European Embedded Control Institute

Graduate School on Control — Spring 2010

The Behavioral Approach to Modeling and Control

Lecture VI

CONTROLLABILITY

Theme

Behavioral controllability, the property of being able to choose the future of a trajectory regardless of the past, is an important and regularizing system property.

In this lecture we obtain

(i) various tests for controllability of LTIDSs,

(ii) special system representations related to controllability,

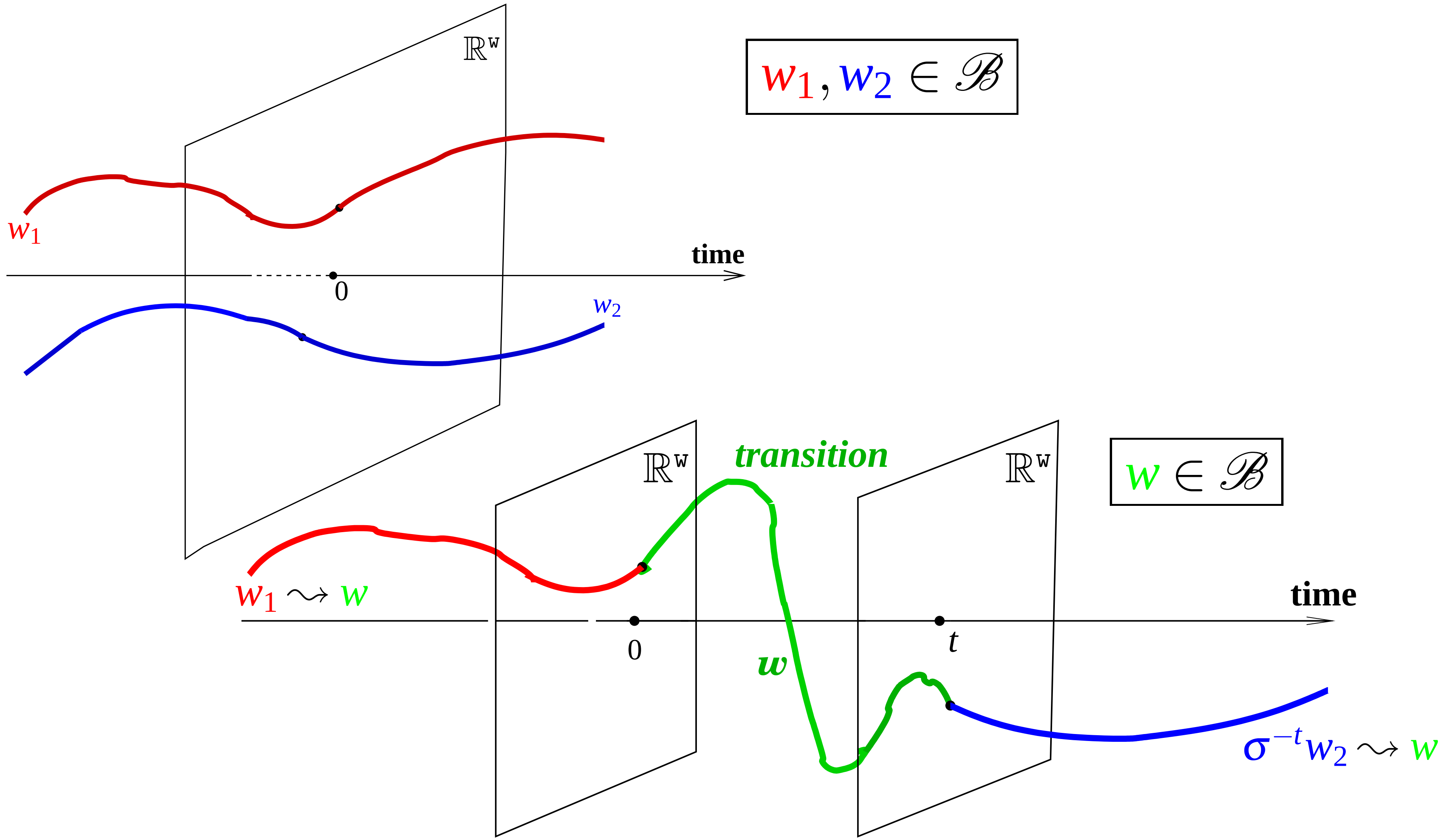
(iii) and generalizations, e.g. to stabilizability.

Outline

- **Controllability of LTIDSs**
- **Image representations**
- **Autonomous LTIDSs**
- **Stability and stabilizability**
- **The controllable part of a behavior**
- **The three main theorems for LTIDSs and their discrete-time and PDE counterparts**
- **Observability**

Controllability

Reminder



controllability :⇔ concatenability of trajectories after a delay

Controllability of LTIDSs

Theorem

The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.

1. $\mathscr{B}$ **is controllable.**
2. **Let** $R\left(\frac{d}{dt}\right) w = 0$ **be a kernel representation** $\mathscr{B}$**.**
   Then $R(\lambda)$ **has the same rank for all** $\lambda \in \mathbb{C}$**.**

Hence,

- $R\left(\frac{d}{dt}\right) w = 0$ **is a minimal kernel representation of a controllable LTIDS if and only if** R **is left prime.**
- $\mathscr{B} \in \mathscr{L}^{\bullet}$ **is controllable if and only if it has a kernel representation** $R\left(\frac{d}{dt}\right) w = 0$ **with** R **left prime over** $\mathbb{R}[\xi]$**.**

Proof in telegram-style

First prove that if $U \in \mathbb{R}[\xi]^{\mathtt{w}\times\mathtt{w}}$ is unimodular, then $\mathscr{B}$ is controllable if and only if $U\left(\frac{d}{dt}\right)\mathscr{B}$ is controllable.

Consequently, we may assume that $\mathscr{B}$ has a minimal kernel representation with R in Smith form,

$$R = \begin{bmatrix} \mathbf{diag}(d_1, d_2, \ldots, d_{\mathtt{r}}) & 0_{\mathtt{r}\times(\mathtt{w}-\mathtt{r})} \end{bmatrix}.$$

1. $\Leftrightarrow$ 2.

Observe, using the theory of autonomous systems, that $\mathscr{B}$ is controllable if and only if all the invariant polynomials $d_1, d_2, \ldots, d_{\mathtt{r}}$ of R are equal to one.
Equivalently, if and only if 2. holds.

Examples

- **Consider the single-input/single output system**

$$p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix},$$

with $p, q \in \mathbb{R}[\xi]$. This system is controllable if and only if p and q are coprime.

The interpretation of common factors in p and q has been a long-standing question in the field. Behavioral controllability demystifies common factors.
We now understand that common factors correspond exactly to lack of controllability, nothing more, nothing less.

In the exercises several algorithms for coprimeness of polynomials are discussed.

Examples

- Applying the controllability theorem and the relation between behavioral and state controllability discussed in Lecture I, shows that the system

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \quad w = \begin{bmatrix} u \\ y \\ x \end{bmatrix}$$

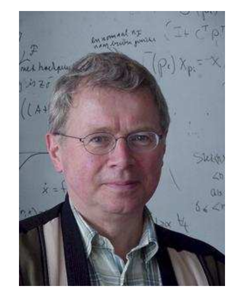
is controllable if and only if

$$\texttt{rank}\left(\left[\begin{array}{c|c} A - I_{\mathtt{n}\times\mathtt{n}}\lambda & B \end{array}\right]\right) = \mathtt{n} \quad \textbf{for all } \lambda \in \mathbb{C}.$$

This state version of controllability is the PBH (Popov-Belevitch-Hautus) test. The controllability theorem is a far-reaching generalization of this classical result.

