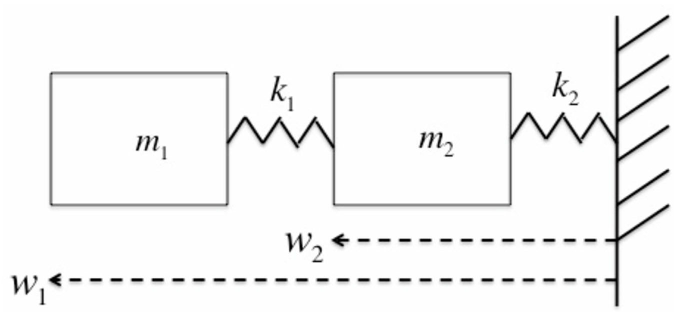
Usually: state-space equations, constant functionals.

But tearing and zooming do not yield state space equations!

¡High-order differential equations!

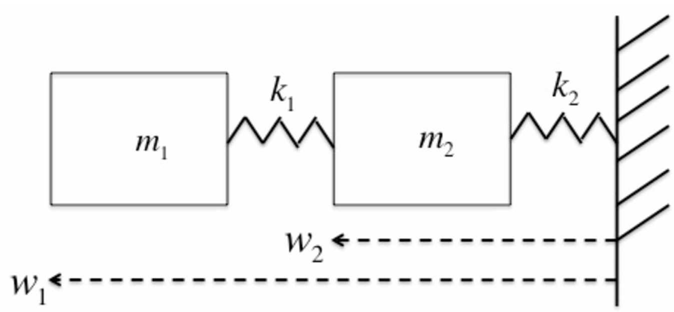
...involving also *latent variables*...

Example : a mechanical system



$$
\begin{aligned}
m_1 \frac{d^2 w_1}{dt^2} + k_1 w_1 - k_2 w_2 &= 0 \\
-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 &= 0
\end{aligned}
$$

Example : a mechanical system

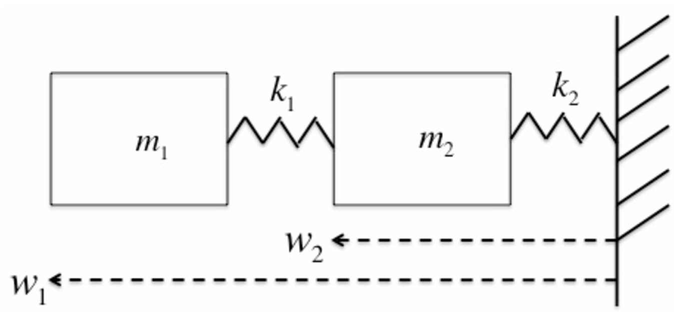


$$\begin{aligned} m_1 \frac{d^2 w_1}{dt^2} + k_1 w_1 - k_2 w_2 &= 0 \\ -k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 &= 0 \end{aligned}$$

Eliminate w_2**:**

$$m_1 m_2 \frac{d^4}{dt^4} w_1 + (k_1 m_1 + k_2 m_1 + k_1 m_2) \frac{d^2}{dt^2} w_1 + k_1 k_2 w_1 = 0$$

Example : a mechanical system



$$
\begin{aligned}
m_1 \frac{d^2 w_1}{dt^2} + k_1 w_1 - k_2 w_2 &= 0 \\
-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 &= 0
\end{aligned}
$$

Eliminate w_2**:**

$$
m_1 m_2 \frac{d^4}{dt^4} w_1 + (k_1 m_1 + k_2 m_1 + k_1 m_2) \frac{d^2}{dt^2} w_1 + k_1 k_2 w_1 = 0
$$

¿Stability, stored energy, conservation laws?

Aim

An effective algebraic representation
of bilinear and quadratic functionals
of the system variables and their derivatives:

Operations/properties of functionals
$\Updownarrow$
algebraic operations/properties of representation

...a calculus of these functionals!

Bilinear and quadratic differential forms

Bilinear differential forms (BDFs)

$$\Phi := \left\{ \Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2} \right\}_{k,\ell=0,\ldots,L}$$

$$L_\Phi : \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}) \times \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2}) \longrightarrow \mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$$

$$L_\Phi(w_1, w_2) := \begin{bmatrix} w_1^\top & \frac{dw_1}{dt}^\top & \ldots \end{bmatrix} \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots \\ \vdots & \vdots & \cdots \\ \Phi_{k,0} & \Phi_{k,1} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} w_2 \\ \frac{dw_2}{dt} \\ \vdots \end{bmatrix}$$

$$= \sum_{k,\ell} \left(\frac{d^k}{dt^k} w_1 \right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dt^\ell} w_2 \right)$$

Quadratic differential forms (QDFs)

