European Embedded Control Institute

Graduate School on Control — Spring 2010

The Behavioral Approach to Modeling and Control

Lecture IV

LATENT VARIABLES

Theme

First-principles models invariably contain latent (auxiliary) variables in addition to the (manifest) variables the model aims at.

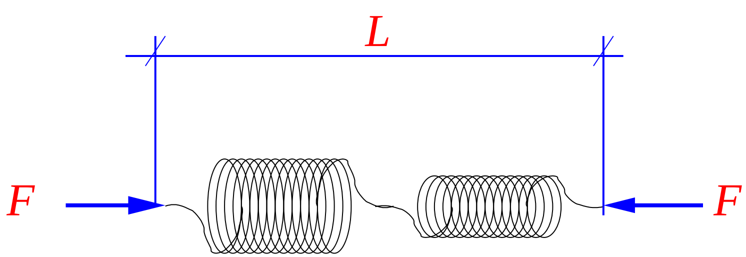
In this lecture we study the emergence of latent variables and their elimination for LTIDSs.

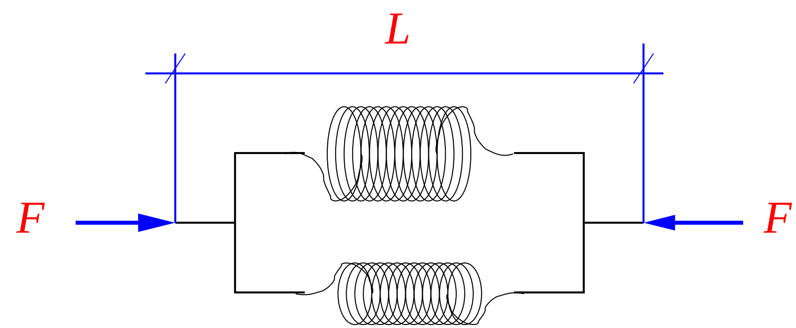
Outline

- The emergence of latent variables in physical models
  - Springs in series and in parallel
  - A mechanical systems
  - An RLC circuit
- The elimination theorem
- Modeling of RLC circuits using MNA (modified nodal analysis)

Springs in series and in parallel

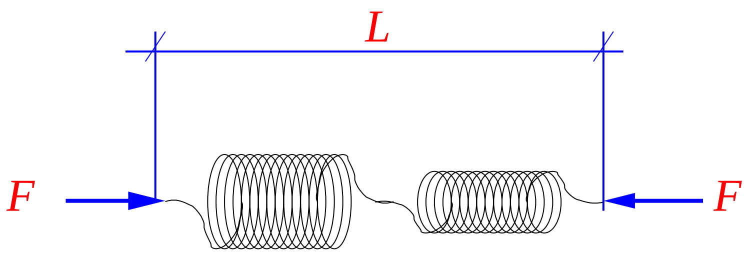
Interconnected springs

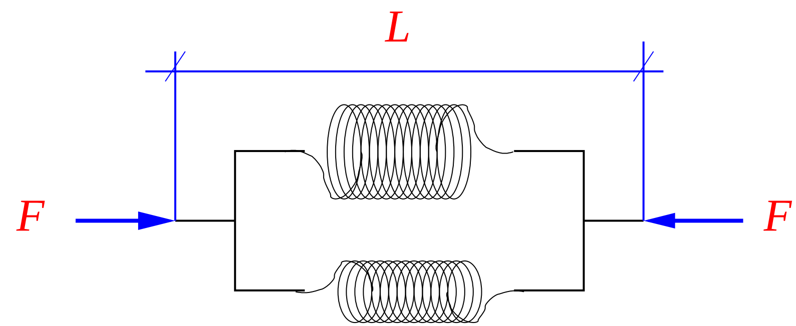




!! Model the relation between L and F !!

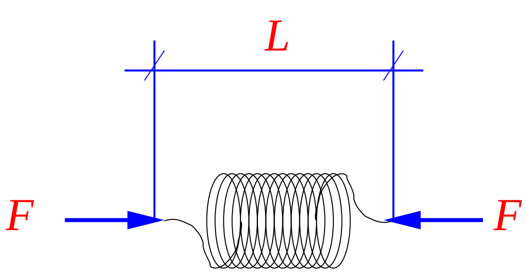
Interconnected springs

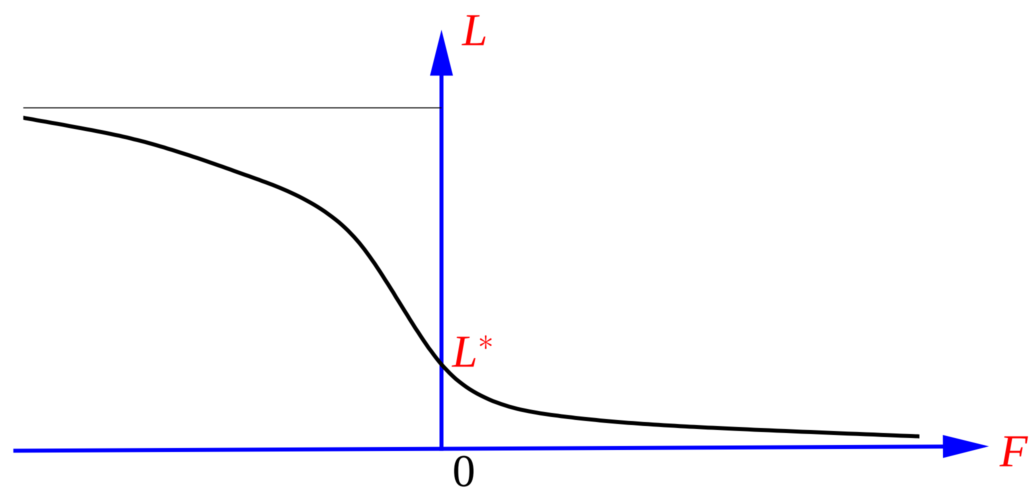




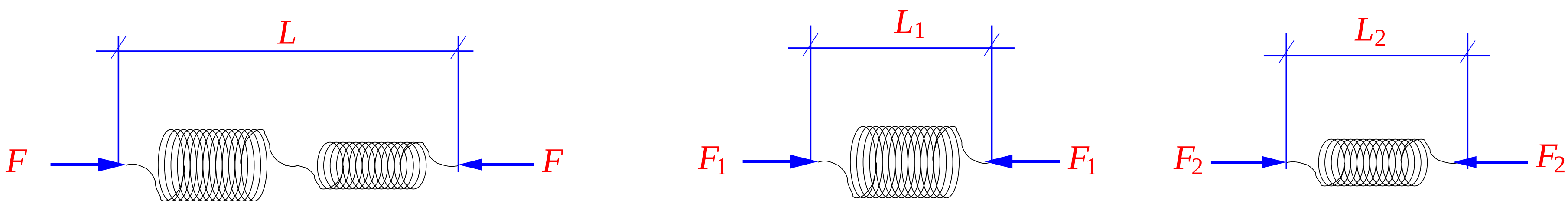
!! Model the relation between L and F !!

Typical force/length characteristic for a simple spring.





Springs in series



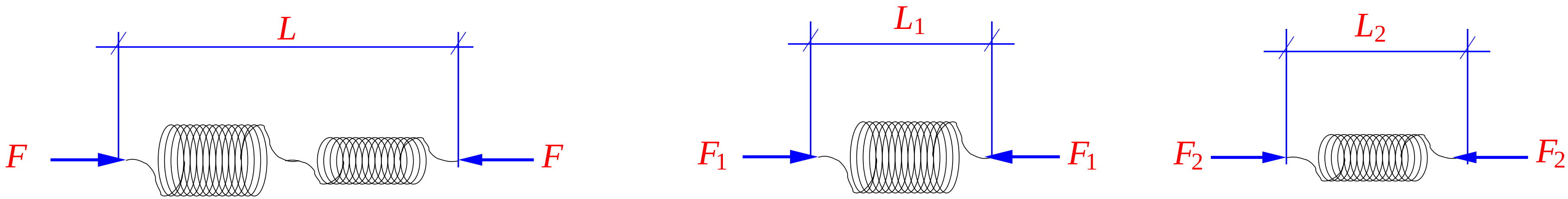
Model for (L, F) (assume that for the individual springs the length is a function of the force):

$$L_1 = \rho_1(F_1), \qquad L_2 = \rho_2(F_2),$$

$$F = F_1 = F_2, \qquad L = L_1 + L_2.$$

(L, F)**: ‘manifest’**, (L_1, F_1, L_2, F_2)**: ‘latent’ variables.**

Springs in series



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(L,F): **'manifest'**, (L_1,F_1,L_2,F_2): **'latent' variables.**

After elimination of the latent variables: $\boxed{L = \rho_1(F) + \rho_2(F)}$**.**

Latent variables are easily eliminated in this case.

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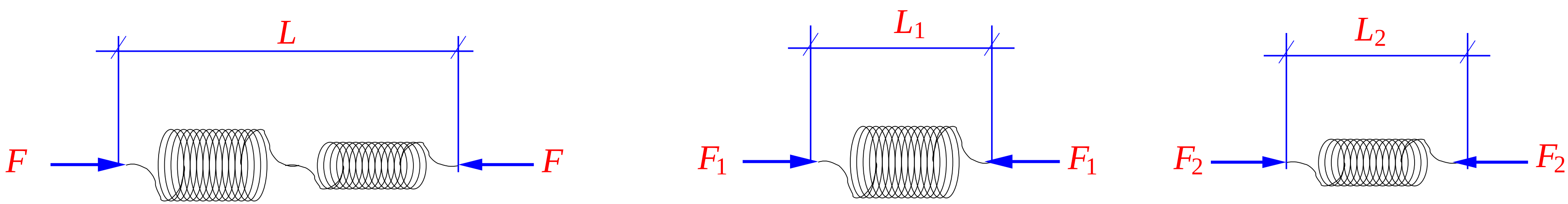

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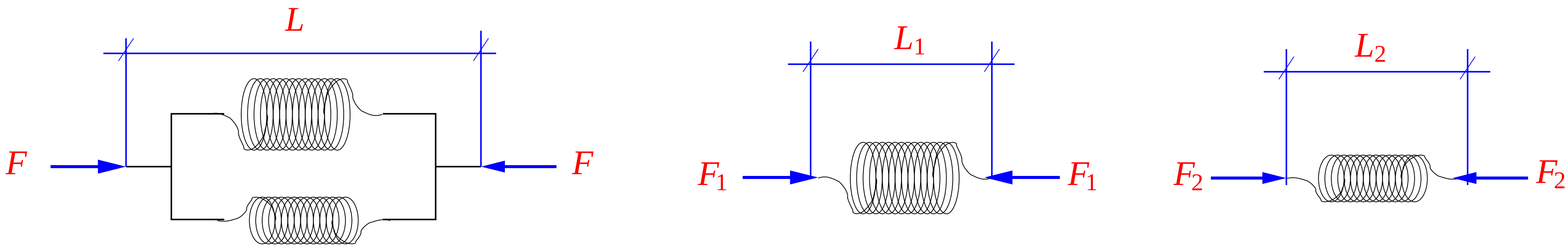
Latent variables are easily eliminated in this case.

Linear springs: $L_1 = L_1^* + \rho_1 F_1, L_2 = L_2^* + \rho_2 F_2,$

$$\rightsquigarrow \qquad L = L_1^* + L_2^* + (\rho_1 + \rho_2)F.$$

Springs in parallel



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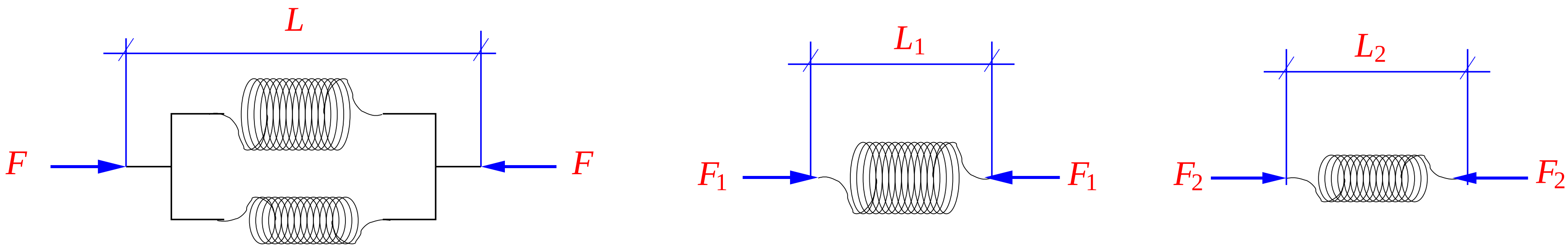

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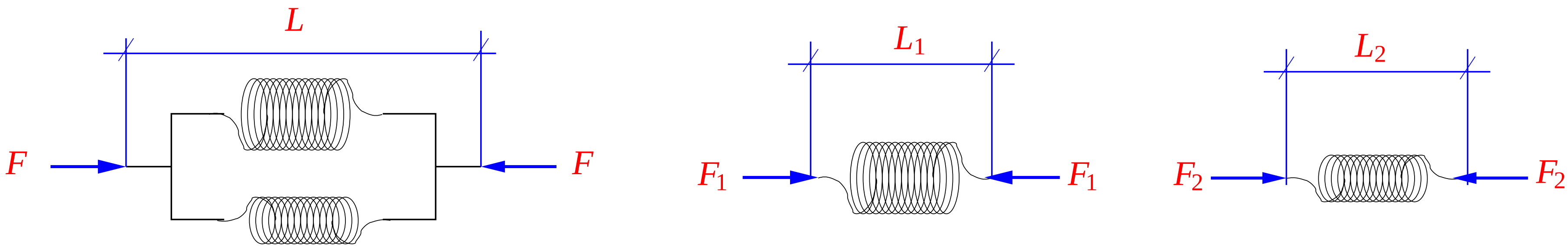
After elimination of the latent variables:

$$\boxed{\mathscr{B} = \{(L, F) \mid \exists\, \alpha : L = \rho_1(\alpha) = \rho_2(F - \alpha)\}}.$$

Latent variables are not easily eliminated in this case.