V.M. Popov (1928–)

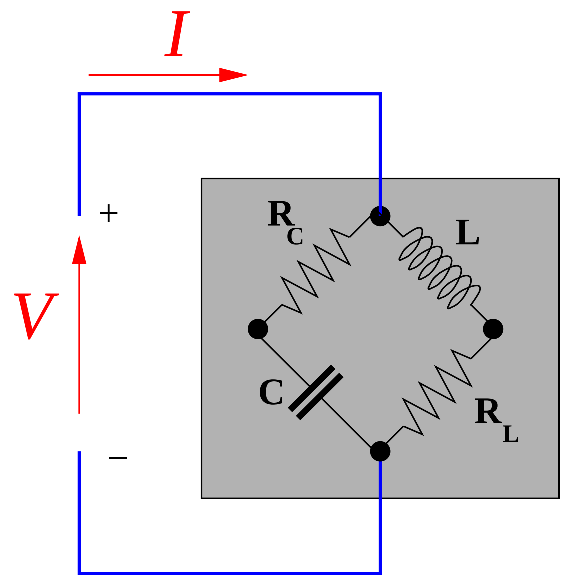
Vitold Belevitch (1921–1999)



Malo Hautus (1940–)

Examples

The port behavior of the RLC circuit (see Lecture IV)



is controllable unless

$$CR_C = \frac{L}{R_L} \quad \textbf{and} \quad R_L = R_C.$$

This shows that lack of controllability can occur in non-degenerate physical systems.

More tests for controllability

Theorem

The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.

1. $\mathscr{B}$ **is controllable.**

3. $\mathscr{N}_{\mathscr{B}}$**, the** $\mathbb{R}[\xi]$**-module of annihilators of** $\mathscr{B}$**, is closed.**

4. $\mathscr{B}$ **has a direct summand, i.e., there exists** $\mathscr{B}' \in \mathscr{L}^{\mathtt{w}}$ **such that** $\mathscr{B} \oplus \mathscr{B}' = \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

The closure of the $\mathbb{R}[\xi]$**-submodule** $\mathscr{M}$ **of** $\mathbb{R}[\xi]^{\mathtt{w}}$ **is defined as**

$$\mathscr{M}^{\text{closure}} := \{\bar{m} \in \mathbb{R}[\xi]^{\mathtt{w}} \mid \exists\, \pi \in \mathbb{R}[\xi], \pi \neq 0, \text{ and } m \in \mathscr{M} \text{ such that } m = \pi \bar{m}\}.$$

$\mathscr{M}$ **is said to be closed if** $\mathscr{M} = \mathscr{M}^{\text{closure}}$.

Proof in telegram-style

First prove that if $U \in \mathbb{R}[\xi]^{\mathtt{w}\times\mathtt{w}}$ **is unimodular, then** $\mathscr{B}$ **is controllable if and only if** $U\left(\frac{d}{dt}\right)\mathscr{B}$ **is controllable.**

Consequently, we may assume that $\mathscr{B}$ **has a minimal kernel representation with** R **in Smith form,**

$$R = \left[\mathbf{diag}(d_1, d_2, \ldots, d_{\mathtt{r}}) \quad 0_{\mathtt{r}\times(\mathtt{w}-\mathtt{r})}\right].$$

2. $\Leftrightarrow$ 3.

Observe that $\mathscr{N}_{\mathscr{B}} = \left[\mathbb{R}[\xi]d_1 \quad \cdots \quad \mathbb{R}[\xi]d_{\mathtt{r}} \quad 0 \quad \cdots \quad 0\right]$**.**
Hence $\mathscr{N}_{\mathscr{B}}$ **is closed if and only if all the invariant polynomials** $d_1, d_2, \ldots, d_{\mathtt{r}}$ **of** R **are equal to one.**

Equivalently, if and only if 2. holds.

Proof in telegram-style

Consequently, we may assume that $\mathscr{B}$ has a minimal kernel representation with R in Smith form,

$$R = \begin{bmatrix} \mathbf{diag}(d_1, d_2, \ldots, d_{\mathtt{r}}) & 0_{\mathtt{r} \times (\mathtt{w}-\mathtt{r})} \end{bmatrix}.$$

3. $\Rightarrow$ 4.

Take for $\mathscr{B}'$ the system with kernel representation

$R'\left(\frac{d}{dt}\right) w = 0$**, with** $R' = \begin{bmatrix} 0_{\mathtt{w}-\mathtt{r} \times \mathtt{r}} & I_{(\mathtt{w}-\mathtt{r}) \times (\mathtt{w}-\mathtt{r})} \end{bmatrix}.$

Proof in telegram-style

Consequently, we may assume that $\mathscr{B}$ has a minimal kernel representation with R in Smith form,

$$R = \begin{bmatrix} \mathbf{diag}(d_1, d_2, \ldots, d_{\mathtt{r}}) & 0_{\mathtt{r}\times(\mathtt{w}-\mathtt{r})} \end{bmatrix}.$$

4. $\Rightarrow$ 3.

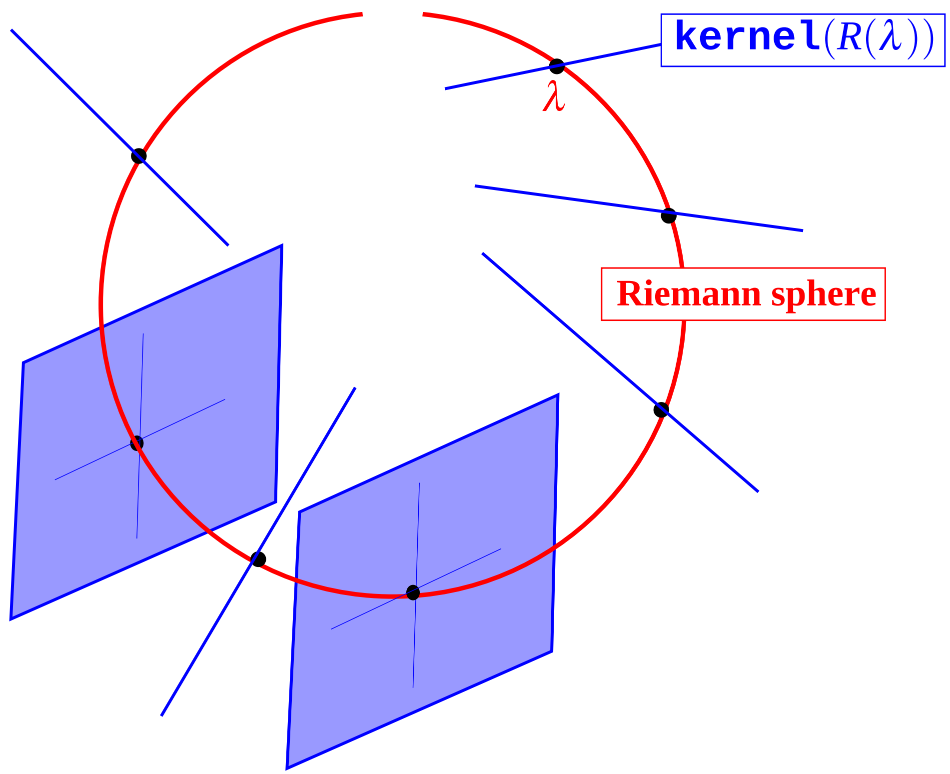
Note that $[\![\mathscr{B} \oplus \mathscr{B}' = \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})]\!] \Leftrightarrow [\![\mathscr{N}_{\mathscr{B}} \oplus \mathscr{N}_{\mathscr{B}'} = \mathbb{R}[\xi]^{1\times\mathtt{w}}]\!]$**.**
Let $R'\left(\frac{d}{dt}\right) w = 0$ **be a minimal kernel representation of** $\mathscr{B}'$**.**

Then $\mathscr{N}_{\mathscr{B}} \oplus \mathscr{N}_{\mathscr{B}'} = \mathbb{R}[\xi]^{\mathtt{w}}$ **implies that the rows of** $\begin{bmatrix} R \\ R' \end{bmatrix}$ **form a basis for** $\mathbb{R}[\xi]^{1\times\mathtt{w}}$**. Equivalently, that** $\begin{bmatrix} R \\ R' \end{bmatrix}$ **is unimodular.**

Hence that the invariant polynomials of R are all equal to one.

Geometric interpretation of controllability

Attach to each point $\lambda \in \mathbb{C}$ of the Riemann sphere (think of the Riemann sphere as $\mathbb{C}$), $\texttt{kernel}(R(\lambda))$. This associates with each $\lambda \in \mathbb{C}$ a linear subspace of $\mathbb{C}^{\texttt{w}}$. In general, this yields a picture shown below. Since the dimension of the subspace attached may change, we obtain a 'sheaf'.