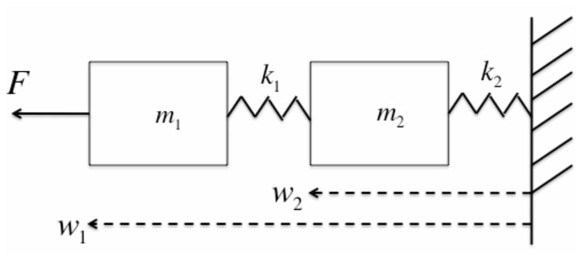
$$\Phi := \left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}\right\}_{k,\ell=0,\ldots,L} \text{ **symmetric, i.e.** } \Phi_{k,\ell} = \Phi_{\ell,k}^{\top}$$

$$Q_\Phi : \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \longrightarrow \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$$

$$Q_\Phi(w) := \begin{bmatrix} w^\top & \frac{dw}{dt}^\top & \ldots \end{bmatrix} \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \ldots \\ \Phi_{1,0} & \Phi_{1,1} & \ldots \\ \vdots & \vdots & \ldots \\ \Phi_{k,0} & \Phi_{k,1} & \ldots \\ \vdots & \vdots & \ldots \end{bmatrix} \begin{bmatrix} w \\ \frac{dw}{dt} \\ \vdots \end{bmatrix}$$

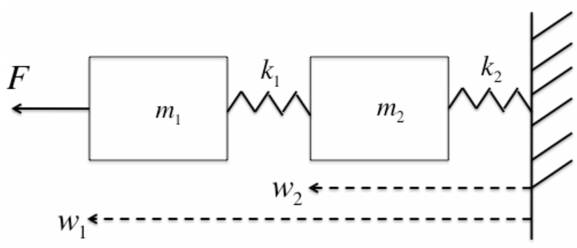
$$= \Sigma_{k,\ell=0}^{L} \left(\frac{d^k w}{dt^k}\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell w}{dt^\ell}\right)$$

Example: total energy in mechanical system



$$
\begin{aligned}
m_1 \frac{d^2 w_1}{dt^2} + k_1(w_1 - w_2) - F &= 0 \\
-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 &= 0
\end{aligned}
$$

Example: total energy in mechanical system



$$
\begin{aligned}
m_1 \frac{d^2 w_1}{dt^2} + k_1(w_1 - w_2) - F &= 0 \\
-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 &= 0
\end{aligned}
$$

Total energy is

$$
\frac{1}{2} m_1 \left(\frac{d}{dt} w_1\right)^2 + \frac{1}{2} m_2 \left(\frac{d}{dt} w_2\right)^2 + \frac{1}{2} k_1 (w_1 - w_2)^2 + \frac{1}{2} k_2 w_2^2
$$

$$
= \begin{bmatrix} w_1 & w_2 & F & \frac{d}{dt} w_1 & \frac{d}{dt} w_2 & \frac{d}{dt} F \end{bmatrix}
\begin{bmatrix}
\frac{1}{2} k_1 & -\frac{1}{2} k_1 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} k_1 & \frac{1}{2}(k_1 + k_2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} m_1 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} m_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} w_1 \\ w_2 \\ F \\ \frac{d}{dt} w_1 \\ \frac{d}{dt} w_2 \\ \frac{d}{dt} F \end{bmatrix}
$$

Two-variable polynomial representation

Two-variable polynomial matrices for BDFs

$$\left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}\right\}_{k,\ell=0,\ldots,L}$$

Two-variable polynomial matrices for BDFs

$$\left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}\right\}_{k,\ell=0,\ldots,L}$$

$$L_\Phi(w_1, w_2) = \sum_{k,\ell=0}^{L} \left(\frac{d^k w_1}{dt^k}\right)^\top \Phi_{k,\ell} \frac{d^\ell w_2}{dt^\ell}$$

Two-variable polynomial matrices for BDFs

$$\left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}\right\}_{k,\ell=0,\ldots,L}$$

$$L_\Phi(w_1, w_2) = \sum_{k,\ell=0}^{L} \left(\frac{d^k w_1}{dt^k}\right)^\top \Phi_{k,\ell} \frac{d^\ell w_2}{dt^\ell}$$

$$\Phi(\zeta, \eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell}\, \zeta^k \eta^\ell$$

Two-variable polynomial matrices for BDFs

$$\left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}\right\}_{k,\ell=0,\ldots,L}$$

$$L_\Phi(w_1, w_2) = \sum_{k,\ell=0}^{L} \left(\frac{d^k w_1}{dt^k}\right)^\top \Phi_{k,\ell} \frac{d^\ell w_2}{dt^\ell}$$

$$\Phi(\zeta, \eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell}\, \zeta^k \eta^\ell$$