Springs in parallel



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Latent variables are not easily eliminated in this case.
Linear springs: $L_1 = L_1^* + \rho_1 F_1, L_2 = L_2^* + \rho_2 F_2,$

$$\rightsquigarrow \qquad L = \frac{\rho_2}{\rho_1+\rho_2} L_1^* + \frac{\rho_1}{\rho_1+\rho_2} L_2^* + \frac{\rho_1\rho_2}{\rho_1+\rho_2} F.$$

What springs teach us

- **First principles models invariably contain latent variables, in addition the manifest variables the model aims at.**
- **It may be impossible to eliminate latent variables, even for simple models.**
- **Be careful about claiming what variable 'causes' what. For a simple spring we may think of the force as causing the length, but this situation is already not robust under parallel connection of two such springs.**

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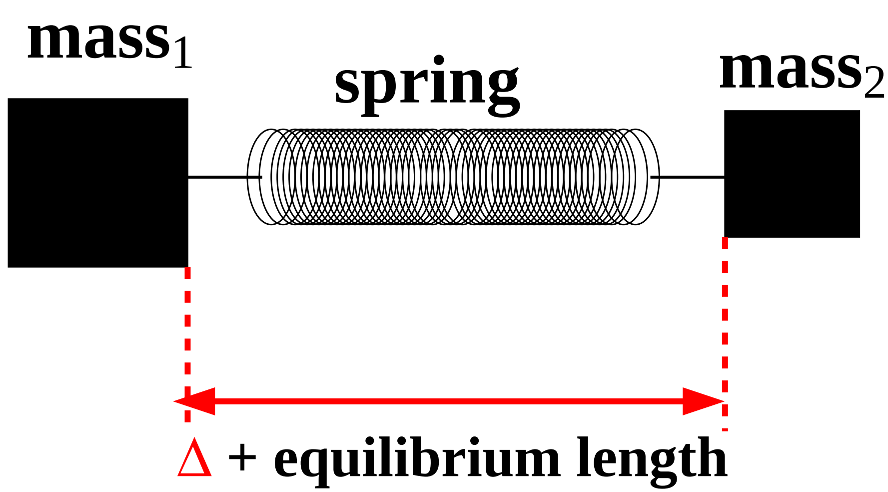
We now illustrate the emergence and elimination of latent variables for a dynamical system.

A mass-spring system

Two masses connected by a spring



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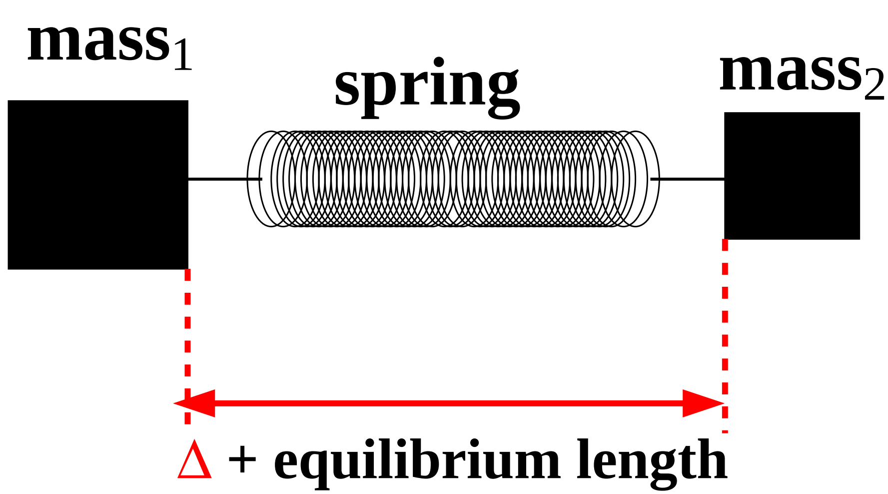

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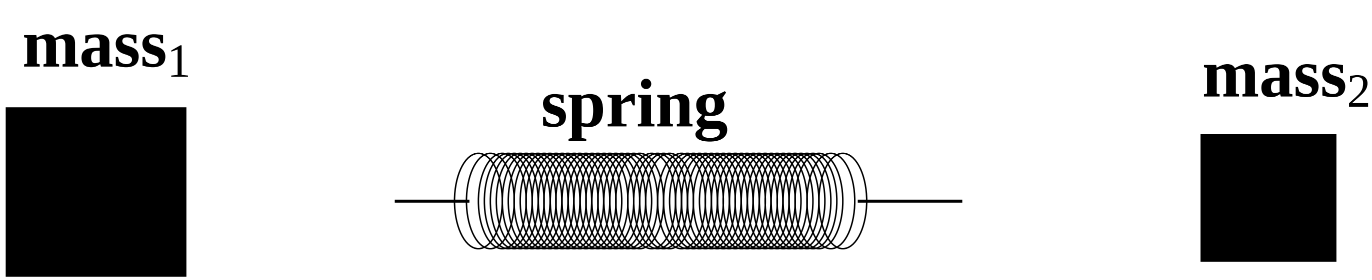
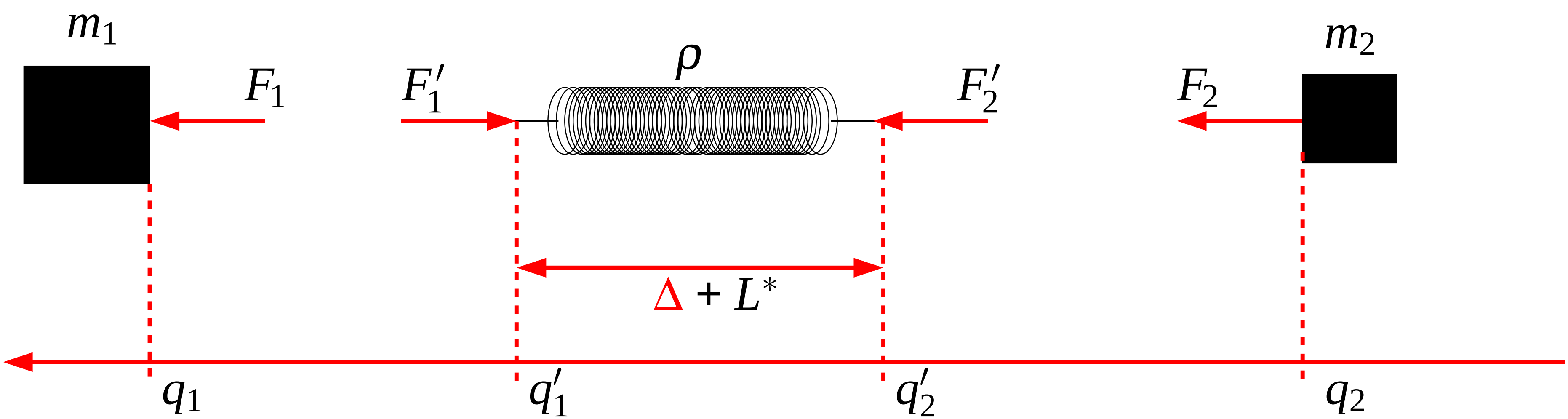
View as interconnection of 3 systems.



Behavioral equations

Now interconnect:



Constitutive equations:

$$m_1 \frac{d^2}{dt^2} q_1 = F_1, \quad m_2 \frac{d^2}{dt^2} q_2 = F_2, \quad q_1' - q_2' = L^* - \rho F_1', \quad F_1' = F_2',$$

with m_1 and m_2 the masses, ρ the elasticity coefficient of the spring, and L^* is equilibrium length.
Assume that the spring operates in its linear regime.

Behavioral equations

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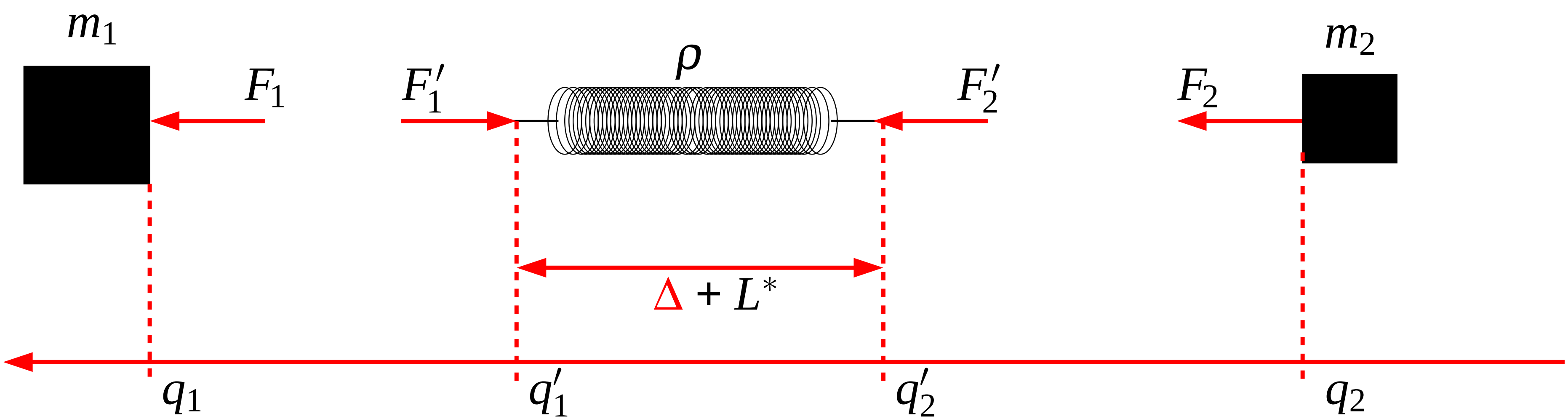

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Interconnection equations:

$$F_1 = F_1', \;\; F_2 + F_2' = 0, \;\; q_1 = q_1' \;\; q_2 = q_2'.$$

Behavioral equations

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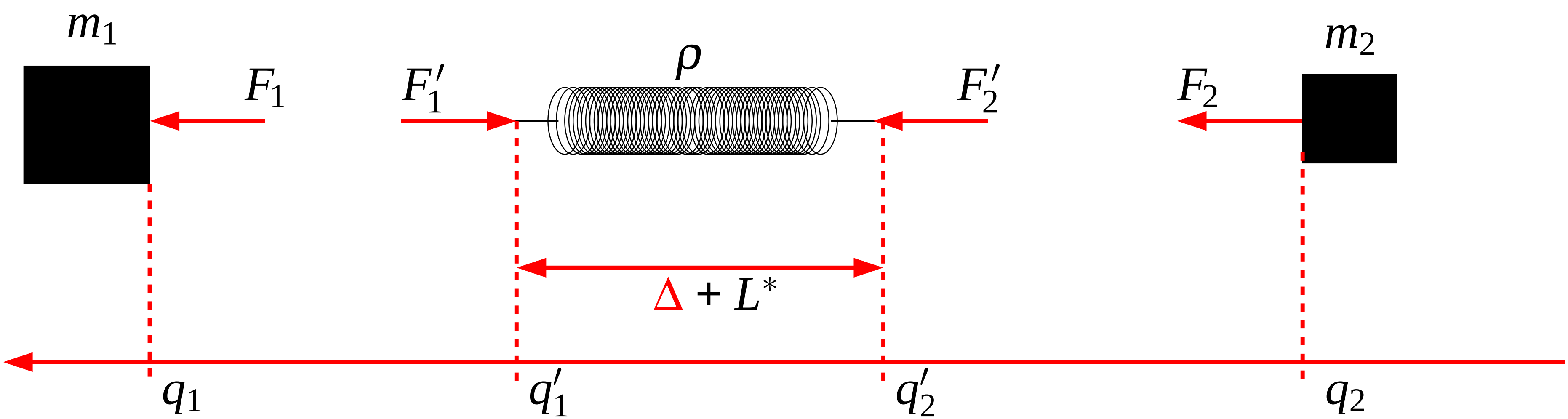