Bernhard Riemann
1826–1866

Geometric interpretation of controllability

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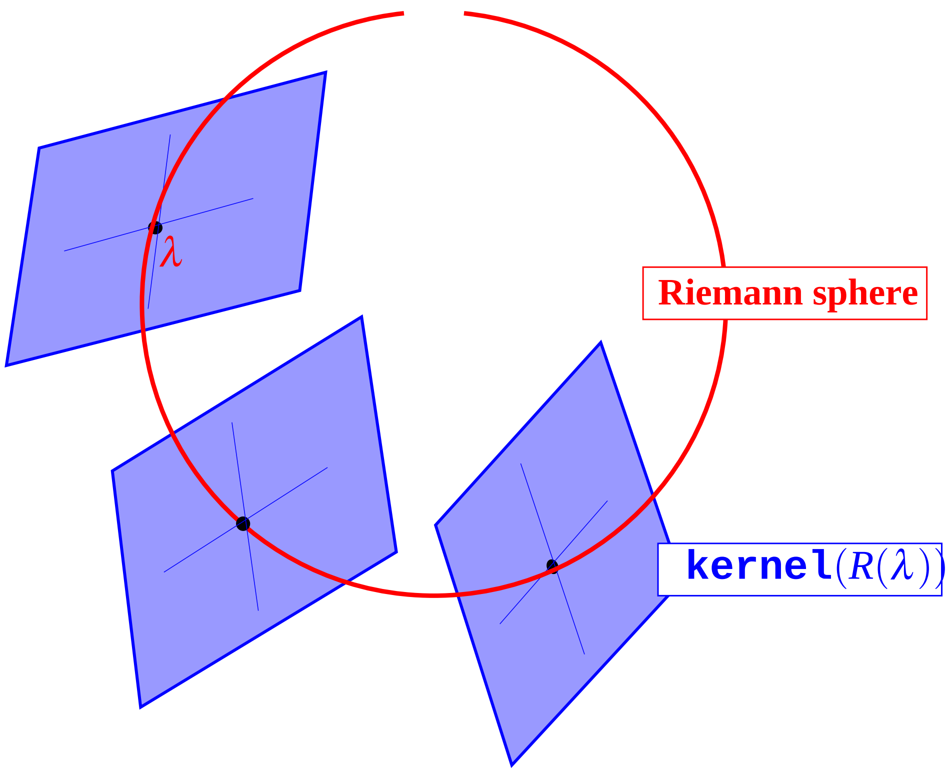

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Representations of behaviors

Kernels, images, and projections

A model $\mathscr{B}$ is a subset of $\mathscr{U}$.
There are many ways to specify a subset. For example,

- **as the set of solutions of equations,**
- **as the image of a map,**
- **as a projection.**

Kernels, images, and projections

A model $\mathscr{B}$ is a subset of $\mathscr{U}$.
There are many ways to specify a subset. For example,

- **as the set of solutions of equations:**

$$f : \mathscr{U} \to \bullet, \qquad \mathscr{B} = \{w \in \mathscr{U} \mid f(w) = 0\},$$

- **as the image of a map:**

$$f : \bullet \to \mathscr{U}, \qquad \mathscr{B} = \{w \in \mathscr{U} \mid \exists\, \ell \text{ such that } w = f(\ell)\},$$

- **as a projection:**

$$\mathscr{B}_{\text{extended}} \subseteq \mathscr{U} \times \mathscr{L},$$

$$\mathscr{B} = \{w \in \mathscr{U} \mid \exists\, \ell \in \mathscr{L} \text{ such that } (w, \ell) \in \mathscr{B}_{\text{extended}}\}.$$

Kernels, images, and projections

A model $\mathscr{B}$ is a subset of $\mathscr{U}$.
There are many ways to specify a subset. For example,

- **as solutions of equations:** **kernel representation**

$$f : \mathscr{U} \to \bullet, \qquad \mathscr{B} = \{w \in \mathscr{U} \mid f(w) = 0\},$$

- **as the image of a map:** **image representation**

$$f : \bullet \to \mathscr{U}, \qquad \mathscr{B} = \{w \in \mathscr{U} \mid \exists\, \ell \text{ such that } w = f(\ell)\},$$

- **as a projection:** **latent variable representation**

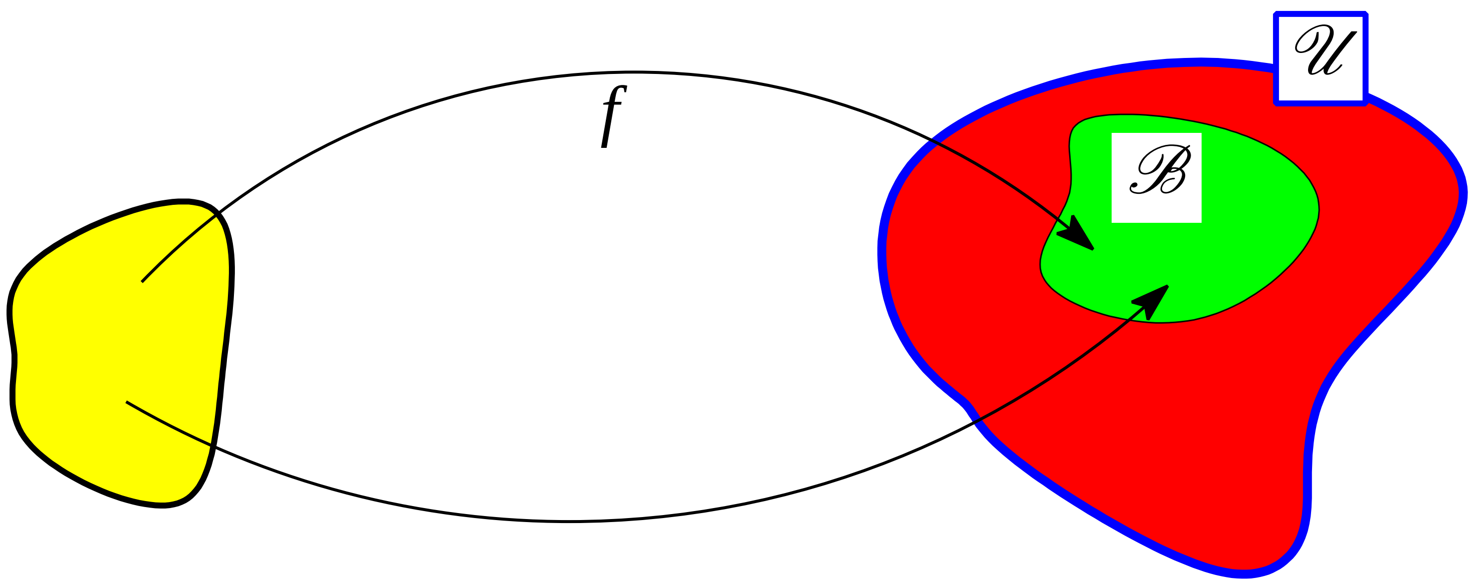
$$\mathscr{B} = \{w \in \mathscr{U} \mid \exists\, \ell \in \mathscr{L} \text{ such that } (w, \ell) \in \mathscr{B}_{\text{extended}}\},$$

w's 'manifest' variables: the variables the model aims at,
ℓ's 'latent' variables: auxiliary variables.

Image

- **as the image of a map:** **image representation**

$$f : \bullet \to \mathscr{U}, \qquad \mathscr{B} = \{w \in \mathscr{U} \mid \exists\, \ell \text{ such that } w = f(\ell)\}.$$