Two-variable polynomial matrices for BDFs

$$\left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}\right\}_{k,\ell=0,\ldots,L}$$

$$L_\Phi(w_1, w_2) = \sum_{k,\ell=0}^{L} \left(\frac{d^k w_1}{dt^k}\right)^\top \Phi_{k,\ell} \, \frac{d^\ell w_2}{dt^\ell}$$

$$\Phi(\zeta, \eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell} \, \zeta^k \, \eta^\ell$$

2-variable polynomial matrix associated with L_Φ

Two-variable polynomial matrices for QDFs

$$\left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}\right\}_{k,\ell=0,\ldots,L} \textbf{ symmetric } (\Phi_{k,\ell} = \Phi_{\ell,k}^\top)$$

$$Q_\Phi(w) = \sum_{k,\ell=0}^{L} \left(\frac{d^k w}{dt^k}\right)^\top \Phi_{k,\ell} \frac{d^\ell w}{dt^\ell}$$

$$\Phi(\zeta,\eta) = \Sigma_{k,\ell=0}^{L} \Phi_{k,\ell}\, \zeta^k\, \eta^\ell$$

***symmetric*:** $\Phi(\zeta,\eta) = \Phi(\eta,\zeta)^\top$

Example: total energy in mechanical system

$$Q_E(w_1, w_2, F) =$$

$$\begin{bmatrix} w_1 & w_2 & F & \frac{d}{dt}w_1 & \frac{d}{dt}w_2 & \frac{d}{dt}F \end{bmatrix} \begin{bmatrix} \frac{1}{2}k_1 & -\frac{1}{2}k_1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}k_1 & \frac{1}{2}(k_1+k_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ F \\ \frac{d}{dt}w_1 \\ \frac{d}{dt}w_2 \\ \frac{d}{dt}F \end{bmatrix}$$

$$E(\zeta, \eta) = \begin{bmatrix} \frac{1}{2}k_1 & -\frac{1}{2}k_1 & 0 \\ -\frac{1}{2}k_1 & \frac{1}{2}(k_1+k_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\zeta\eta & 0 & 0 \\ 0 & \frac{1}{2}\zeta\eta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example: total energy in mechanical system

$$Q_E(w_1, w_2, F) =$$

$$\begin{bmatrix} w_1 & w_2 & F & \frac{d}{dt}w_1 & \frac{d}{dt}w_2 & \frac{d}{dt}F \end{bmatrix} \begin{bmatrix} \frac{1}{2}k_1 & -\frac{1}{2}k_1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}k_1 & \frac{1}{2}(k_1+k_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ F \\ \frac{d}{dt}w_1 \\ \frac{d}{dt}w_2 \\ \frac{d}{dt}F \end{bmatrix}$$

$$E(\zeta, \eta) = \begin{bmatrix} \frac{1}{2}k_1 & -\frac{1}{2}k_1 & 0 \\ -\frac{1}{2}k_1 & \frac{1}{2}(k_1+k_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\zeta\eta & 0 & 0 \\ 0 & \frac{1}{2}\zeta\eta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example: total energy in mechanical system

$Q_E(w_1, w_2, F) =$

$$\begin{bmatrix} w_1 & w_2 & F & \frac{d}{dt}w_1 & \frac{d}{dt}w_2 & \frac{d}{dt}F \end{bmatrix} \begin{bmatrix} \frac{1}{2}k_1 & -\frac{1}{2}k_1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}k_1 & \frac{1}{2}(k_1+k_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ F \\ \frac{d}{dt}w_1 \\ \frac{d}{dt}w_2 \\ \frac{d}{dt}F \end{bmatrix}$$

$$E(\zeta, \eta) = \begin{bmatrix} \frac{1}{2}k_1 & -\frac{1}{2}k_1 & 0 \\ -\frac{1}{2}k_1 & \frac{1}{2}(k_1+k_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\zeta\eta & 0 & 0 \\ 0 & \frac{1}{2}\zeta\eta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The calculus of B/QDFs

The calculus of B/QDFs

Using powers of ζ and η as placeholders,

B/QDF $\leftrightsquigarrow$ two-variable polynomial matrix

The calculus of B/QDFs

Using powers of ζ and η as placeholders,

B/QDF $\leftrightsquigarrow$ two-variable polynomial matrix

Operations and properties of B/QDF	$\leftrightsquigarrow$	**algebraic operations/properties on two-variable matrix**

Differentiation

$\Phi \in \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$. **$\overset{\bullet}{\Phi}$ derivative of Q_Φ:**

$$Q_{\overset{\bullet}{\Phi}} : \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$$

$$Q_{\overset{\bullet}{\Phi}}(w) := \frac{d}{dt}(Q_\Phi(w))$$

...a QDF, evidently... what is its two-variable representation?