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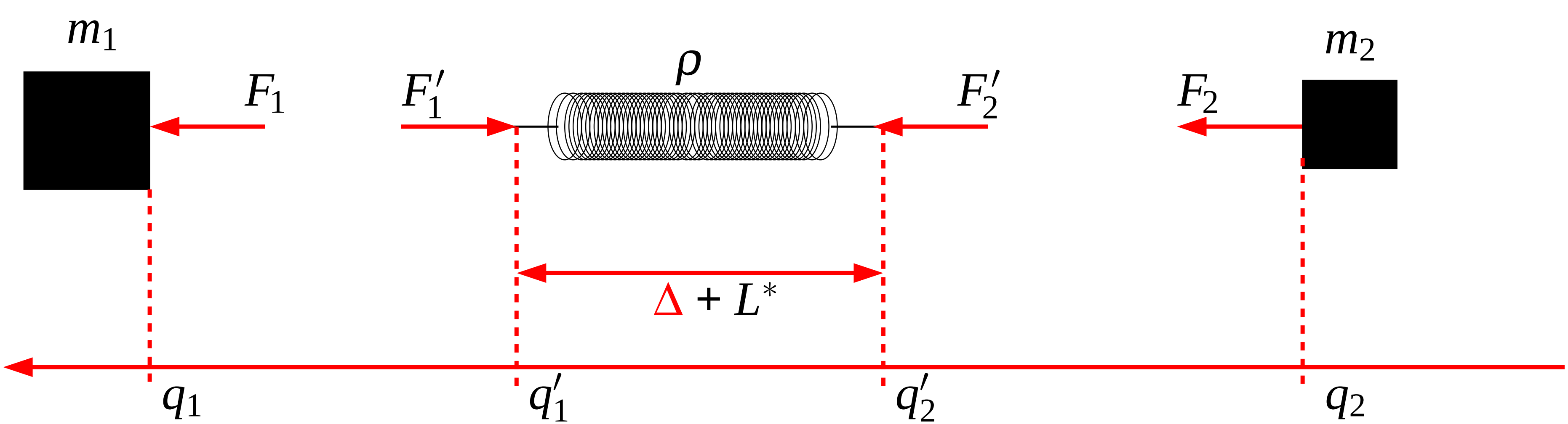
Interconnection equations:

$$F_1 = F_1', \quad F_2 + F_2' = 0, \quad q_1 = q_1' \quad q_2 = q_2'.$$

Manifest variable:

$$\Delta = q_1 - q_2 - L^*.$$

Manifest behavior

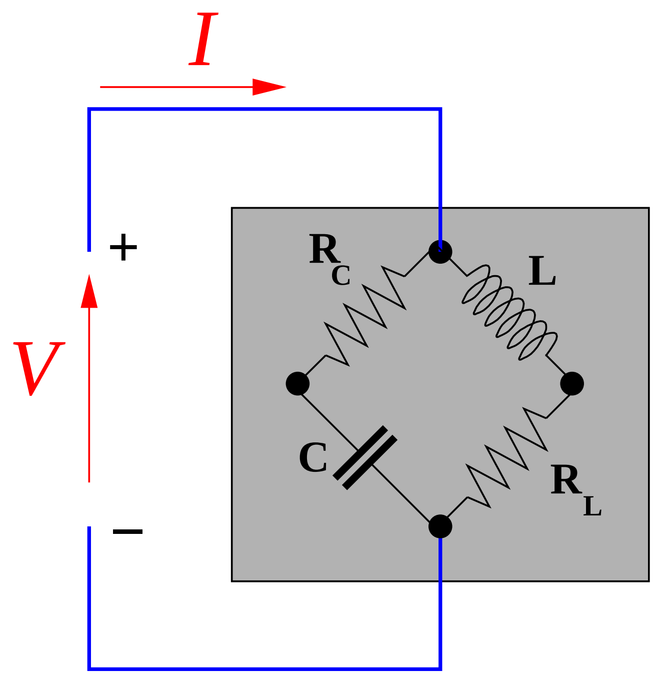


After elimination of the latent variables $F_1, F_2, F_1', F_2', q_1, q_2, q_1', q_2'$, the following equation is obtained for the manifest variable Δ

$$\frac{m_1 m_2}{m_1 + m_2} \frac{d^2}{dt^2}\Delta + \frac{1}{\rho}\Delta = 0.$$

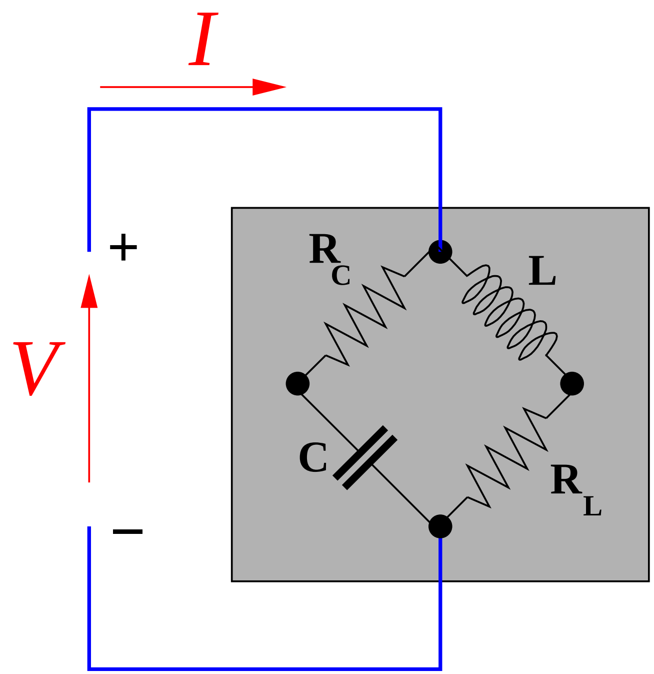
An RLC circuit

RLC circuit



Model the port behavior of this circuit!

RLC circuit



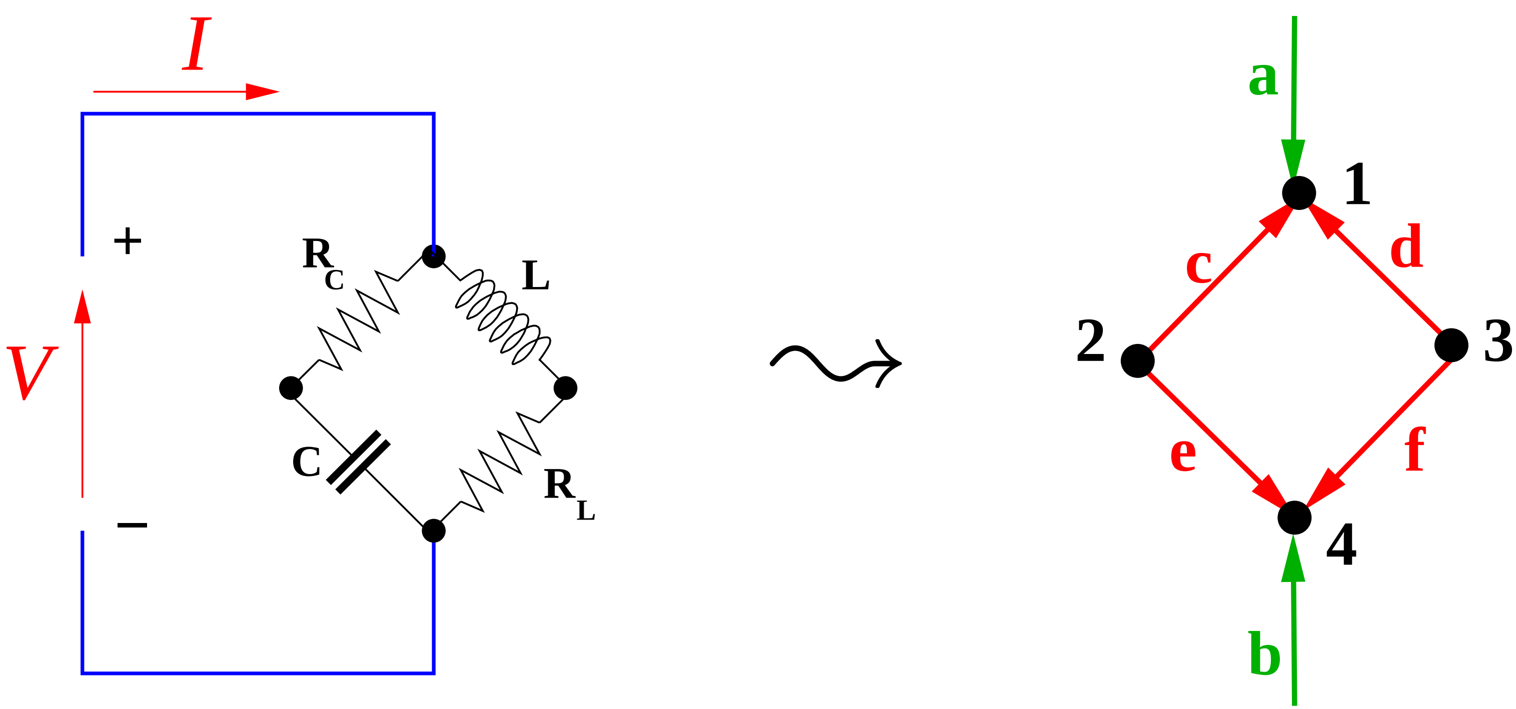
Model the port behavior of this circuit!

Manifest variables: V, the port voltage, and I, the port current.

$$\mathbb{T} = \mathbb{R}, \mathbb{W} = \mathbb{R}^2, \; w = \begin{bmatrix} V \\ I \end{bmatrix}.$$

Choice of latent variables

To model this circuit, we use nodal analysis.
Associate a digraph with the circuit:



Latent variables: potentials of vertices, currents in edges:

$$(E_1, E_2, E_3, E_4), (I_a, I_b, I_c, I_d, I_e, I_f).$$

Behavioral equations

KCL:

$$
\begin{aligned}
\textbf{vertex 1:} &\quad I_a + I_c + I_d = 0, \\
\textbf{vertex 2:} &\quad I_c + I_e = 0, \\
\textbf{vertex 3:} &\quad I_d + I_f = 0, \\
\textbf{vertex 4:} &\quad I_b + I_e + I_f = 0.
\end{aligned}
$$

Behavioral equations

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vertex 1: $I_a + I_c + I_d = 0,$

vertex 2: $I_c + I_e = 0,$

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Constitutive equations:

edge c: $E_2 - E_1 = R_C I_c,$

edge d: $E_3 - E_1 = L \frac{d}{dt} I_d,$

edge e: $C \frac{d}{dt}(E_2 - E_4) = I_e,$

edge f: $E_3 - E_4 = R_L I_f.$

Behavioral equations

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Manifest variables:

port voltage: $V = E_1 - E_4,$

port current: $I = I_a.$

Behavioral equations

In total, 10 latent variables: $(E_1, E_2, E_3, E_4, I_a, I_b, I_c, I_d, I_e, I_f)$**,**

2 manifest variables: (V, I),

and 10 equations.

Behavioral equations

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2 manifest variables: (V, I),
and 10 equations.

Which equations govern (V, I) **?**

A straightforward calculation (left as an exercise) leads to the following answer.

The port behavior

The port behavior is described by the following ODE:

Case 1: $CR_C \neq \frac{L}{R_L}$

$$\left(\frac{R_C}{R_L} + \left(1 + \frac{R_C}{R_L}\right) CR_C \frac{d}{dt} + CR_C \frac{L}{R_L} \frac{d^2}{dt^2}\right) V = \left(1 + CR_C \frac{d}{dt}\right)\left(1 + \frac{L}{R_L}\frac{d}{dt}\right) R_C\, I$$

Case 2: $CR_C = \frac{L}{R_L}$

$$\left(\frac{R_C}{R_L} + CR_C \frac{d}{dt}\right) V = \left(1 + CR_C \frac{d}{dt}\right) R_C\, I$$

The port behavior

- **The behavioral equations after elimination tell *exactly* what the port behavior is.**
  There are no hidden assumptions.
- **Next, we prove that complete elimination of the latent variables is always possible in the class of linear constant coefficient differential equations.**
  It is a theorem!
  The RLC circuit illustrates this in a particular example.
- **The different cases show that elimination is not a trivial matter. The order of the differential equation may chance with the element values, etc.**

Representations of behaviors

Kernels and projections

A model $\mathscr{B}$ is a subset of a universum $\mathscr{U}$.
There are many ways to specify a subset. For example,

- **as the set of solutions of equations,**
- **as a projection.**

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- **as the set of solutions of equations:**

$$f : \mathscr{U} \to \bullet, \qquad \mathscr{B} = \{w \in \mathscr{U} \mid f(w) = 0\},$$

- **as a projection:**

$$\mathscr{B}_{\text{extended}} \subseteq \mathscr{U} \times \mathscr{L},$$

$$\mathscr{B} = \{w \in \mathscr{U} \mid \exists\, \ell \in \mathscr{L} \textbf{ such that } (w,\ell) \in \mathscr{B}_{\text{extended}}\}.$$

Kernels and projections

A model $\mathscr{B}$ is a subset of a universum $\mathscr{U}$.
There are many ways to specify a subset. For example,

- **as solutions of equations:** **kernel representation**

$$f : \mathscr{U} \to \bullet, \qquad \mathscr{B} = \{ w \in \mathscr{U} \mid f(w) = 0 \},$$

- **as a projection:** **latent variable representation**

$$\mathscr{B} = \{ w \in \mathscr{U} \mid \exists\, \ell \in \mathscr{L} \text{ such that } (w, \ell) \in \mathscr{B}_{\text{extended}} \},$$

w's 'manifest' variables: the variables the model aims at,
ℓ's 'latent' variables: auxiliary variables.