Controllability

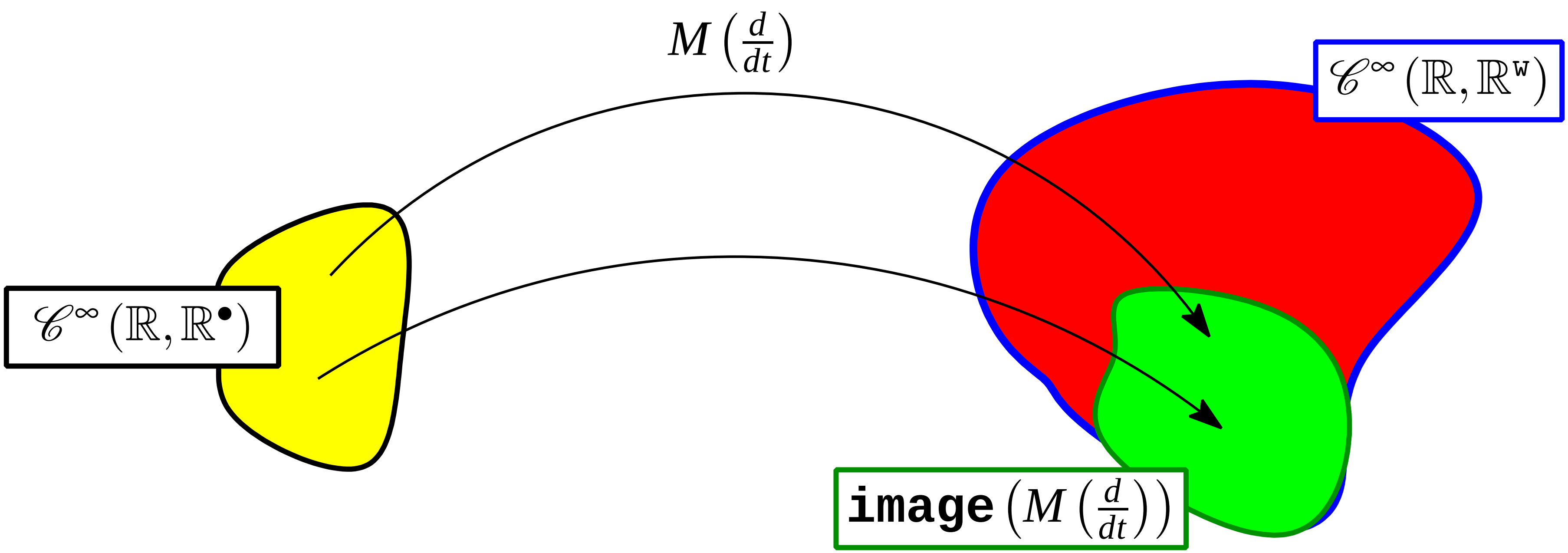
and

image representations

Images

We have seen kernels and projections of linear constant-coefficient differential operators.
It is time for images!

$$w = M\left(\frac{d}{dt}\right)\ell.$$



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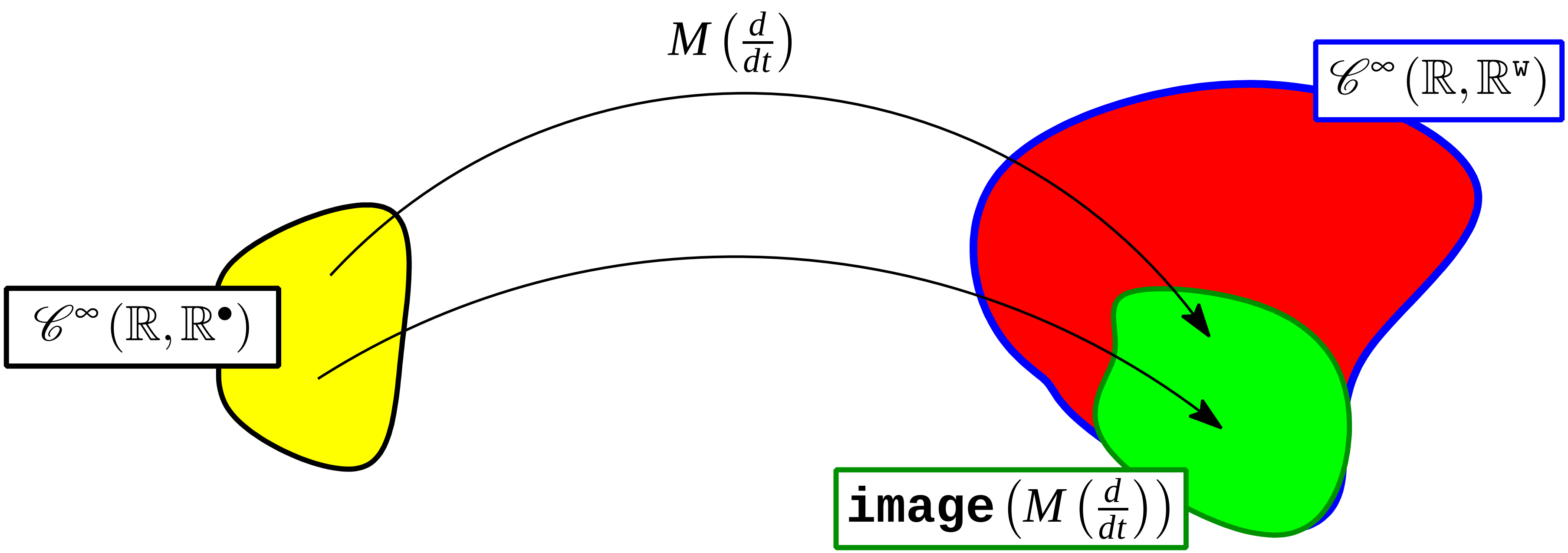

Elimination theorem (see Lecture IV) $\Rightarrow$ an image is a kernel.

What is special about images ?

The image representation theorem

Theorem

The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.

1. $\mathscr{B}$ **is controllable.**
5. $\mathscr{B}$ **has a image representation**

$$w = M\left(\frac{d}{dt}\right)\ell.$$

Images have a nice system theoretic interpretation!

Proof in telegram-style

1. ⇒ 5.

By controllability, the invariant polynomials of R, with $R\left(\frac{d}{dt}\right)w=0$ a minimal kernel representation of $\mathscr{B}$, are equal to one. Therefore, $R=V\begin{bmatrix}I_{\mathtt{r}\times\mathtt{r}} & 0_{\mathtt{r}\times(\mathtt{w}-\mathtt{r})}\end{bmatrix}U$, with U,V unimodular. It follows that $w=U^{-1}\left(\frac{d}{dt}\right)\begin{bmatrix}0_{\mathtt{r}\times(\mathtt{w}-\mathtt{r})}\\ I_{(\mathtt{w}-\mathtt{r})\times(\mathtt{w}-\mathtt{r})}\end{bmatrix}\ell$ is an image representation of $\mathscr{B}$.

5. ⇒ 1.

The extended behavior $\{(w,\ell)\mid w=M\left(\frac{d}{dt}\right)\ell\}$ is controllable, since $\begin{bmatrix}I_{\mathtt{w}\times\mathtt{w}} & -M(\lambda)\end{bmatrix}$ has rank $\mathtt{w}$ for all $\lambda\in\mathbb{C}$.
This implies that the projection is controllable.

Algorithm for controllability

Algorithm

- **Start with** $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$ **parametrizing the LTIDS** $R\left(\frac{d}{dt}\right) w = 0$.

Problem: **Verify if this system is controllable.**

Algorithm

- **Start with** $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$

Problem: **Verify if this system is controllable.**

- **The set**

$$\{ f \in \mathbb{R}[\xi]^{\mathtt{w}} \mid Rf = 0 \}$$

is called the *right syzygy* of R. It is obviously an $\mathbb{R}[\xi]$-module.

Compute a basis for the right syzygy of R. Let F be a matrix whose rows form a basis for this syzygy.

Algorithm

- **Start with $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$**

  Problem: **Verify if this system is controllable.**

- **Compute a basis for the right syzygy of R. Let F be a matrix whose rows form a basis for this syzygy.**

- **Compute a basis for the left syzygy of F. Let R' be a matrix whose rows form a basis for this syzygy.**