Differentiation

$\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. $\overset{\bullet}{\Phi}$ **derivative** **of** Q_Φ**:**

$$Q_{\overset{\bullet}{\Phi}} : \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \to \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$$

$$Q_{\overset{\bullet}{\Phi}}(w) := \frac{d}{dt}(Q_\Phi(w))$$

...a QDF, evidently... what is its two-variable representation?

$$\overset{\bullet}{\Phi}(\zeta,\eta) = (\zeta+\eta)\Phi(\zeta,\eta)$$

Two-variable version of Leibniz's rule

Integration

$\mathscr{D}(\mathbb{R},\mathbb{R}^\bullet)$: set of $\mathscr{C}^\infty$-compact-support trajectories

$$L_\Phi : \mathscr{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}_1}) \times \mathscr{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}_2}) \longrightarrow \mathscr{D}(\mathbb{R},\mathbb{R})$$

$$\int L_\Phi : \mathscr{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}_1}) \times \mathscr{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}_2}) \longrightarrow \mathbb{R}$$

$$\int L_\Phi(w_1,w_2) := \int_{-\infty}^{+\infty} L_\Phi(w_1,w_2)dt$$

Analogous for QDFs

Combining dynamics and functionals: B/QDFs zero along behaviors

Q_Φ **zero on** $\mathfrak{B}$ **(denoted** $Q_\Phi \stackrel{\mathfrak{B}}{=} 0$**) if**

$$Q_\Phi(w) = 0 \textbf{ for all } w \in \mathfrak{B}$$

Combining dynamics and functionals: B/QDFs zero along behaviors

Q_Φ **zero on** $\mathfrak{B}$ **(denoted** $Q_\Phi \stackrel{\mathfrak{B}}{=} 0$**) if**

$$Q_\Phi(w) = 0 \textbf{ for all } w \in \mathfrak{B}$$

<u>Theorem</u>: Let $\mathfrak{B} = \texttt{kernel}(R(\frac{d}{dt}))$**. Then**

$$Q_\Phi \stackrel{\mathfrak{B}}{=} 0 \Longleftrightarrow \exists\, F \in \mathbb{R}^{\bullet\times\bullet}[\zeta,\eta] \textbf{ such that}$$

$$\Phi(\zeta,\eta) = R(\zeta)^\top F(\zeta,\eta) + F(\eta,\zeta)^\top R(\eta)$$

Combining dynamics and functionals: B/QDFs zero along behaviors

Q_Φ **zero on** $\mathfrak{B}$ **(denoted** $Q_\Phi \overset{\mathfrak{B}}{=} 0$**) if**

$$Q_\Phi(w) = 0 \text{ for all } w \in \mathfrak{B}$$

Theorem: Let $\mathfrak{B} = \texttt{kernel}(R(\frac{d}{dt}))$**. Then**

$$Q_\Phi \overset{\mathfrak{B}}{=} 0 \Longleftrightarrow \exists\, F \in \mathbb{R}^{\bullet\times\bullet}[\zeta,\eta] \text{ such that}$$

$$\Phi(\zeta,\eta) = R(\zeta)^\top F(\zeta,\eta) + F(\eta,\zeta)^\top R(\eta)$$

QDF induced by the 2-variable polynomial matrix on RHS is instantaneously zero for $w \in \mathscr{B} = \texttt{kernel}\, R\left(\frac{d}{dt}\right)$**.**

Example: conservation laws

Oscillator: $m\frac{d^2}{dt^2}w + kw = 0 \rightsquigarrow r(\xi) = m\xi^2 + k$

Total energy is $Q_E(w) = \frac{1}{2}m\left(\frac{dw}{dt}\right)^2 + \frac{1}{2}kw^2$**. A QDF.**

Example: conservation laws

Oscillator: $m\frac{d^2}{dt^2}w + kw = 0 \rightsquigarrow r(\xi) = m\xi^2 + k$

Total energy is $Q_E(w) = \frac{1}{2}m\left(\frac{dw}{dt}\right)^2 + \frac{1}{2}kw^2$**. A QDF.**

Two-variable representation is $E(\zeta,\eta) = \frac{1}{2}m\zeta\eta + \frac{1}{2}k$**.**

Example: conservation laws

Oscillator: $m\frac{d^2}{dt^2}w + kw = 0 \rightsquigarrow r(\xi) = m\xi^2 + k$

Total energy is $Q_E(w) = \frac{1}{2}m\left(\frac{dw}{dt}\right)^2 + \frac{1}{2}kw^2$**. A QDF.**

Two-variable representation is $E(\zeta,\eta) = \frac{1}{2}m\zeta\eta + \frac{1}{2}k$**.**

Physics: total energy is conserved

$$\frac{d}{dt}E(w) = 0 \text{ for all } w \in \mathscr{B}$$

Example: conservation laws

Oscillator: $m\frac{d^2}{dt^2}w + kw = 0 \rightsquigarrow r(\xi) = m\xi^2 + k$

Total energy is $Q_E(w) = \frac{1}{2}m\left(\frac{dw}{dt}\right)^2 + \frac{1}{2}kw^2$**. A QDF.**