Projection

- **as a projection:** **latent variable representation**

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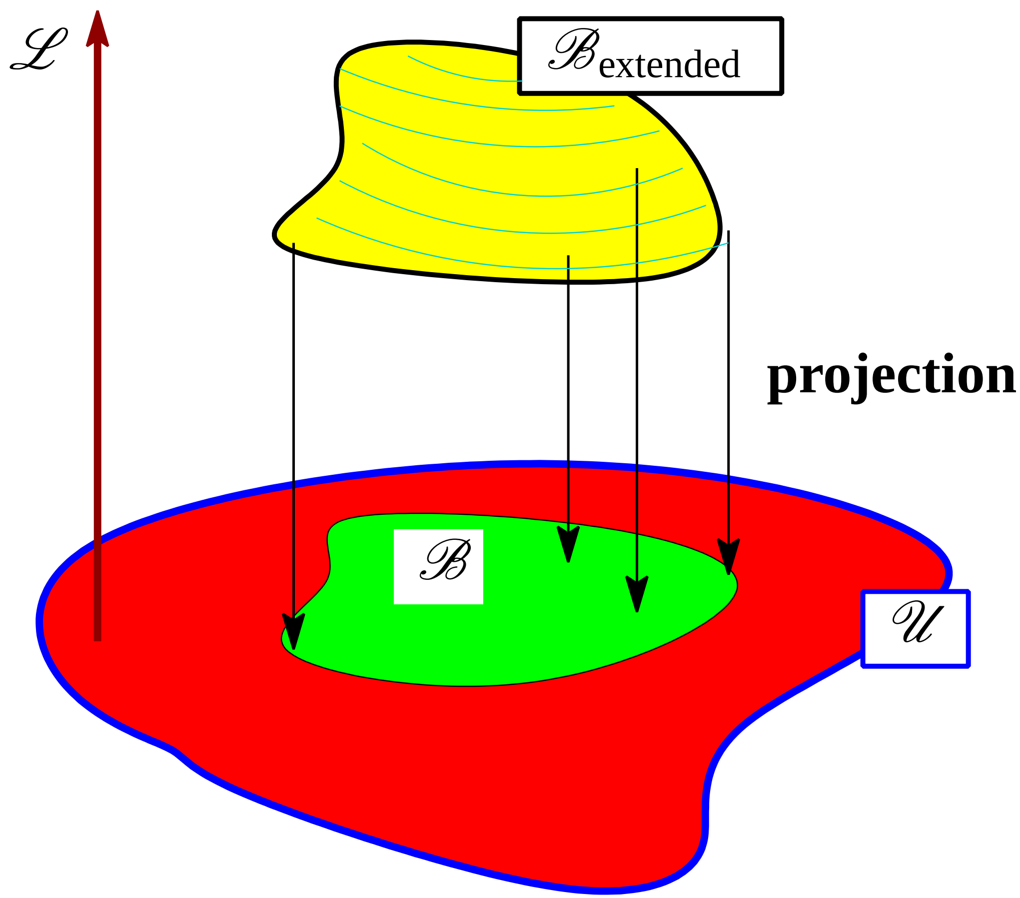

The elimination theorem

Elimination problem

Assume that the (equations specifying the) extended behavior $\mathscr{B}_{\text{extended}}$ has a certain structure.

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We have illustrated the emergence of latent variables, and their elimination in a few examples.

Examples

- $\mathscr{B}_{\text{extended}}$ **open** $\Rightarrow$ $\mathscr{B}$ **open.**
  $\mathscr{B}_{\text{extended}}$ **closed** $\not\Rightarrow$ $\mathscr{B}$ **closed.**
  $\mathscr{B}_{\text{extended}}$ **algebraic variety** $\not\Rightarrow$ $\mathscr{B}$ **algebraic variety.**
  $\mathscr{B}_{\text{extended}}$ **semi-algebraic** $\Rightarrow$ $\mathscr{B}$ **semi-algebraic.**
  $\mathscr{B}_{\text{extended}}$ **polyhedron** $\Rightarrow$ $\mathscr{B}$ **polyhedron.**

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- $\mathscr{B}_{\text{extended}}$ **differential** $\Rightarrow$ $\mathscr{B}$ **differential ?**

- $\mathscr{B}_{\text{extended}}$ **LTIDS** $\Rightarrow$ $\mathscr{B}$ **LTIDS ?**

Projection

Consider the dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathscr{B})$**.**

Define the projection $\Sigma_1 = (\mathbb{T}, \mathbb{W}_1, \mathscr{B}_1)$ **with**

$$\mathscr{B}_1 = \{w_1 : \mathbb{T} \to \mathbb{W}_1 \mid \exists\, w_2 : \mathbb{T} \to \mathbb{W}_2 \textbf{ such that } (w_1, w_2) \in \mathscr{B}\}.$$

In the LTIDS case, $\mathscr{B} \in \mathscr{L}^{\mathtt{w}_1+\mathtt{w}_2}$**,** $\mathscr{B} \subseteq \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1+\mathtt{w}_2})$.
Therefore, $\mathscr{B}_1 \subseteq \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1})$**.**

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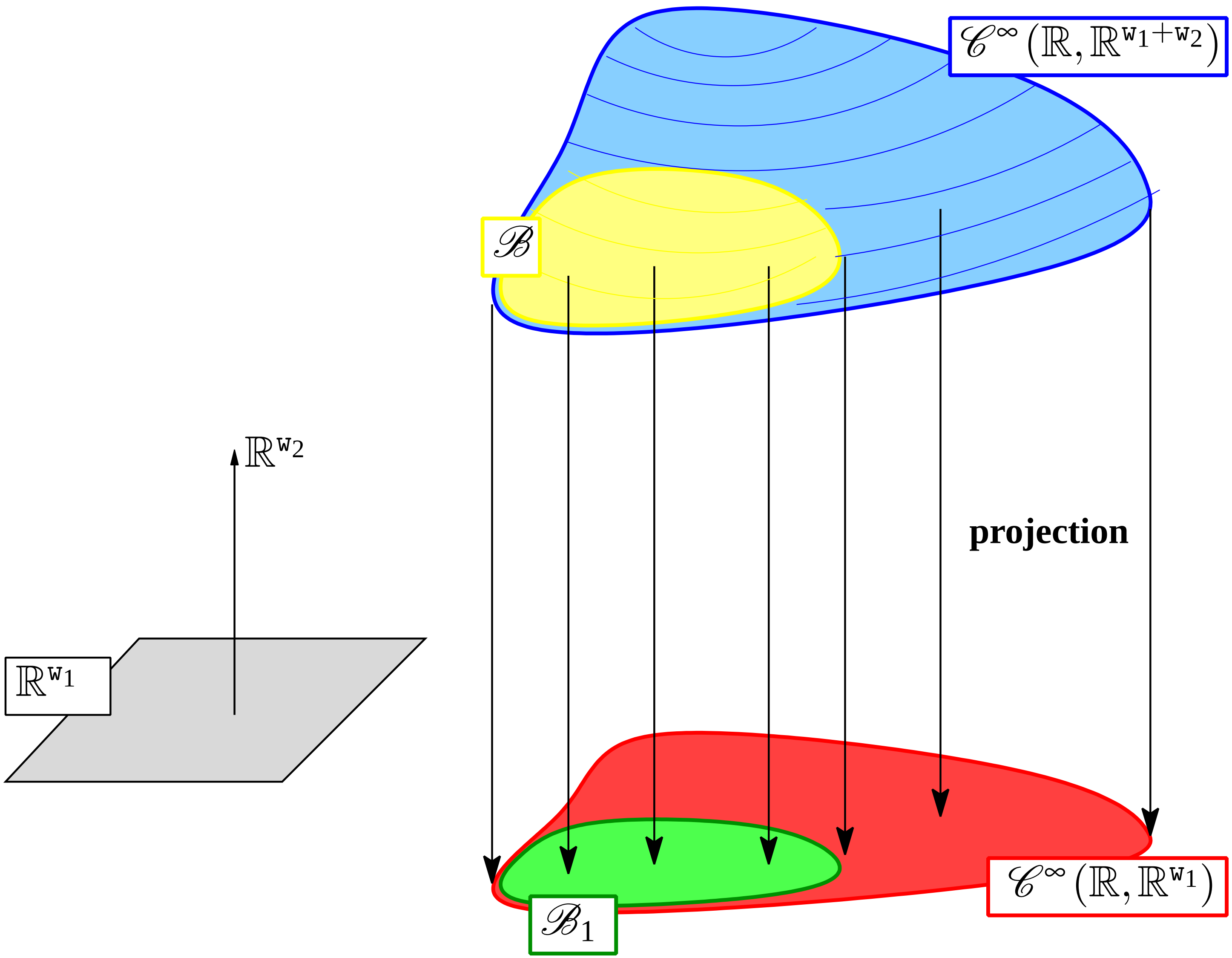
Therefore, $\mathscr{B}_1 \subseteq \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1})$**.**

The question which we consider is if, when Σ **is a LTIDS,** Σ_1 **is also a LTIDS. In other words,**

$$[\![\mathscr{B} \in \mathscr{L}^{\mathtt{w}_1+\mathtt{w}_2}]\!] \Rightarrow [\![\mathscr{B}_1 \in \mathscr{L}^{\mathtt{w}_1}]\!] \,?$$

In a picture



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Elimination theorem

Theorem

$\mathscr{L}^\bullet$ is closed under projection, that is,

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Elimination theorem

Theorem

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With

$$R_1\left(\frac{d}{dt}\right)w_1 = R_2\left(\frac{d}{dt}\right)w_2$$

a kernel representation of $\mathscr{B}$, and

$$R\left(\frac{d}{dt}\right)w_1 = 0$$

a kernel representation of $\mathscr{B}_1$, we think of this theorem as *'elimination'* of the variables w_2 from the equations.

Proof in telegram-style

- **Let** $R_1\left(\frac{d}{dt}\right)w_1 = R_2\left(\frac{d}{dt}\right)w_2$ **be a kernel representation of** $\mathscr{B}$**.**
- **Note that it can be assumed, without loss of generality, that** R_2 **is in Smith form,**

$$R_2 = \begin{bmatrix} R_2' \\ R_2'' \end{bmatrix} = \begin{bmatrix} \mathtt{diag}(d_1, d_2, \ldots, d_\mathtt{r}) & 0_{\mathtt{r}\times(\mathtt{n}_2-\mathtt{r})} \\ 0_{(\mathtt{n}_1-\mathtt{r})\times\mathtt{r}} & 0_{(\mathtt{n}_1-\mathtt{r})\times(\mathtt{n}_2-\mathtt{r})} \end{bmatrix},$$

with $d_1, d_2, , \ldots, d_\mathtt{r} \neq 0$**.**

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with $d_1, d_2, , \ldots, d_{\mathtt{r}} \neq 0$**.**

- **Observe that** $R_2'\left(\frac{d}{dt}\right)$ **is a surjective operator (see Proposition 4 of the section on differential operators).**

Proof in telegram-style

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with $d_1, d_2, , \ldots, d_{\mathtt{r}} \neq 0$**.**

- **Observe that** $R_2'\left(\frac{d}{dt}\right)$ **is a surjective operator (see Proposition 4 of the section on differential operators).**
- **Partition** $R_1 = \begin{bmatrix} R_1' \\ R_1'' \end{bmatrix}$ **conformably to** $R_2 = \begin{bmatrix} R_2' \\ R_2'' \end{bmatrix}$**.**

Then $R_1''\left(\frac{d}{dt}\right)w_1 = 0$ **is a kernel representation of** $\mathscr{B}_1$**.**

Applications of the elimination theorem

- **Elimination of state variables (x) in input/state/output systems:**

$$\frac{d}{dt}x = Ax + Bu, y = Cx + Du, \quad \rightsquigarrow \quad P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u.$$

- **Elimination of nuisance variables (x) in DAEs:**

$$E\frac{d}{dt}x = Ax + Bw \quad \rightsquigarrow \quad R\left(\frac{d}{dt}\right)w = 0.$$

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- $\mathscr{L}^\bullet$ **is closed under**

intersection, addition (see Exercise III.3), and projection .

Applications of the elimination theorem

For the RLC circuit, KCL, the constitutive equations, and the manifest variable assignment: all linear constant-coefficient differential equations — most of them algebraic equations (zero-th order), but linear constant-coefficient differential equations nevertheless.

Elimination theorem $\Rightarrow$ the latent variables (the potentials of the vertices and the currents in the edges) can be completely eliminated. $\Rightarrow$ the port behavior is described by linear constant-coefficient differential equations.

Since there are 2 real port variables, there could be 0, 1, or 2 differential equations that govern the port behavior. We derived that the behavior is described by *one* the differential equation. To prove that there is exactly one for a minimal kernel representation for the port behavior requires use of the passivity properties of the circuit elements.

Elimination algorithm

Algorithm

- **Start with** $(R_1, R_2) \in \mathbb{R}[\xi]^{\bullet\times\bullet}$**,**
  parametrizing the LTIDS $R_1\left(\frac{d}{dt}\right)w_1 = R_2\left(\frac{d}{dt}\right)w_2$**.**

  Problem: compute $R_1' \in \mathbb{R}[\xi]^{\bullet\times\bullet}$**,**
  parametrizing the projected LTIDS $R_1'\left(\frac{d}{dt}\right)w_1 = 0$,
  with w_2 **eliminated.**

Algorithm

- **Start with** $(R_1, R_2) \in \mathbb{R}[\xi]^{\bullet\times\bullet}$**,**

  Problem: compute $R_1' \in \mathbb{R}[\xi]^{\bullet\times\bullet}$**,**

- **The set**

$$\{f \in \mathbb{R}[\xi]^{1\times\texttt{rowdimension}(R_2)} \mid fR_2 = 0\}$$

is called the *left syzygy* of R_2.
It is obviously an $\mathbb{R}[\xi]$-module.

Compute a basis for the left syzygy of R_2. Let F be a matrix whose rows form a basis for this syzygy. Computing such a basis is a standard problem in computer algebra.