  Computing such a basis is a standard problem in computer algebra

Algorithm

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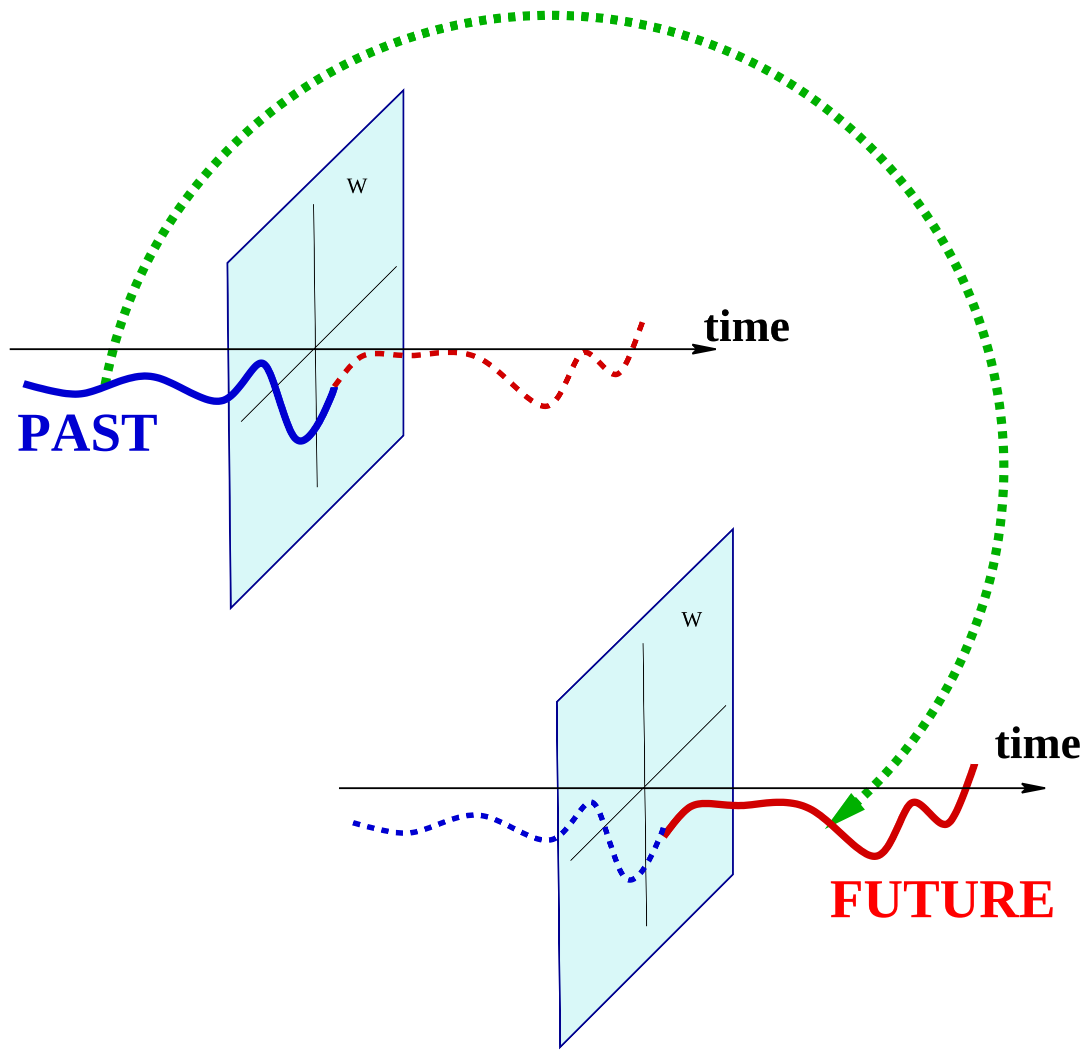
- **Compute a basis for the left syzygy of F. Let R' be a matrix whose rows form a basis for this syzygy.**

- **The system is controllable $\Leftrightarrow$ $\langle R' \rangle = \langle R \rangle$.**

  Verifying this equality is a standard problem in computer algebra.

Autonomous LTIDSs

Reminder



autonomous :⇔ the past implies the future.

Characterization of autonomous LTIDSs

Theorem

The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.

- $\mathscr{B}$ **is autonomous.**
- $\mathscr{B}$ **is a finite-dimensional subspace of** $\mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.
- $\mathscr{B}$ **has a minimal kernel representation** $R\left(\frac{d}{dt}\right)w = 0$ **with** R **square and** $\mathtt{determinant}(R) \neq 0$.
- $\mathtt{m}(\mathscr{B}) = 0$, **equivalently,** $\mathtt{p}(\mathscr{B}) = \mathtt{w}(\mathscr{B})$.

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The proof follows readily from the Smith form and Propositions 1 and 3 of the section on differential operators (see Lecture II).

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- $\mathtt{m}(\mathscr{B}) = 0$, **equivalently,** $\mathtt{p}(\mathscr{B}) = \mathtt{w}(\mathscr{B})$.

With R **minimal, there holds,**
for $\mathtt{w} = 1$, $\mathtt{dimension}(\mathscr{B}) = \mathtt{degree}(R)$,
for $\mathtt{w} > 1$, $\mathtt{dimension}(\mathscr{B}) = \mathtt{degree}(\mathtt{determinant}(R))$.

Autonomous LTIDSs

Each trajectory w **of an autonomous** $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **is a sum of products of a polynomial and an exponential in the complex case,**

$$w(t) = \pi_1(t)e^{\lambda_1 t} + \pi_2(t)e^{\lambda_2 t} + \cdots + \pi_{\mathtt{r}}(t)e^{\lambda_{\mathtt{r}} t},$$

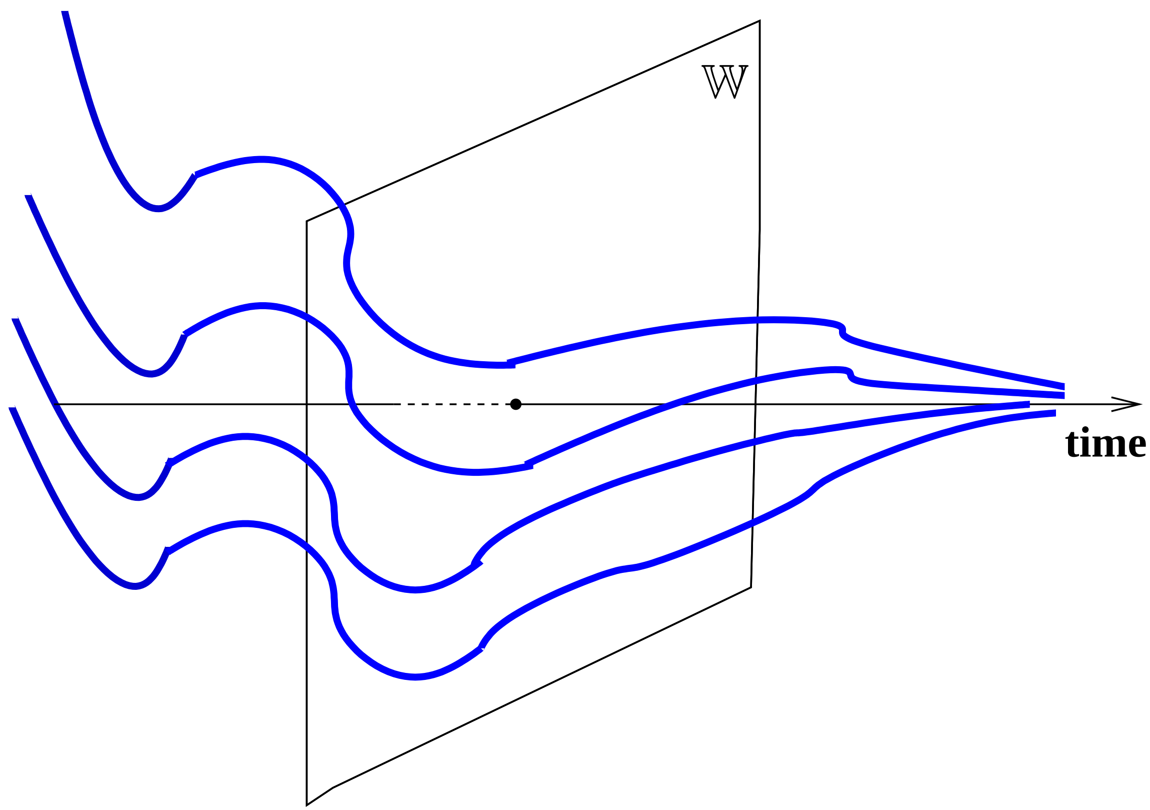
with $\pi_{\mathtt{k}} \in \mathbb{C}[\xi]^{\mathtt{w}}$ **and** $\lambda_{\mathtt{k}} \in \mathbb{C}$. **In the real case, it is a sum of products of a polynomial, an exponential, and a trigonometric function,**

$$\begin{aligned} w(t) = \pi_1'(t)e^{\lambda_1 t}\mathtt{cos}(\omega_1 t) + \pi_1''(t)e^{\lambda_1 t}\mathtt{sin}(\omega_1 t) \\ + \pi_2'(t)e^{\lambda_2 t}\mathtt{cos}(\omega_2 t) + \pi_2''(t)e^{\lambda_2 t}\mathtt{sin}(\omega_2 t) \\ + \cdots + \pi_{\mathtt{r}}'(t)e^{\lambda_{\mathtt{r}} t}\mathtt{cos}(\omega_{\mathtt{r}} t) + \pi_{\mathtt{r}}''(t)e^{\lambda_{\mathtt{r}} t}\mathtt{sin}(\omega_{\mathtt{r}} t), \end{aligned}$$

with $\pi_{\mathtt{k}}', \pi_{\mathtt{k}}'' \in \mathbb{R}[\xi]^{\mathtt{w}}, \lambda_{\mathtt{k}} \in \mathbb{R}$, **and** $\omega_{\mathtt{k}} \in \mathbb{R}$**.**