Two-variable representation is $E(\zeta,\eta) = \frac{1}{2}m\zeta\eta + \frac{1}{2}k$**.**

Physics: total energy is conserved

$$\frac{d}{dt}E(w) = 0 \text{ for all } w \in \mathscr{B}$$

Indeed,

$$(\zeta+\eta)E(\zeta,\eta) = \frac{1}{2}\left(m\zeta^2+k\right)\eta + \left(m\eta^2+k\right)\zeta \;\; = \;\; \zeta r(\eta) + r(\zeta)\eta$$

zero along $\mathscr{B}$**.**

Equivalence of QDFs

$$Q_{\Phi_1} \stackrel{\mathfrak{B}}{=} Q_{\Phi_2} \textbf{ if } Q_{\Phi_1}(w) = Q_{\Phi_2}(w) \textbf{ for all } w \in \mathfrak{B}$$

If $\mathfrak{B} = \mathtt{kernel}(R(\frac{d}{dt}))$**,** $Q_{\Phi_1} \stackrel{\mathfrak{B}}{=} Q_{\Phi_2}$ **equivalent with**

$$\Phi_1(\zeta,\eta) - \Phi_2(\zeta,\eta) = R(\zeta)^\top F(\zeta,\eta) + F(\eta,\zeta)^\top R(\eta)$$

¿Canonical representative ?

Preliminary: R-canonical polynomial differential operators

$$\mathscr{B} = \texttt{kernel}(R(\tfrac{d}{dt}))$$ **autonomous:** $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ $\det(R) \neq 0$.

$D(\frac{d}{dt}) \stackrel{\mathscr{B}}{=} P(\frac{d}{dt})$ **if** $D(\frac{d}{dt})w = P(\frac{d}{dt})w$ **for all** $w \in \mathscr{B}$.

$D(\frac{d}{dt})$ **is** ***R*-canonical** **if** DR^{-1} **is strictly proper.**

Every $D(\frac{d}{dt})$ **is equivalent along** $\mathscr{B}$ **to an** ***R*-canonical polynomial differential operator:**

$$DR^{-1} = \underbrace{P}_{\text{polynomial}} + \underbrace{S}_{\text{strictly proper}} \Longrightarrow D \stackrel{\mathscr{B}}{=} D - PR$$

R-canonical quadratic differential forms

Q_Φ **R-canonical** **if** $R(\zeta)^{-\top}\Phi(\zeta,\eta)R(\eta)^{-1}$ **strictly proper.**

R-canonical quadratic differential forms

Q_Φ R-canonical if $R(\zeta)^{-\top}\Phi(\zeta,\eta)R(\eta)^{-1}$ strictly proper.

Theorem: Every Q_Φ is $\mathscr{B}$-equivalent to a R-canonical QDF.

R-canonical quadratic differential forms

Q_Φ **R-canonical if** $R(\zeta)^{-\top}\Phi(\zeta,\eta)R(\eta)^{-1}$ **strictly proper.**

Theorem: Every Q_Φ is $\mathscr{B}$-equivalent to a R-canonical QDF.

1. **Compute factorization** $\Phi(\zeta,\eta) = N(\zeta)^\top M(\eta)$**;**
2. **Compute R-canonical repr. M' for M, and N' for N;**
3. **R-canonical repr. of Φ is** $\Phi'(\zeta,\eta) := N'(\zeta)^\top M'(\eta)$

R-canonical quadratic differential forms

Q_Φ R-canonical if $R(\zeta)^{-\top}\Phi(\zeta,\eta)R(\eta)^{-1}$ strictly proper.

Theorem: Every Q_Φ is $\mathscr{B}$-equivalent to a R-canonical QDF.

1. **Compute factorization $\Phi(\zeta,\eta)=N(\zeta)^\top M(\eta)$;**
2. **Compute R-canonical repr. M' for M, and N' for N;**
3. **R-canonical repr. of Φ is $\Phi'(\zeta,\eta):=N'(\zeta)^\top M'(\eta)$**

Factorization of step 1: factorize coefficient matrix

$$\begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} = \begin{bmatrix} N'_0 \\ N'_1 \\ \vdots \end{bmatrix} \begin{bmatrix} M'_0 & M'_1 & \cdots \end{bmatrix}$$

Example: the scalar case

$$r_0 w + r_1 \frac{dw}{dt} + \cdots + \frac{d^n w}{dt^n} w = 0$$

Example: the scalar case

$$r_0 w + r_1 \frac{dw}{dt} + \cdots + \frac{d^n w}{dt^n} w = 0$$

Any quadratic expression of w **and its derivatives with terms containing** $\frac{d^k w}{dt^k}$ **for** $k \geq n$ **rewritable in terms of** w **and its derivatives up to** $\frac{d^{n-1} w}{dt^{n-1}}$**.**