Algorithm

- **Start with** $(R_1, R_2) \in \mathbb{R}[\xi]^{\bullet\times\bullet}$**,**

  Problem: compute $R_1' \in \mathbb{R}[\xi]^{\bullet\times\bullet}$**,**

- **Compute a basis for the left syzygy of** R_2**. Let** F **be a matrix whose rows form a basis for this syzygy. Computing such a basis is a standard problem in computer algebra.**

- $R_1' = FR_1$**.**

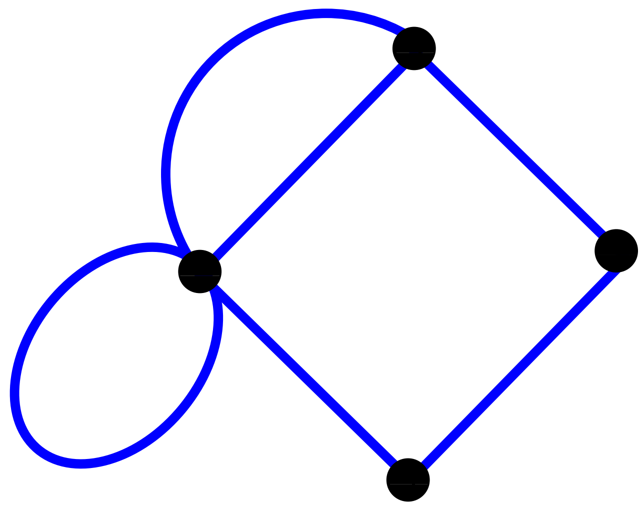
Modeling RLC circuits

Notions from graph theory

Graphs

A **graph** is one of the most useful notions from mathematics, with applications in almost every applied area.

It is instructive to think of a graph as a set of points, called **vertices,** and lines, called **edges,** connecting pairs of points.



The formal definition is a follows.

Graphs

A graph $\mathscr{G}$ is defined as

$$\mathscr{G} = (\mathbb{V}, \mathbb{E}, f)$$

with $\mathbb{V}$ the finite set of vertices, $\mathbb{E}$ the finite set of edges,

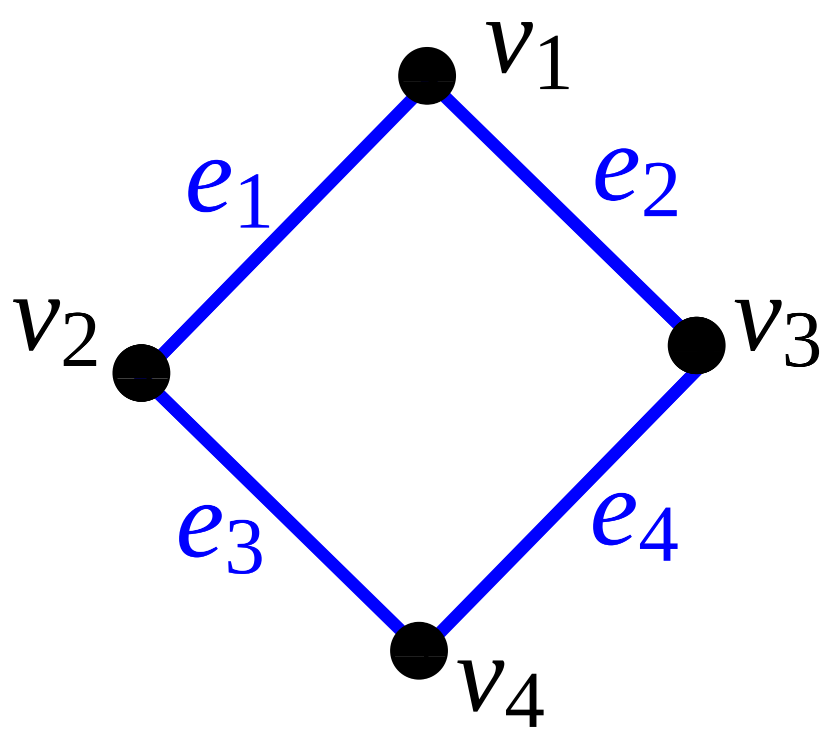
f the incidence map;

f maps each element $e \in \mathbb{E}$ into an
unordered pair $f(e) = [v_1, v_2]$ with $v_1, v_2 \in \mathbb{V}$.

If $f(e) = [v_1, v_2]$, then we call v_1 and v_2 *incident* to $e \in \mathbb{E}$.

Notation: $\{a, b\} =$ the set with elements a and b;
$\{a, b\} = \{b, a\}$ and, if $a = b$, $\{a, b\} = \{a\}$.
$(a, b) =$ the ordered pair of elements a and b;
$(a, b) \neq (b, a)$ unless $a = b$.
$[a, b] =$ the unordered pair of elements a, b; $[a, b] = [b, a]$.

Example



$\mathbb{V} = \{v_1, v_2, v_3, v_4\}$,
$\mathbb{E} = \{e_1, e_2, e_3, e_4\}$,
$f : e_1 \mapsto [v_1, v_2], e_2 \mapsto [v_1, v_3], e_3 \mapsto [v_2, v_4], e_4 \mapsto [v_3, v_4]$.

Incidence matrix

The edge e is called a *self-loop* if $f(e) = [v, v]$.

A convenient way of specifying a graph without self-loops in mathematical notation is by its incidence matrix.

The incidence matrix is a matrix of 0's and 1's
having $|\mathbb{V}|$ rows and $|\mathbb{E}|$ columns,
with $(\mathtt{k}, \ell)$-th element
$= 1$ if the ℓ-th edge is incident to the $\mathtt{k}$-th vertex,
$= 0$ otherwise.

<u>Notation</u>: $|\mathbb{S}| =$ the *cardinality* of the set $\mathbb{S}$.
If $\mathbb{S}$ is finite, then the cardinality = the number of elements.

Incidence matrix

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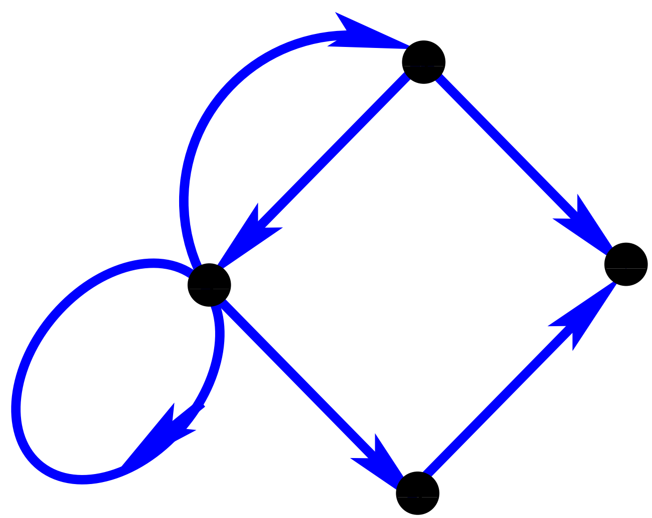
with $(\mathtt{k}, \ell)$-th element
$= 1$ if the ℓ-th edge is incident to the $\mathtt{k}$-th vertex,
$= 0$ otherwise.

For our example, the incidence matrix equals

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Digraphs

A directed graph is a graph in which each edge is assigned a direction. Think of a digraph as a set of points and lines with arrows pointing from one edge to another.



The formal definition is as follows.

Digraphs

A directed graph, or digraph, $\mathscr{G}$ is defined as

$$\mathscr{G} = (\mathbb{V}, \mathbb{E}, f)$$

with $\mathbb{V}$ the finite set of vertices, $\mathbb{E}$ the finite set of edges,

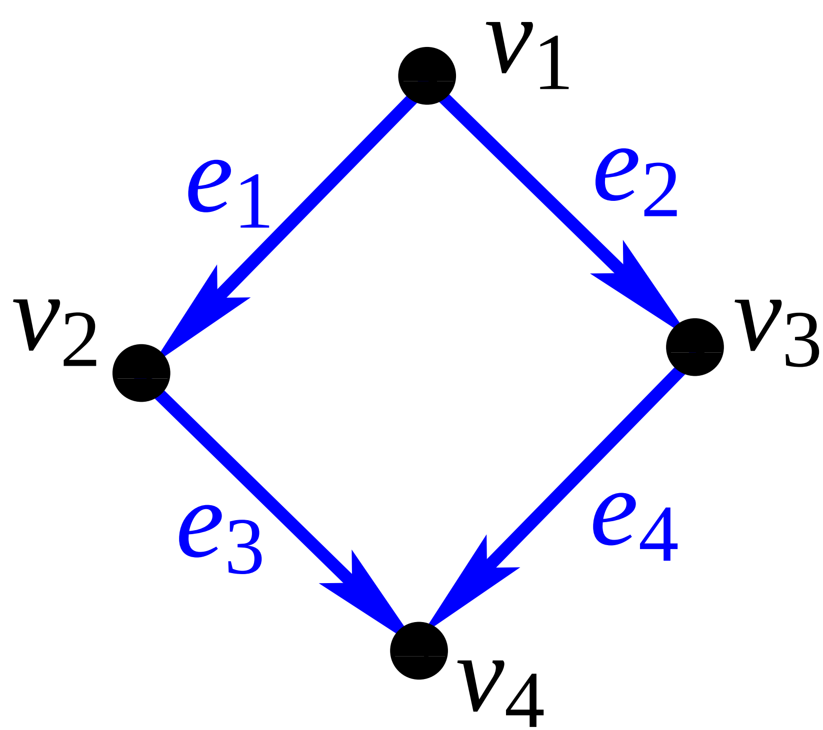
f the incidence map;

f maps each element $e \in \mathbb{E}$ into
an ordered pair $f(e) = (v_1, v_2)$ with $v_1, v_2 \in \mathbb{V}$.

If $f(e) = (v_1, v_2)$, then we call v_1 and v_2 *incident* to $e \in \mathbb{E}$.
v_1 is the *source* of e and v_2 is the *sink* of e.
We think of e as being directed from v_1 to v_2.

The edge e is called a *self-loop* if $f(e) = (v, v)$.

Example



$\mathbb{V} = \{v_1, v_2, v_3, v_4\}$,
$\mathbb{E} = \{e_1, e_2, e_3, e_4\}$,
$f : e_1 \mapsto (v_1, v_2), e_2 \mapsto (v_1, v_3), e_3 \mapsto (v_2, v_4), e_4 \mapsto (v_3, v_4)$.

Incidence matrix

A convenient way of specifying a digraph without self-loops is by its incidence matrix.

The incidence matrix is a matrix of 0's, $+1$'s, and -1's having $|\mathbb{V}|$ rows and $|\mathbb{E}|$ columns, with $(\mathtt{k}, \ell)$-th element

$= +1$ **if the $\mathtt{k}$-th vertex is the source for the ℓ-th edge,**
$= -1$ **if the $\mathtt{k}$-th vertex is the sink for the ℓ-th edge,**
$= \ 0$ **otherwise.**

<u>Caveat</u>:
Sometimes the opposite convention for $+1$ and -1 is used!

Incidence matrix

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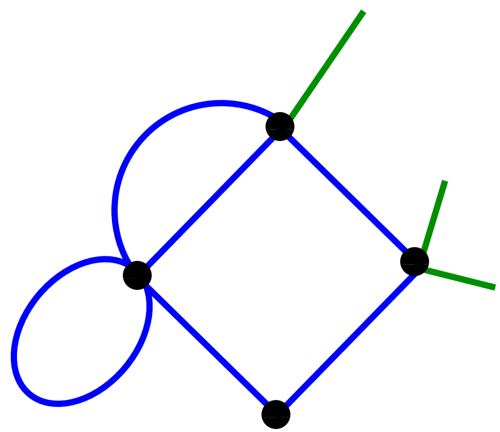
$= \ 0$ **otherwise.**

For our example, the incidence matrix equals

$$\begin{bmatrix} +1 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & -1 \end{bmatrix}.$$

Graph with leaves

A graph with leaves is like an ordinary graph except that some of the edges are incident to only one vertex. Think of a graph with leaves as a set of points, called vertices, lines, called edges, connecting pairs of points, and leaves, that connect to one point only.



The formal definition is a follows.

Graph with leaves

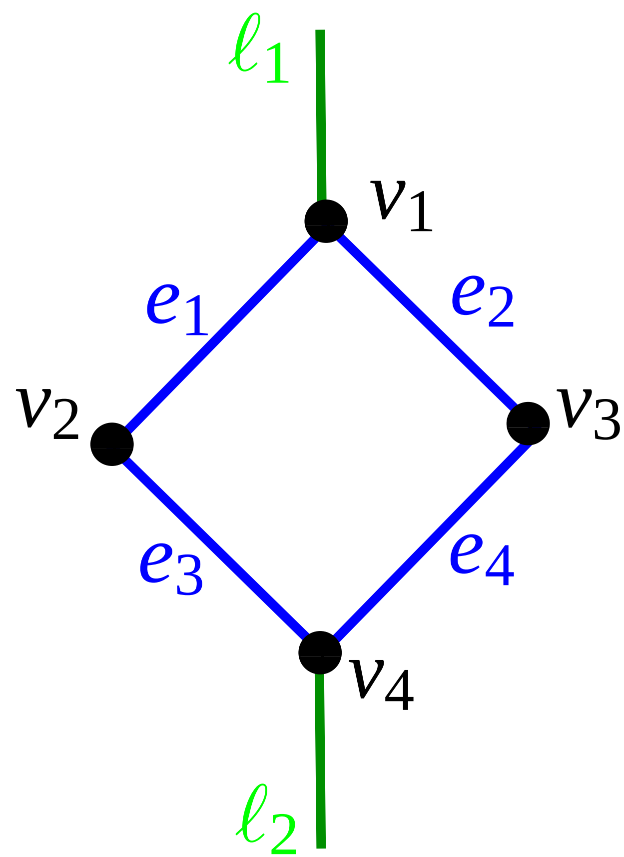
A graph with leaves $\mathscr{G}$ is defined as

$$\mathscr{G} = (\mathbb{V}, \mathbb{E}, \mathbb{L}, f_{\mathbb{E}}, f_{\mathbb{L}})$$

with $\mathbb{V}, \mathbb{E}$, and $f_{\mathbb{E}}$, the edge incidence map, defined as for graphs,
$\mathbb{L}$ the finite set of edges,
$f_{\mathbb{L}}$, the edge incidence map, maps each element $\ell \in \mathbb{L}$ into an element $f_{\mathbb{L}}(\ell) \in \mathbb{V}$.