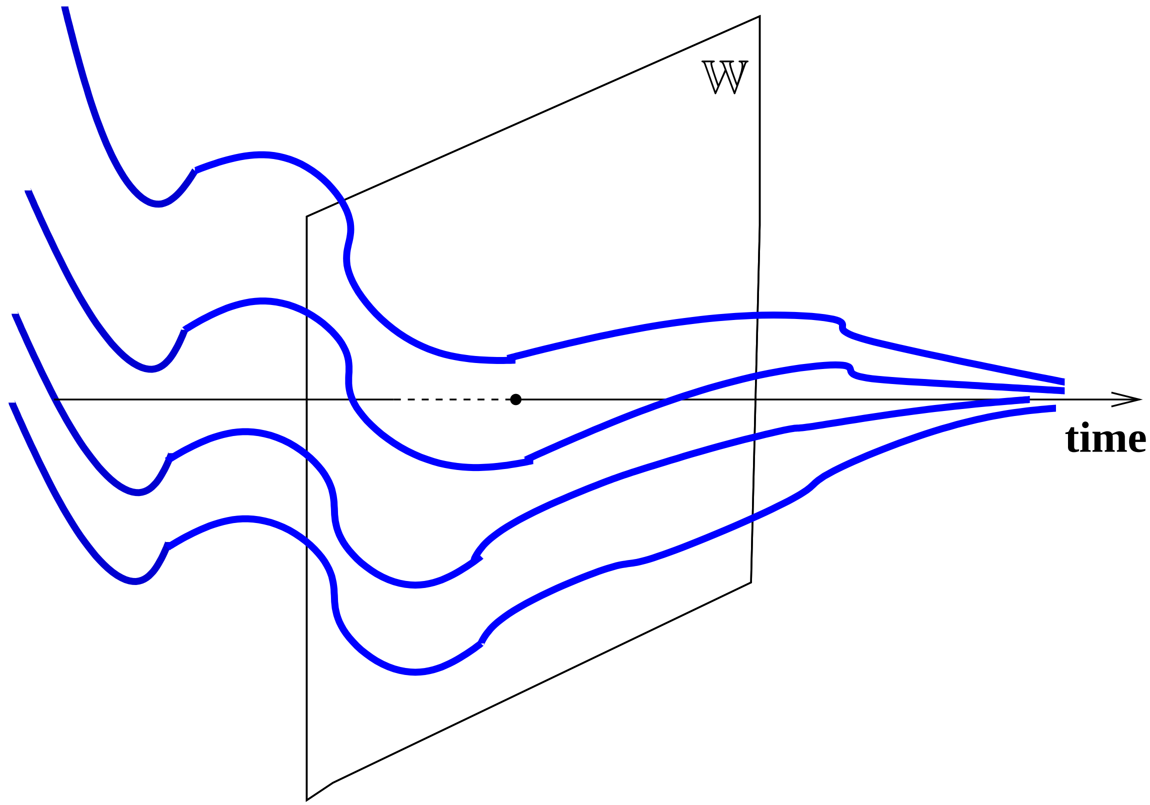
Stability

Reminder



stability :⇔ all trajectories go to 0.

Reminder



stability :⇔ all trajectories go to 0.

For $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$, there holds $[\![\mathscr{B}$ stable$]\!] \Rightarrow [\![\mathscr{B}$ autonomous$]\!]$.

Stability of LTIDSs

Theorem

The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.

- $\mathscr{B}$ **is stable.**
- **Every exponential trajectory** $t \mapsto e^{\lambda t} a, \ a \in \mathbb{C}^{\mathtt{w}}$, **in** $\mathscr{B}$ **(complexified) has** $\mathtt{Real}(\lambda) < 0$**.**
- $\mathscr{B}$ **has a minimal kernel representation** $R\left(\frac{d}{dt}\right) w = 0$ **with** R **Hurwitz.**

A polynomial $\in \mathbb{C}[\xi]$ **is said to be Hurwitz if all its roots are in** $\{\lambda \in \mathbb{C} \mid \mathtt{Real}(\lambda) < 0\}$**.** $P \in \mathbb{C}[\xi]^{\mathtt{n} \times \mathtt{n}}$ **is said to be Hurwitz if it is square and** $\mathtt{determinant}(P)$ **is Hurwitz.**

Stability of LTIDSs

Theorem

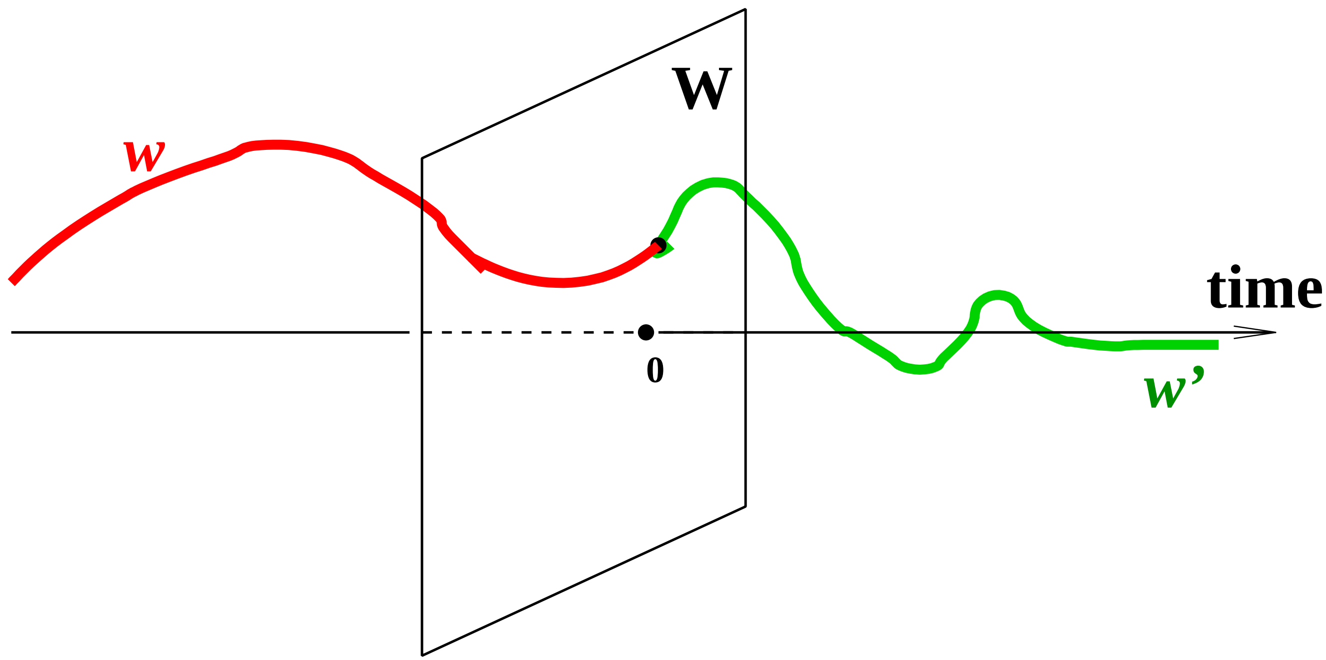
The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.

- $\mathscr{B}$ **is stable.**
- **Every exponential trajectory** $t \mapsto e^{\lambda t} a, \ a \in \mathbb{C}^{\mathtt{w}}$, **in $\mathscr{B}$ (complexified) has** $\mathtt{Real}(\lambda) < 0$.
- $\mathscr{B}$ **has a minimal kernel representation** $R\left(\frac{d}{dt}\right) w = 0$ **with** R **Hurwitz.**

The proof follows readily from the Smith form and Propositions 1 and 3 of the section on differential operators (see Lecture II).