Example: the scalar case

$$r_0 w + r_1 \frac{dw}{dt} + \cdots + \frac{d^n w}{dt^n} w = 0$$

Any quadratic expression of w **and its derivatives with terms containing** $\frac{d^k w}{dt^k}$ **for** $k \geq n$ **rewritable in terms of** w **and its derivatives up to** $\frac{d^{n-1} w}{dt^{n-1}}$**.**

E.g. since for $w \in \texttt{kernel}\, r\left(\frac{d}{dt}\right)$

$$\frac{d^n w}{dt^n} = -\left(r_0 w + r_1 \frac{dw}{dt} + \cdots + \frac{d^{n-1} w}{dt^{n-1}} w \right)$$

it holds

$$\left(\frac{d^n w}{dt^n}\right)^2 = \left(r_0 w + r_1 \frac{dw}{dt} + \cdots + \frac{d^{n-1} w}{dt^{n-1}} w \right)^2$$

Example: the scalar case

$$r_0 w + r_1 \frac{dw}{dt} + \cdots + \frac{d^n w}{dt^n} w = 0$$

Any quadratic expression of w and its derivatives with terms containing $\frac{d^k w}{dt^k}$ for $k \geq n$ rewritable in terms of w and its derivatives up to $\frac{d^{n-1} w}{dt^{n-1}}$.

Same for terms containing $\frac{d^{n+1} w}{dt^{n+1}}$, etc.

Example: the scalar case

$$r_0 w + r_1 \frac{dw}{dt} + \cdots + \frac{d^n w}{dt^n} w = 0$$

Any quadratic expression of w and its derivatives with terms containing $\frac{d^k w}{dt^k}$ for $k \geq n$ rewritable in terms of w and its derivatives up to $\frac{d^{n-1} w}{dt^{n-1}}$.

"Rewriting in terms of lower order derivatives" equivalent to "taking the r-canonical representative".

Nonnegativity and positivity

$Q_\Phi \geq 0$ **if** $Q_\Phi(w) \geq 0$ **for all** $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$

Prop.: $Q_\Phi \geq 0$ **if and only if exists** $D \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ **s.t.**

$$\Phi(\zeta, \eta) = D(\zeta)^\top D(\eta)$$

Nonnegativity and positivity

$Q_\Phi \geq 0$ **if** $Q_\Phi(w) \geq 0$ **for all** $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$

Prop.: $Q_\Phi \geq 0$ **if and only if exists** $D \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ **s.t.**

$$\Phi(\zeta, \eta) = D(\zeta)^\top D(\eta)$$

$Q_\Phi > 0$ **if** $Q_\Phi \geq 0$**, and** $[Q_\Phi(w) = 0] \Longrightarrow [w = 0]$

Prop.: $Q_\Phi > 0$ **if and only if exists** $D \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ **s.t.**

$$\Phi(\zeta, \eta) = D(\zeta)^\top D(\eta)$$

$$\mathrm{rank}(D(\lambda)) = \mathtt{w} \textbf{ for all } \lambda \in \mathbb{C}$$

Nonnegativity and positivity along a behavior

$Q_\Phi \overset{\mathscr{B}}{\geq} 0$ **if** $Q_\Phi(w) \geq 0 \ \forall \ w \in \mathscr{B}$

Prop.: $Q_\Phi \overset{\mathscr{B}}{\geq} 0$ **iff** $\exists \ \Phi' \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ **such that**

$\Phi(\zeta,\eta) \overset{\mathscr{B}}{=} \Phi'(\zeta,\eta)$ **and** $Q_{\Phi'} \geq 0$

Nonnegativity and positivity along a behavior

$Q_\Phi \overset{\mathscr{B}}{\geq} 0$ **if** $Q_\Phi(w) \geq 0 \;\forall\, w \in \mathscr{B}$

Prop.: $Q_\Phi \overset{\mathscr{B}}{\geq} 0$ **iff** $\exists\, \Phi' \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ **such that**
$\Phi(\zeta,\eta) \overset{\mathscr{B}}{=} \Phi'(\zeta,\eta)$ **and** $Q_{\Phi'} \geq 0$

Prop.: **Let** $\mathscr{B} = \texttt{kernel}\, R(\frac{d}{dt})$**. Then** $Q_\Phi \overset{\mathscr{B}}{\geq} 0$ **iff there exist** $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\zeta,\eta]$ **such that**

$$\Phi(\zeta,\eta) = \underbrace{D(\zeta)^\top D(\eta)}_{\geq 0 \text{ for all } w} + \underbrace{R(\zeta)^\top X(\zeta,\eta) + X(\eta,\zeta)^\top R(\eta)}_{= 0 \text{ if evaluated on } \mathscr{B}}$$