If $f_{\mathbb{L}}(\ell) = v$, then we call $\ell \in \mathbb{L}$ *incident* to $e \in \mathbb{E}$.

Example



$$\mathbb{V} = \{v_1, v_2, v_3, v_4\}, \mathbb{E} = \{e_1, e_2, e_3, e_4\}, \mathbb{L} = \{\ell_1, \ell_2\},$$
$$f_{\mathbb{E}} : e_1 \mapsto (v_1, v_2), e_2 \mapsto (v_1, v_3), e_3 \mapsto (v_2, v_4), e_4 \mapsto (v_3, v_4),$$
$$f_{\mathbb{L}} : \ell_1 \mapsto v_1, \ell_2 \mapsto v_4.$$

Incidence matrices

A convenient way of specifying a graph with leaves without self-loops is by its incidence matrices.

The edge incidence matrix $\mathbb{A}_{\mathbb{E}}$ **is defined as for graphs,**

the leaf incidence matrix $\mathbb{A}_{\mathbb{L}}$ **is a matrix of** 0**'s and** 1**'s having** $|\mathbb{V}|$ **rows and** $|\mathbb{L}|$ **columns, with** $(\mathtt{k}, \ell)$**-th element**

$= 1$ **if the** ℓ**-th leaf is incident to the** $\mathtt{k}$**-th vertex,**

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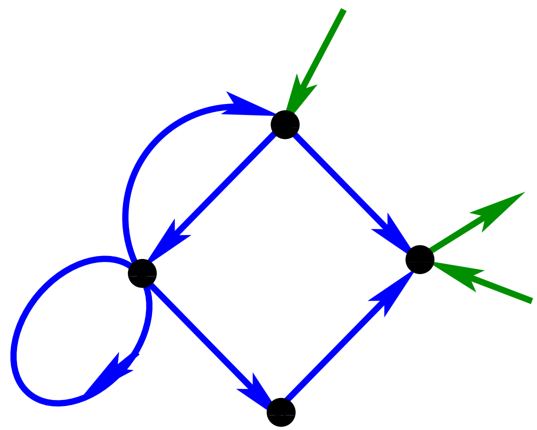
For our example, the incidence matrices equal

$$\mathbb{A}_{\mathbb{E}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbb{A}_{\mathbb{L}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Digraph with leaves

A **digraph with leaves** is like an ordinary graph with leaves except that each edge and leaf is assigned a direction.



The formal definition is a follows.

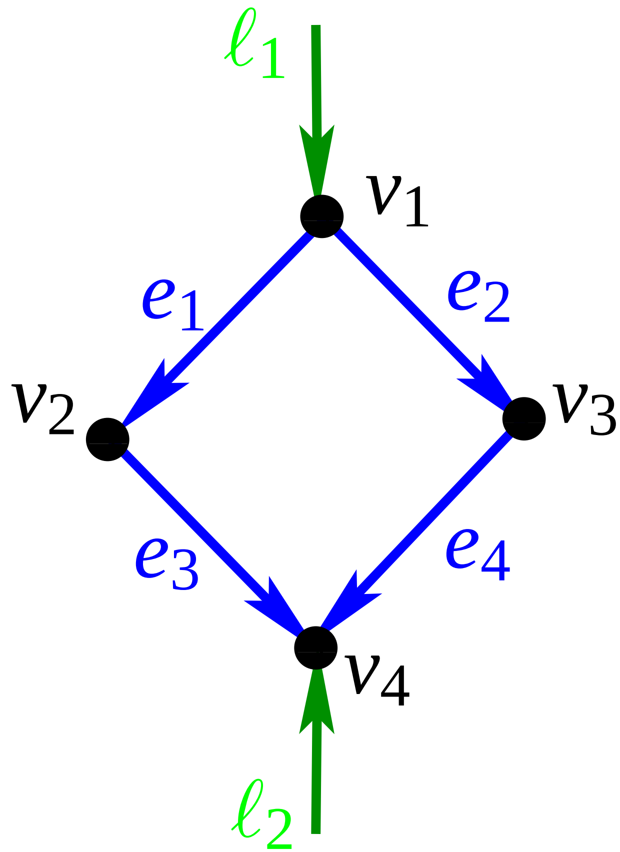
Digraph with leaves

A **digraph with leaves** $\mathscr{G}$ is defined as

$$\mathscr{G} = (\mathbb{V}, \mathbb{E}, \mathbb{L}, f_{\mathbb{E}}, f_{\mathbb{L}})$$

with $\mathbb{V}, \mathbb{E}$, and $f_{\mathbb{E}}$, the **edge incidence map,** defined as for graphs,
$f_{\mathbb{L}}$, the **edge incidence map,** maps each element $\ell \in \mathbb{L}$ into an element $f_{\mathbb{L}}(\ell) \in \mathbb{V}$, and assigns to each leaf ℓ a direction, either away from the vertex $f_{\mathbb{L}}(\ell)$, in which case the vertex is called the *source* of ℓ, or into the vertex $f_{\mathbb{L}}(\ell)$, in which case the vertex is called the *sink* of ℓ.

Example



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$f_{\mathbb{L}} : \ell_1 \mapsto v_1$ **(sink),** $\ell_2 \mapsto v_4$ **(sink).**

Incidence matrices

A convenient way of specifying a digraph with leaves without self-loops is by its incidence matrices.

The edge incidence matrix $\mathbb{A}_{\mathbb{E}}$ is defined as for digraphs,

the leaf incidence matrix $\mathbb{A}_{\mathbb{L}}$ is a matrix of 0's, $+1$'s, and -1's, having $|\mathbb{V}|$ rows and $|\mathbb{L}|$ columns, with $(\mathtt{k}, \ell)$-th element

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$= \ \ 0$ **otherwise.**

Incidence matrices

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The edge incidence matrix $\mathbb{A}_\mathbb{E}$ **is defined as for digraphs, the leaf incidence matrix** $\mathbb{A}_\mathbb{L}$ **is a matrix of** 0**'s,** $+1$**'s, and** -1**'s, having** $|\mathbb{V}|$ **rows and** $|\mathbb{L}|$ **columns, with** $(\mathtt{k},\ell)$**-th element**

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$= \ \ 0$ **otherwise.**

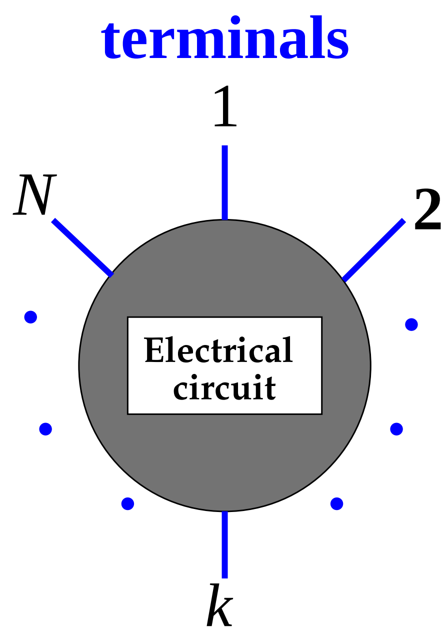
For our example, the incidence matrices equal

$$\mathbb{A}_\mathbb{E} = \begin{bmatrix} +1 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \quad \mathbb{A}_\mathbb{L} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

The interaction variables

Circuits

We view an electrical as a device, with a finite number of wires, called *terminals*, sticking out of it. The electrical circuit interacts with its environment through these terminals.



Modeling the electrical circuit means coming up with a specification of this interaction.

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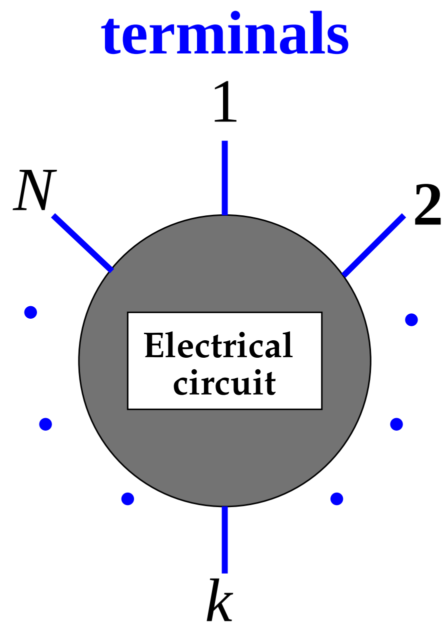
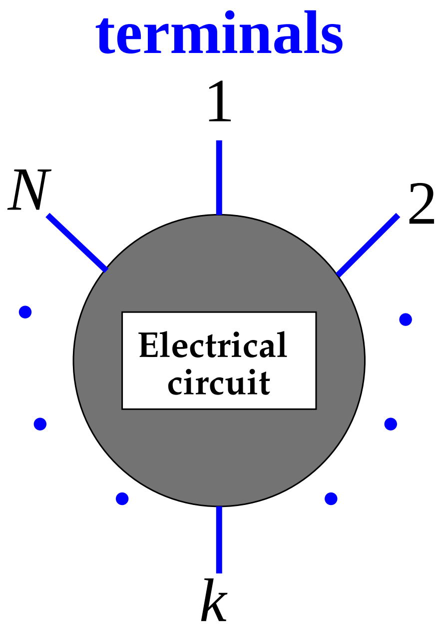


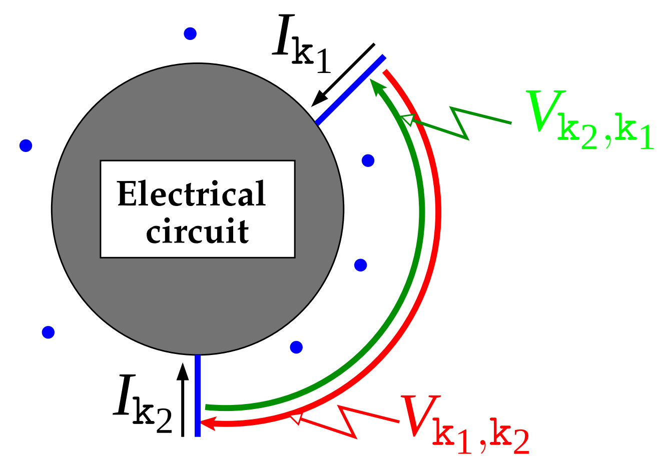
Modeling the electrical circuit means coming up with a specification of this interaction.

How do we describe this interaction?

What are the interaction variables?

Voltages and currents

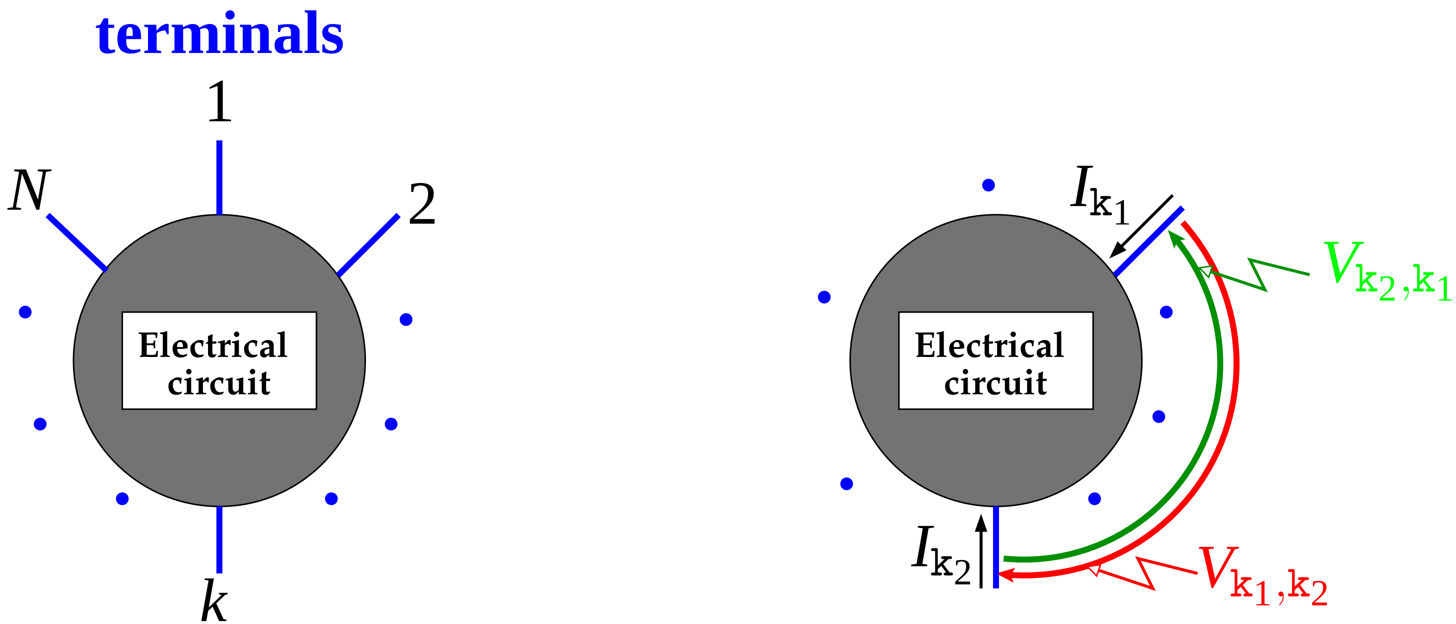




The natural choice is to take for the interaction variables

- **the currents into the circuit along the terminals,**
- **the voltages across the terminals.**