Stabilizability

Reminder



stabilizability :⇔ all trajectories can be steered to 0.

Stabilizability of LTIDSs

Theorem

The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.

1. **$\mathscr{B}$ is stabilizable.**
2. **If $R\left(\frac{d}{dt}\right)w = 0$ is a kernel representation of $\mathscr{B}$, then**

$$[\![\lambda \in \mathbb{C}, \mathtt{rank}(R(\lambda)) < \mathtt{rank}(R)]\!] \Rightarrow [\![\mathtt{Real}(\lambda) \geq 0]\!].$$

Proof in telegram-style

2. $\Rightarrow$ 1.

First assume that R is in Smith form. Prove that if 2. is satisfied, then the first $\mathtt{r}$ components of w are polynomial exponentials, with exponentials having negative real part, while the remaining components of w are free. Conclude stabilizability.

1. $\Rightarrow$ 2.

Conversely, if 2. is not satisfied, then there is a solution whose first component is an exponential with real part ≥ 0.
This exponential cannot be steered to zero, and the system is not stabilizable.

Proof in telegram-style

Next, consider general R's. The solutions are now of the form $w = U\left(\frac{d}{dt}\right)w'$ with $U \in \mathbb{R}[\xi]^{\mathtt{w}\times\mathtt{w}}$ unimodular, and w' a solution corresponding to the Smith form of R. The arguments extend, since polynomial exponentials are converted by $U\left(\frac{d}{dt}\right)$ to polynomial exponentials with the same exponential coefficients.

More on controllability

The controllable part of a LTIDS

The controllable part of $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$**, denoted by** $\mathscr{B}_{\text{controllable}}$**,**
is defined as the largest controllable LTIDS contained in $\mathscr{B}$**.**
That is,

1. $\mathscr{B}_{\text{controllable}} \in \mathscr{L}^{\mathtt{w}}$,
2. $\mathscr{B}_{\text{controllable}} \subseteq \mathscr{B}$,
3. $[\![\mathscr{B}' \in \mathscr{L}^{\mathtt{w}}, \mathscr{B}' \subseteq \mathscr{B}$, **and** $\mathscr{B}'$ **controllable** $]\!]$
$$\Rightarrow [\![\mathscr{B}' \subseteq \mathscr{B}_{\text{controllable}}]\!].$$

The controllable part of a LTIDS

The controllable part of $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$**, denoted by** $\mathscr{B}_{\text{controllable}}$**, is defined as the largest controllable LTIDS contained in** $\mathscr{B}$**.**

Let $R\left(\frac{d}{dt}\right)w = 0$ **be a minimal kernel representation of** $\mathscr{B}$**. The polynomial matrix** R **can be factored as** $R = FR',$ **with** $F \in \mathbb{R}[\xi]^{\mathtt{p}(\mathscr{B})\times\mathtt{p}(\mathscr{B})}$ **and with** $R' \in \mathbb{R}[\xi]^{\mathtt{p}(\mathscr{B})\times\mathtt{w}(\mathscr{B})}$ **left prime:** $R'(\lambda)$ **has the same rank for all** $\lambda \in \mathbb{C}$**. Then** $R'\left(\frac{d}{dt}\right)w = 0$ **is a kernel representation of** $\mathscr{B}_{\text{controllable}}$**.**

The controllable part of a LTIDS

The controllable part of $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$, denoted by $\mathscr{B}_{\text{controllable}}$, is defined as the largest controllable LTIDS contained in $\mathscr{B}$.

There holds

$$\mathscr{B}_{\text{controllable}} = \overline{\mathscr{B}_{\text{compact}}}^{\mathscr{C}^\infty\left(\mathbb{R},\mathbb{R}^{\mathtt{w}(\mathscr{B})}\right)},$$

where $\mathscr{B}_{\text{compact}}$ denotes the set of compact support trajectories in $\mathscr{B}$, and $\overline{\mathscr{B}_{\text{compact}}}^{\mathscr{C}^\infty\left(\mathbb{R},\mathbb{R}^{\mathtt{w}(\mathscr{B})}\right)}$ the closure of $\mathscr{B}_{\text{compact}}$ in the $\mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}(\mathscr{B})})$-topology.

The controllable part of a LTIDS

The controllable part of $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$**, denoted by** $\mathscr{B}_{\text{controllable}}$**, is defined as the largest controllable LTIDS contained in** $\mathscr{B}$**.**

Every $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **admits a decomposition** $\mathscr{B} = \mathscr{B}_1 \oplus \mathscr{B}_2,$ **with** $\mathscr{B}_1 \in \mathscr{L}^{\mathtt{w}}$ **controllable and** $\mathscr{B}_2 \in \mathscr{L}^{\mathtt{w}}$ **autonomous**.

In every such a decomposition, $\mathscr{B}_1 = \mathscr{B}_{\text{controllable}}$**.**

Controllability and rational symbols

$\mathscr{B}$ is controllable if and only if it admits a representation

$$w = M\left(\frac{d}{dt}\right)\ell$$

with $M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$. In particular, therefore, the system

$$y = G\left(\frac{d}{dt}\right)u,$$

with $G \in \mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{m}}$, is always controllable.

Controllability and transfer functions

Consider the input/output systems

$$P_1\left(\frac{d}{dt}\right)y_1 = Q_1\left(\frac{d}{dt}\right)u_1, \quad P_2\left(\frac{d}{dt}\right)y_2 = Q_2\left(\frac{d}{dt}\right)u_2,$$

with $\mathtt{determinant}(P_1) \neq 0$, **and** $\mathtt{determinant}(P_2) \neq 0$**.**

These two systems have the same transfer function,

$$P_1^{-1}Q_1 = P_2^{-1}Q_2,$$

if and only if they have the same controllable part.

Therefore, ***the transfer function determines the controllable part of a system, but only the controllable part.***

The three theorems

LTIDSs

The following are the three main theorems for LTIDSs.

Theorem 1: There exists a $1 \leftrightarrow 1$ relation between $\mathscr{L}^{\mathtt{w}}$ and the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{\mathtt{w}}$.

Theorem 2: Elimination. $\mathscr{L}^{\bullet}$ is closed under projection:

$$[\![\mathscr{B} \in \mathscr{L}^{\mathtt{w}_1+\mathtt{w}_2}]\!] \Rightarrow [\![\Pi_1 \mathscr{B} \in \mathscr{L}^{\mathtt{w}_1}]\!].$$

Π_{w_1} **defines the projection onto the first** $\mathtt{w}_1$ **components.**

Theorem 3: A LTIDS is controllable if and only if its behavior is the image of a linear constant-coefficient differential operator.

Discrete-time LTIDSs

These theorems remain valid for discrete-systems.

The relevant ring for the case $\mathbb{T} = \mathbb{Z}$ **is** $\mathbb{R}[\xi, \xi^{-1}]$**.**
For $\mathbb{T} = \mathbb{Z}_+$**, it is** $\mathbb{R}[\xi]$**.**

PDEs

These theorems also remain valid for systems described by linear constant-coefficient PDEs with

- **the relevant ring** $\mathbb{R}[\xi_1, \xi_2, \ldots, \xi_n]$**,**
- **the appropriate notion of controllability.**