Nonnegativity and positivity along a behavior

$Q_\Phi \overset{\mathscr{B}}{>} 0$ **if** $Q_\Phi \overset{\mathscr{B}}{\geq} 0$**, and** $[Q_\Phi(w) = 0] \Longrightarrow [w = 0]$

Prop.: $Q_\Phi \overset{\mathscr{B}}{>} 0$ **iff** $\exists\ \Phi' \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ **such that**

$\Phi(\zeta,\eta) \overset{\mathscr{B}}{=} \Phi'(\zeta,\eta)$ **and** $\Phi'(\zeta,\eta) = D(\zeta)^\top D(\eta)$ **with**
rank col$(R(\lambda), D(\lambda)) = \mathtt{w}$ **for all** $\lambda \in \mathbb{C}$

Nonnegativity and positivity along a behavior

$Q_\Phi \overset{\mathscr{B}}{>} 0$ **if** $Q_\Phi \overset{\mathscr{B}}{\geq} 0$**, and** $[Q_\Phi(w) = 0] \implies [w = 0]$

Prop.: $Q_\Phi \overset{\mathscr{B}}{>} 0$ **iff** $\exists\ \Phi' \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ **such that**

$\Phi(\zeta,\eta) \overset{\mathscr{B}}{=} \Phi'(\zeta,\eta)$ **and** $\Phi'(\zeta,\eta) = D(\zeta)^\top D(\eta)$ **with**
$\text{rank col}(R(\lambda),D(\lambda)) = \mathtt{w}$ **for all** $\lambda \in \mathbb{C}$

Prop.: **Let** $\mathscr{B} = \mathbf{kernel}\, R(\frac{d}{dt})$**. Then** $Q_\Phi \overset{\mathscr{B}}{>} 0$ **iff there exist**
$D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\zeta,\eta]$ **such that**

$$\Phi(\zeta,\eta) = \underbrace{D(\zeta)^\top D(\eta)}_{\geq 0 \text{ for all } w} + \underbrace{R(\zeta)^\top X(\zeta,\eta) + X(\eta,\zeta)^\top R(\eta)}_{=0 \text{ if evaluated on } \mathscr{B}}$$

and moreover $\text{rank col}(R(\lambda),D(\lambda)) = \mathtt{w}$ **for all** $\lambda \in \mathbb{C}$

Remarks

- **Property of functional (positivity, nonnegativity along a behavior, etc.) translated into algebraic properties of its two-variable representation;**

Remarks

- **Property of functional (positivity, nonnegativity along a behavior, etc.) translated into algebraic properties of its two-variable representation;**
- **Standard polynomial computations to obtain R-canonical representative $\Longrightarrow$ positivity, negativity along behaviors easy to check.**

Lyapunov theory

Lyapunov theory

$\mathscr{B}$ **is *asymptotically stable*** $\iff \lim_{t\to\infty} w(t) = 0 \; \forall \, w \in \mathscr{B}$

This implies $\mathscr{B}$ is autonomous.

Lyapunov theory

$\mathscr{B}$ **is *asymptotically stable*** $\iff \lim_{t\to\infty} w(t) = 0 \ \forall \ w \in \mathscr{B}$

This implies $\mathscr{B}$ **is autonomous.**

Algebraic characterization:

<u>Theorem</u>: $\mathscr{B} = \texttt{kernel}\, R(\frac{d}{dt})$ **stable,**
$R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$
$\iff \det(R)$ **Hurwitz**

Lyapunov theory

$\mathscr{B}$ **is *asymptotically stable*** $\iff \lim_{t\to\infty} w(t) = 0 \ \forall\, w \in \mathscr{B}$

This implies $\mathscr{B}$ is autonomous.

Algebraic characterization:

<u>Theorem</u>: $\mathscr{B} = \mathtt{kernel}\, R(\frac{d}{dt})$ **stable,**
$R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$
$\iff \det(R)$ **Hurwitz**

Lyapunov functional characterization:

<u>Theorem</u>: $\mathscr{B}$ **asymptotically stable iff**
exists Q_Φ such that $Q_\Phi \overset{\mathscr{B}}{\geq} 0$ and$Q_{\overset{\bullet}{\Phi}} \overset{\mathscr{B}}{<} 0$

Example

$$\mathscr{B} = \texttt{kernel}\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) \rightsquigarrow r(\xi) = \xi^2 + 3\xi + 2$$

Example

$$\mathscr{B} = \texttt{kernel}\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) \rightsquigarrow r(\xi) = \xi^2 + 3\xi + 2$$

Choose $\Psi(\zeta, \eta)$ **s.t.** $Q_\Psi \overset{\mathscr{B}}{<} 0$**, e.g.** $\Psi(\zeta, \eta) = -\zeta\eta$**;**

Example

$$\mathscr{B} = \mathtt{kernel}\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) \rightsquigarrow r(\xi) = \xi^2 + 3\xi + 2$$