Voltages and currents



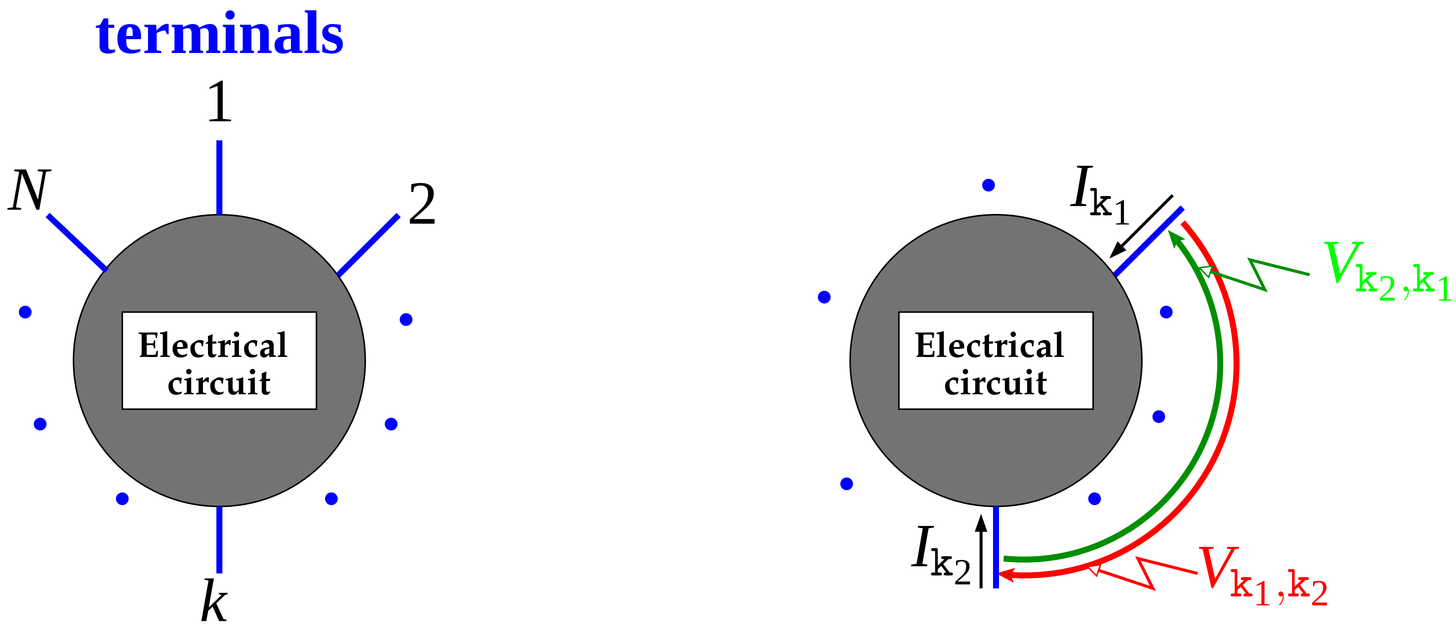
This leads to the following terminal variables

$$I_1, I_2, \ldots, I_N,$$

$$V_{1,1}, V_{1,2}, \ldots, V_{1,N}, V_{2,1}, V_{2,2}, \ldots, V_{2,N}, \ldots, V_{N,1}, V_{N,2}, \ldots, V_{N,N},$$

with I_k = the current flowing into the circuit along terminal k,
V_{k_1,k_2} = the voltage between terminal k_1 and k_2.

Voltages and currents



As sign convention we take

$I_k > 0$ **if along terminal** k **the current flows into the circuit,**
$V_{k_1,k_2} > 0$ **if the voltage drop from terminal** k_1 **to** k_2 **is positive.**

The voltage/current behavior

Organizing these variables as vectors and matrices leads to

$$I = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}, \quad V = \begin{bmatrix} V_{1,1} & V_{1,2} & \cdots & V_{1,N} \\ V_{2,1} & V_{2,2} & \cdots & V_{2,N} \\ \vdots & \vdots & \cdots & \vdots \\ V_{N,1} & V_{N,2} & \cdots & V_{N,N} \end{bmatrix}.$$

$\rightsquigarrow$ **the dynamical system** $\Sigma = \left(\mathbb{R}, \mathbb{R}^N \times \mathbb{R}^{N\times N}, \mathscr{B}_{IV}\right)$**.**

$(I,V) \in \mathscr{B}_{IV}$ **means that the trajectory of currents and voltages** $(I,V) : \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}^{N\times N}$ **is compatible with the circuit architecture and its element values.**

The aim of circuit modeling is to specify $\mathscr{B}_{IV}$**.**

The subscript in $\mathscr{B}_{IV}$ **refers to the choice of currents and voltages as the variables that describe the interaction. We will see other choices of terminal variables soon.**

Kirchhoff's laws

The behavior $\mathscr{B}_{IV}$ is said to satisfy Kirchhoff's current law (KCL) if

$$\llbracket (I,V) \in \mathscr{B}_{IV} \rrbracket \Rightarrow \llbracket I_1 + I_2 + \cdots + I_N = 0 \rrbracket,$$

Gustav Kirchhoff (1824-1887)

and Kirchhoff's voltage law (KVL) if

$$\llbracket (I,V) \in \mathscr{B}_{IV} \text{ and } \mathtt{k}_1, \mathtt{k}_2, \ldots, \mathtt{k}_\mathtt{n} \in \{1,2,\ldots,N\} \rrbracket$$

$$\Rightarrow \llbracket V_{\mathtt{k}_1,\mathtt{k}_2} + V_{\mathtt{k}_2,\mathtt{k}_3} + \cdots + V_{\mathtt{k}_{\mathtt{n}-1},\mathtt{k}_\mathtt{n}} + V_{\mathtt{k}_\mathtt{n},\mathtt{k}_1} = 0 \rrbracket.$$

KCL means that the circuit stores no net charge, while KVL means that the sum of the voltage drops across a cycle is zero.

Potentials

KVL allows to reduce the number of interaction variables greatly by introducing potentials.
The underlying idea follows from the proposition below.

Let $\mathscr{K} \subset \mathbb{R}^{N\times N}$ **and** $e \in \mathbb{R}^N$ **be defined as follows**

$$\mathscr{K} = \{M \in \mathbb{R}^{N\times N} \mid M_{\mathtt{k}_1,\mathtt{k}_2} + M_{\mathtt{k}_2,\mathtt{k}_3} + \cdots + M_{\mathtt{k}_{\mathtt{n}-1},\mathtt{k}_\mathtt{n}} + M_{\mathtt{k}_\mathtt{n},\mathtt{k}_1} = 0 \quad \textbf{for all } \mathtt{k}_1, \mathtt{k}_2, \ldots, \mathtt{k}_\mathtt{n} \in \{1, 2, \ldots, N\} \ \},$$

$$e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Potentials

KVL allows to reduce the number of interaction variables greatly by introducing potentials.
The underlying idea follows from the proposition below.

Proposition: Define the map $L : \mathbb{R}^N \rightarrow \mathbb{R}^{N\times N}$ **by**

$$L : P \mapsto Pe^\top - eP^\top.$$

There holds

$$\texttt{image}(L) = \mathscr{K} \quad \text{and} \quad \texttt{kernel}(L) = \texttt{span}(e).$$

This proposition implies that for each $M \in \mathscr{K}$**, there exists** $P \in \mathbb{R}^{\mathbb{N}}$ **such that**

$$M = Pe^\top - eP^\top, \qquad \text{i.e., } M_{\mathtt{k},\ell} = P_{\mathtt{k}} - P_\ell$$

Potentials

KVL allows to reduce the number of interaction variables greatly by introducing potentials.
The underlying idea follows from the proposition below.

Assume that $\mathscr{B}_{IV}$ **satisfies KVL. Then we can express** $V(t) : \mathbb{R} \longrightarrow \mathbb{R}^{N \times N}$ **as** $V(t) = P(t)e^\top - eP(t)^\top$ **for some** $P : \mathbb{R} \longrightarrow \mathbb{R}^N$**. Call** $P_\mathtt{k}$ **the potential of terminak** $\mathtt{k}$**. The voltages are related to the potentials by**

$$V_{\mathtt{k}_1,\mathtt{k}_2} = P_{\mathtt{k}_1} - P_{\mathtt{k}_2}.$$

Potentials

KVL allows to reduce the number of interaction variables greatly by introducing potentials.
The underlying idea follows from the proposition below.

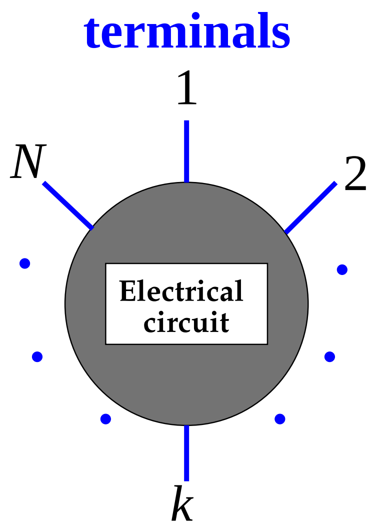
It follows that (assuming KVL) we can take for the interaction variables

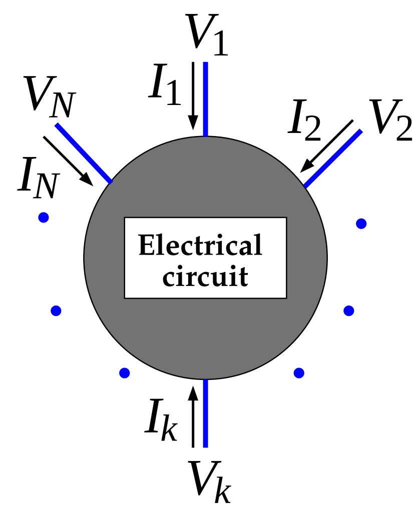
- **the currents into the circuit along the terminals,**
- **the potentials of the terminals.**

This leads to the dynamical system $\Sigma = \left(\mathbb{R}, \mathbb{R}^N \times \mathbb{R}^N, \mathscr{B}\right)$**.**
$(I,P) \in \mathscr{B}_{IV}$ **means that the trajectory of currents and potentials** $(I,P) : \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}^N$ **is compatible with the circuit architecture and its element values.**
The aim of circuit modeling is to specify $\mathscr{B}$**.**

Electrical circuit

Summarizing, we arrive at the following alternative definition of a circuit behavior. We use this definition in the sequel.





<u>At each terminal</u>:

a potential (!) and a current (counted > 0 into the circuit),

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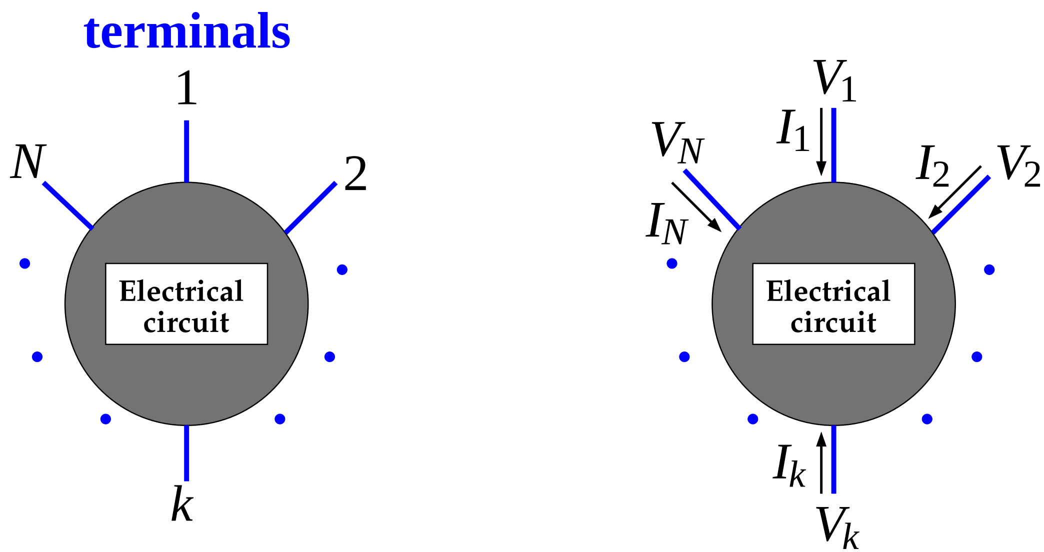

At each terminal:

a potential (!) and a current (counted > 0 into the circuit),

$$\rightsquigarrow \textbf{behavior } \mathscr{B} \subseteq \left(\mathbb{R}^N \times \mathbb{R}^N\right)^{\mathbb{R}}.$$

$(V_1, V_2, \ldots, V_N, I_1, I_2, \ldots, I_N) \in \mathscr{B}$ **means:**

this potential/current trajectory is compatible with the circuit architecture and its element values.