$\mathtt{n}$-D controllabilty

$[\![(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ **is controllable** $]\!] :\Leftrightarrow [\![\forall\, w_1, w_2 \in \mathscr{B}$, **and** $\forall$ **open subsets** $O_1, O_2 \subset \mathbb{R}^{\mathtt{n}}$ **with non-intersecting closures,** $\exists\, w \in \mathscr{B}$ **such that**

$$w|_{O_1} = w_1|_{O_1} \quad \textbf{and} \quad w|_{O_2} = w_1|_{O_2}]\!].$$

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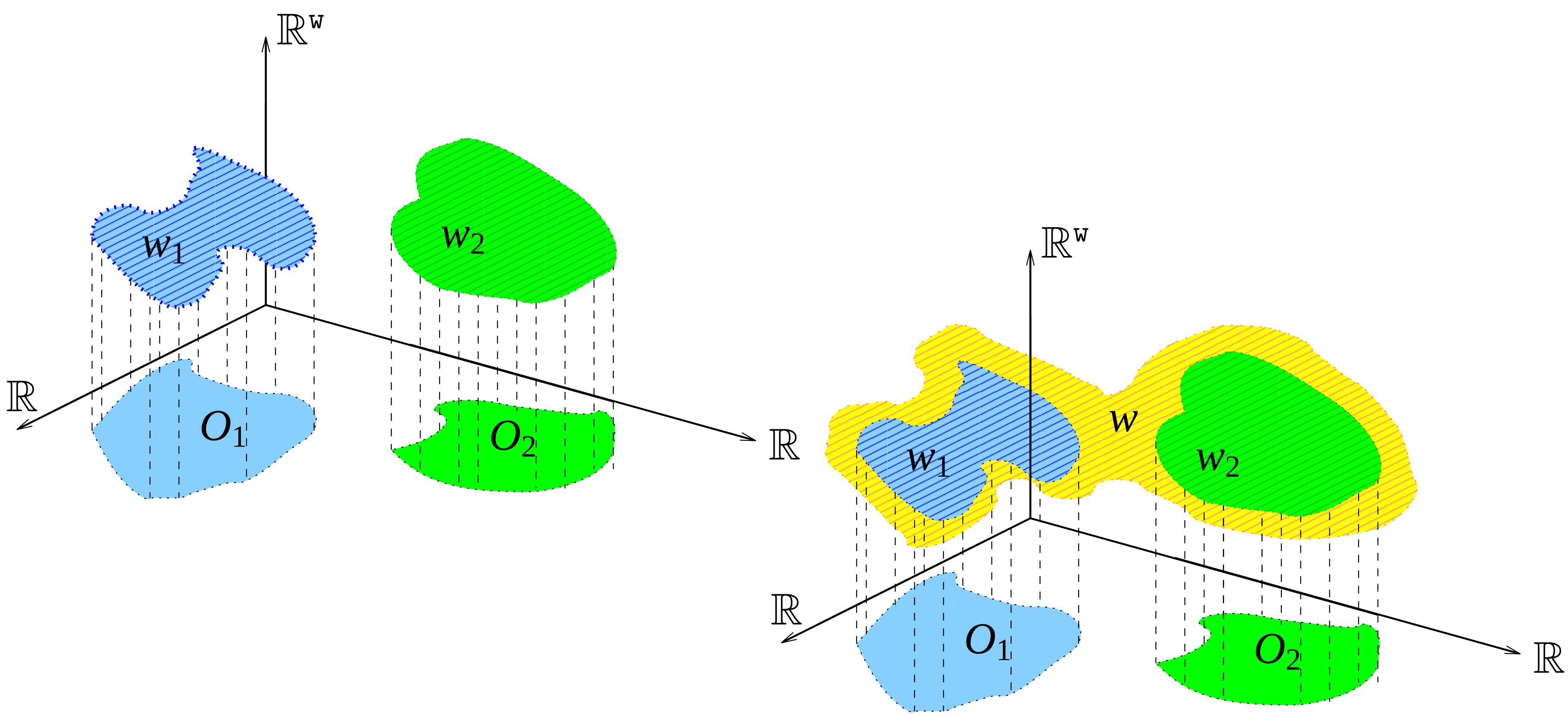
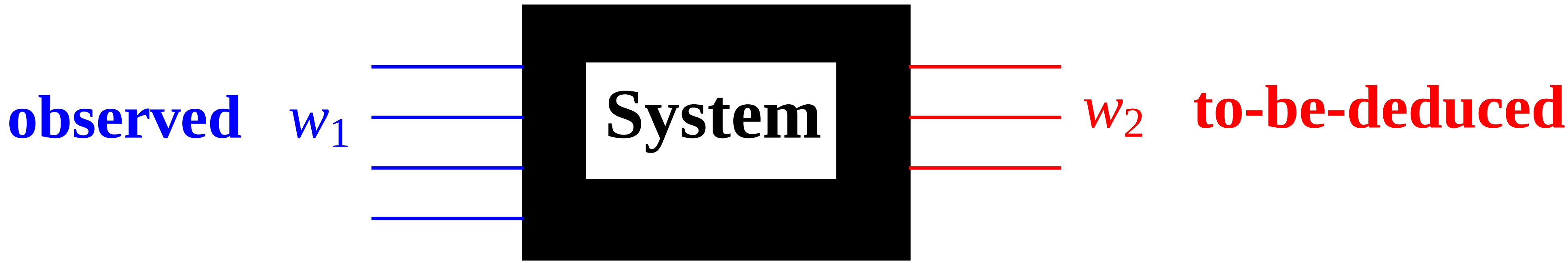
$$w|_{O_1} = w_1|_{O_1} \quad \textbf{and} \quad w|_{O_2} = w_1|_{O_2}]\!].$$

This definition is illustrated in the picture below.



Observability and detectability

Reminder



observability :⇔ w_2 may be deduced from w_1.

!!! Knowing the laws of the system !!!

The observability theorem

Theorem

The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}_1+\mathtt{w}_2}$, $\mathscr{B} \subseteq \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1} \times \mathbb{R}^{\mathtt{w}_2})$.

1. w_2 **is observable from** w_1 **in** $\mathscr{B}$.
2. $\mathscr{B}$ **has a kernel representation** $R_1\left(\frac{d}{dt}\right)w_1 = R_2\left(\frac{d}{dt}\right)w_2$, **with** $\mathtt{rank}\left(R_2(\lambda)\right) = \mathtt{w}_2$ **for all** $\lambda \in \mathbb{C}$.
3. $\mathscr{B}$ **has a minimal kernel representation**

$$w_2 = F\left(\frac{d}{dt}\right)w_1, \quad R\left(\frac{d}{dt}\right)w_1 = 0.$$

Proof in telegram-style

1. $\Leftrightarrow$ 2.

$$[\![\textbf{Observability}]\!] \Leftrightarrow [\![R_2\left(\frac{d}{dt}\right) \textbf{ is injective}]\!] \Leftrightarrow [\![\textbf{2.}]\!]$$

Proof in telegram-style

1. $\Leftrightarrow$ 2.

$[\![$**Observability**$]\!] \Leftrightarrow [\![R_2\left(\frac{d}{dt}\right)$ **is injective**$]\!] \Leftrightarrow [\![$**2.**$]\!]$

2. $\Leftrightarrow$ 3.

$(\Leftarrow)$ **is obvious. To prove** $(\Rightarrow)$**, observe that 2. implies that** R_2 **is of the form** $R_2 = V \begin{bmatrix} I_{\mathtt{w}_2 \times \mathtt{w}_2} \\ 0_{\bullet \times \mathtt{w}_2} \end{bmatrix} U$**, with** V, U **unimodular.**
Therefore $\mathscr{B}$ **admits the kernel representation**

$$R_1'\left(\frac{d}{dt}\right) w_1 = U\left(\frac{d}{dt}\right) w_2, \quad R_1''\left(\frac{d}{dt}\right) w_1 = 0,$$

leading to 3.

The detectability theorem

Theorem

The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}_1+\mathtt{w}_2}$, $\mathscr{B} \subseteq \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1} \times \mathbb{R}^{\mathtt{w}_2})$.

1. **w_2 is detectable from w_1 in $\mathscr{B}$.**
2. **$\mathscr{B}$ has a kernel representation $R_1\left(\frac{d}{dt}\right) w_1 = R_2\left(\frac{d}{dt}\right) w_2$, with $\mathtt{rank}\,(R_2(\lambda)) = \mathtt{w}_2$ for $\lambda \in \{\lambda' \in \mathbb{C} \mid \mathtt{Real}(\lambda') \geq 0\}$.**
3. **$\mathscr{B}$ has a minimal kernel representation**

$$H\left(\frac{d}{dt}\right) w_2 = F\left(\frac{d}{dt}\right) w_1, \quad R\left(\frac{d}{dt}\right) w_1 = 0,$$

with H Hurwitz.

The proof is analogous to that of the observability theorem.

Recapitulation

Summary

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Summary

- There exists tests for verifying the controllability of a LTIDS.
- A LTIDS is controllable if and only if it allows an image representation.
- There exists tests for verifying the stabilizability of a LTIDS.
- Every LTIDS admits a decomposition as controllable $\oplus$ autonomous.
- The transfer function determines the controllable part only.
- There exists tests for verifying the observability and the detectability of a LTIDS.

End of Lecture VI