Choose $\Psi(\zeta,\eta)$ **s.t.** $Q_\Psi \overset{\mathscr{B}}{<} 0$**, e.g.** $\Psi(\zeta,\eta) = -\zeta\eta$**;**

Find $\Phi(\zeta,\eta)$ **s.t.** $\frac{d}{dt}Q_\Phi(w) = Q_\Psi(w)$ **for all** $w \in \mathscr{B}$**:**

$$(\zeta+\eta)\Phi(\zeta,\eta) = \Psi(\zeta,\eta) + \underbrace{r(\zeta)x(\eta) + x(\zeta)r(\eta)}_{=0 \textbf{ on } \mathscr{B}}$$

Example

$$\mathscr{B} = \texttt{kernel}\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) \rightsquigarrow r(\xi) = \xi^2 + 3\xi + 2$$

Choose $\Psi(\zeta,\eta)$ **s.t.** $Q_\Psi \overset{\mathscr{B}}{<} 0$**, e.g.** $\Psi(\zeta,\eta) = -\zeta\eta$**;**

Find $\Phi(\zeta,\eta)$ **s.t.** $\frac{d}{dt}Q_\Phi(w) = Q_\Psi(w)$ **for all** $w \in \mathscr{B}$**:**

$$(\zeta+\eta)\Phi(\zeta,\eta) = \Psi(\zeta,\eta) + \underbrace{r(\zeta)x(\eta) + x(\zeta)r(\eta)}_{=0 \textbf{ on } \mathscr{B}}$$

$$\frac{d}{dt}Q_\Phi(w) = Q_\Psi(w) \textbf{ for all } w \in \mathscr{B}$$

Example

$$\mathscr{B} = \texttt{kernel}\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) \rightsquigarrow r(\xi) = \xi^2 + 3\xi + 2$$

Choose $\Psi(\zeta,\eta)$ **s.t.** $Q_\Psi \overset{\mathscr{B}}{<} 0$**, e.g.** $\Psi(\zeta,\eta) = -\zeta\eta$**;**

Find $\Phi(\zeta,\eta)$ **s.t.** $\frac{d}{dt}Q_\Phi(w) = Q_\Psi(w)$ **for all** $w \in \mathscr{B}$**:**

$$(\zeta+\eta)\Phi(\zeta,\eta) = \Psi(\zeta,\eta) + \underbrace{r(\zeta)x(\eta) + x(\zeta)r(\eta)}_{=0 \textbf{ on } \mathscr{B}}$$

Equivalent to solving polynomial Lyapunov equation

$$0 = \underbrace{\Psi(-\xi,\xi)}_{\xi^2} + \underbrace{r(-\xi)}_{\xi^2-3\xi+2}\, x(\xi) + x(-\xi)\, \underbrace{r(\xi)}_{\xi^2+3\xi+2}$$

$$\rightsquigarrow x(\xi) = \frac{1}{6}\xi$$

Example

$$\mathscr{B} = \texttt{kernel}\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) \rightsquigarrow r(\xi) = \xi^2 + 3\xi + 2$$

Choose $\Psi(\zeta,\eta)$ **s.t.** $Q_\Psi \overset{\mathscr{B}}{<} 0$**, e.g.** $\Psi(\zeta,\eta) = -\zeta\eta$**;**

Find $\Phi(\zeta,\eta)$ **s.t.** $\frac{d}{dt}Q_\Phi(w) = Q_\Psi(w)$ **for all** $w \in \mathscr{B}$**:**

$$(\zeta+\eta)\Phi(\zeta,\eta) = \Psi(\zeta,\eta) + \underbrace{r(\zeta)x(\eta) + x(\zeta)r(\eta)}_{=0 \textbf{ on } \mathscr{B}}$$

$$\begin{aligned}\Phi(\zeta,\eta) &= \frac{-\zeta\eta + (\zeta^2+3\zeta+2)\frac{1}{6}\eta + \frac{1}{6}\zeta(\eta^2+3\eta+2)}{\zeta+\eta} \\ &= \frac{1}{6}\zeta\eta + \frac{1}{3} > 0\end{aligned}$$

Main points

- **Functionals of system variables and derivatives thereof;**

Main points

- **Functionals of system variables and derivatives thereof;**
- **Two-variable polynomial representation;**

Main points

- **Functionals of system variables and derivatives thereof;**
- **Two-variable polynomial representation;**
- **Operations/properties in time domain $\rightsquigarrow$ algebraic operations on two-variable representation.**

Main points

- **Functionals of system variables and derivatives thereof;**
- **Two-variable polynomial representation;**
- **Operations/properties in time domain $\rightsquigarrow$ algebraic operations on two-variable representation;**
- **Lyapunov theory for higher-order systems.**

End of Lecture VII