Kirchhoff's laws

Gustav Kirchhoff (1824-1887)

Kirchhoff's laws now take the following form.
The behavior $\mathscr{B}$ satisfies
Kirchhoff's current law (KCL) if

$$[\![(I,P) \in \mathscr{B}]\!] \Rightarrow [\![I_1 + I_2 + \cdots + I_N = 0]\!],$$

and Kirchhoff's voltage law (KVL) if

$$[\![(I,P) \in \mathscr{B} \text{ and } \alpha : \mathbb{R} \to \mathbb{R}]\!] \Rightarrow [\![(I, P + \alpha e) \in \mathscr{B}]\!].$$

KCL means that the circuit stores no net charge, while KVL means that the potentials are defined only up to an additive constant (that may change in time).

Circuit specification

Circuit architecture

We now explain how one can formally describe a circuit.

Some of the aspects of our formalization take into account that we are only interested in describing linear passive RLC circuits. The ideas are applicable to more general situations, but some details have to be adapted.

Circuit architecture

We now explain how one can formally describe a circuit.

An RLC circuit is defined through its architecture, a digraph with leaves,

$$\mathscr{G} = (\mathbb{V}, \mathbb{E}, \mathbb{L}, f_{\mathbb{E}}, f_{\mathbb{L}}) .$$

Assume that the leaves are all sinks, in the sense that they are incident towards a vertex.

The circuit elements (R's, L's, and C's) are imbedded in the edges, the vertices correspond to connectors, and the leaves correspond to the terminals by which the circuit interacts with its environment.

Circuit architecture

We now explain how one can formally describe a circuit.

An RLC circuit is defined through its architecture, a digraph with leaves,

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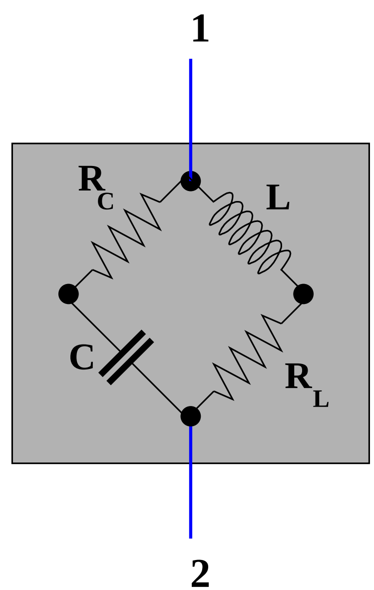
Assume that the leaves are all sinks, in the sense that they are incident towards a vertex.

and its element specification. This assigns to each vertex,

either a resistance value $R \geq 0$**,**
or an inductance value $L > 0$**,**
or a capacitance value $C > 0$**.**

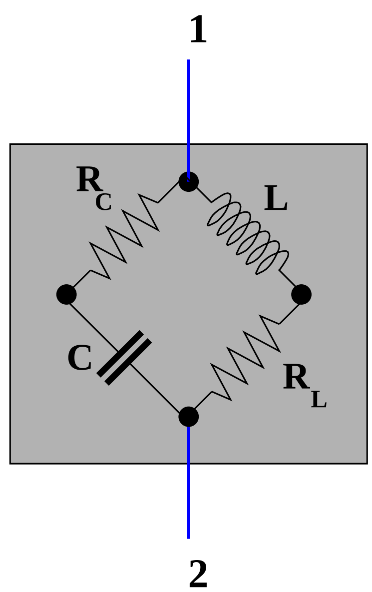
Example

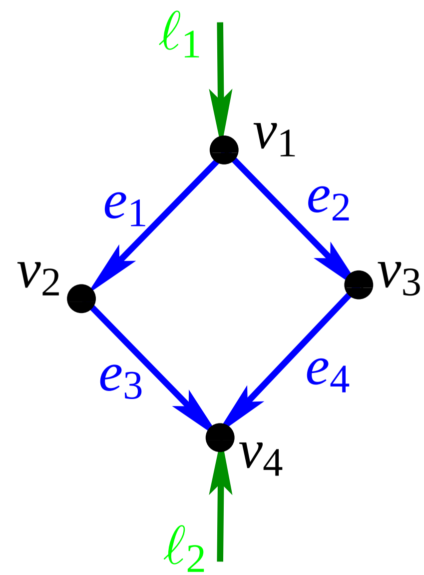
For the 2-terminal circuit



Example

For the 2-terminal circuit, the circuit architecture is





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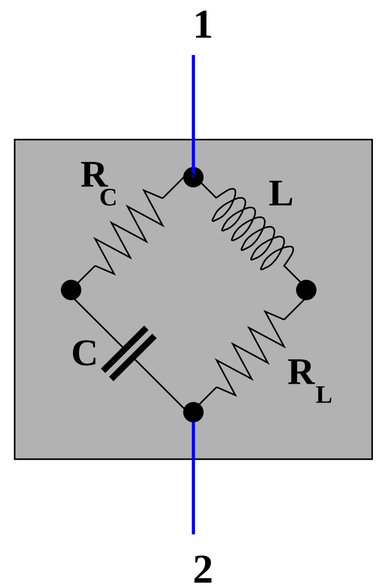

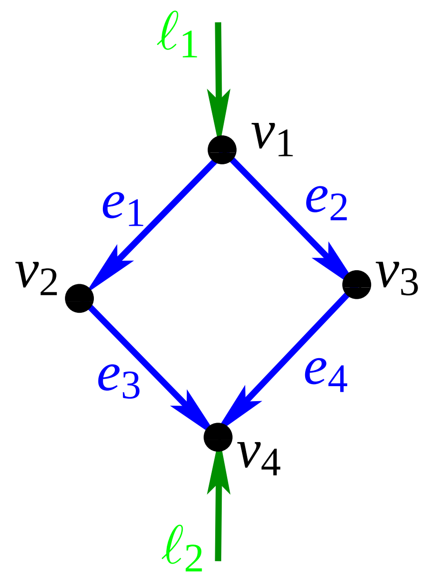

and the element specification is

$$e_1 \mapsto \textbf{resistance } R_C,$$
$$e_2 \mapsto \textbf{inductance } L,$$
$$e_3 \mapsto \textbf{capacitance } C,$$
$$e_4 \mapsto \textbf{resistance } R_L.$$

Circuit equations

Mathematical circuit specification

Obtain 4 matrices from the circuit description:

- $\mathbb{A}_{\mathbb{E}}$, the edge incidence matrix, a $|\mathbb{V}| \times |\mathbb{E}|$ matrix having values $+1$, -1, and 0,
- $\mathbb{A}_{\mathbb{L}}$, the leaf incidence matrix, a $|\mathbb{V}| \times |\mathbb{L}|$ matrix, having values $+1$ and 0,
- Z, the impedance matrix, a $|\mathbb{E}| \times |\mathbb{E}|$ diagonal polynomial matrix with

$$Z(\xi)_{\mathrm{k},\mathrm{k}} = \begin{cases} R_{\mathrm{k}} \text{ if a resistor with value } R_{\mathrm{k}} \text{ is in edge } e_{\mathrm{k}}, \\ L_k\xi \text{ if an inductor with value } L_{\mathrm{k}} \text{ is in edge } e_{\mathrm{k}}, \\ 1 \text{ otherwise}, \end{cases}$$

- Y, the admittance matrix, a $|\mathbb{E}| \times |\mathbb{E}|$ diagonal polynomial matrix with

$$Y(\xi)_{\mathrm{k},\mathrm{k}} = \begin{cases} C_k\xi \text{ if an capacitor with value } C_{\mathrm{k}} \text{ is in edge } e_{\mathrm{k}}, \\ 1 \text{ otherwise}. \end{cases}$$

Mathematical circuit specification

From these 4 matrices, we obtain directly the circuit equations. These involve as manifest variables, the terminal currents and potentials

$$I = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{bmatrix},$$

and as latent variables, the edge currents, the edge voltages, and the vertex potentials

$$I_{\mathbb{E}} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{|\mathbb{E}|} \end{bmatrix}, \quad V_{\mathbb{E}} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{|\mathbb{E}|} \end{bmatrix}, \quad \text{and} \quad P_{\mathbb{V}} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{|\mathbb{V}|} \end{bmatrix}.$$

MNA

The circuit equations are

$$V_{\mathbb{E}} = \mathbb{A}_{\mathbb{E}}^{\top} P_{\mathbb{V}},$$

$$Z\left(\frac{d}{dt}\right) I_{\mathbb{E}} = Y\left(\frac{d}{dt}\right) V_{\mathbb{E}},$$

$$\mathbb{A}_{\mathbb{E}} I_{\mathbb{E}} + \mathbb{A}_{\mathbb{L}} I = 0,$$

$$\mathbb{A}_{\mathbb{L}}^{\top} P_{\mathbb{V}} + P = 0.$$

MNA

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$$V_{\mathbb{E}} = \mathbb{A}_{\mathbb{E}}^{\top} P_{\mathbb{V}},$$
$$Z\left(\frac{d}{dt}\right) I_{\mathbb{E}} = Y\left(\frac{d}{dt}\right) V_{\mathbb{E}},$$
$$\mathbb{A}_{\mathbb{E}} I_{\mathbb{E}} + \mathbb{A}_{\mathbb{L}} I = 0,$$
$$\mathbb{A}_{\mathbb{L}}^{\top} P_{\mathbb{V}} + P = 0.$$

The variables $V_{\mathbb{E}}$ can be eliminated immediately, leading to the 'modified nodal analysis' circuit equations

$$Z\left(\tfrac{d}{dt}\right) I_{\mathbb{E}} = Y\left(\tfrac{d}{dt}\right) \mathbb{A}_{\mathbb{E}}^{\top} P_{\mathbb{V}}, \mathbb{A}_{\mathbb{E}} I_{\mathbb{E}} + \mathbb{A}_{\mathbb{L}} I = 0, \mathbb{A}_{\mathbb{L}}^{\top} P_{\mathbb{V}} + P = 0,$$

with (I, P) as manifest and $(I_{\mathbb{E}}, P_{\mathbb{V}})$ as latent variables.

MNA

Equation

$$V_{\mathbb{E}} = \mathbb{A}_{\mathbb{E}}^{\top} P_{\mathbb{V}}$$

relates the vertex potentials to the voltages across the edges;

$$Z\left(\frac{d}{dt}\right) I_{\mathbb{E}} = Y\left(\frac{d}{dt}\right) V_{\mathbb{E}}$$

expresses the constitutive laws of the resistors, inductors, and capacitors in the edges;

$$\mathbb{A}_{\mathbb{E}} I_{\mathbb{E}} + \mathbb{A}_{\mathbb{L}} I = 0$$

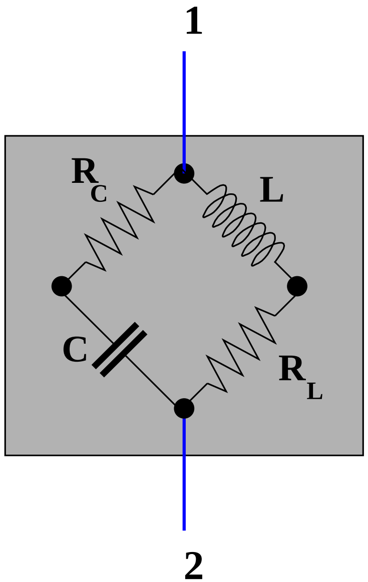
is KCL for each of the vertices; and

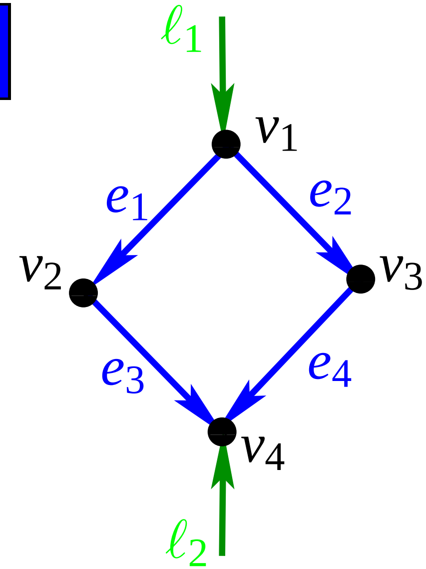
$$\mathbb{A}_{\mathbb{L}}^{\top} P_{\mathbb{V}} + P = 0$$

assigns the terminal potentials to the corresponding vertex potentials.

Example

For





we have

$$\mathbb{A}_{\mathbb{E}} = \begin{bmatrix} +1 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \quad \mathbb{A}_{\mathbb{L}} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$Z(\xi) = \begin{bmatrix} R_C & 0 & 0 & 0 \\ 0 & L\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & R_L \end{bmatrix}, \quad Y(\xi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C\xi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example

Leading to the MNA equations

$$R_C I_{e_1} = P_{v_1} - P_{v_2}, \quad L\frac{d}{dt} I_{e_2} = P_{v_1} - P_{v_3},$$

$$I_{e_1} = C\frac{d}{dt}\left(P_{v_2} - P_{v_4}\right), \quad R_L I_{e_4} = P_{v_3} - P_{v_4};$$

$$I_1 = I_{e_1} + I_{e_2}, \quad I_{e_2} = I_{e_3}, \quad I_{e_4} = I_{e_4}, \quad I_2 = I_{e_1} + I_{e_2};$$

$$P_1 = P_{e_1}, \quad P_2 = P_{e_4}.$$

Example

Leading to the MNA equations

$$R_C I_{e_1} = P_{v_1} - P_{v_2}, \quad L\frac{d}{dt} I_{e_2} = P_{v_1} - P_{v_3},$$

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$$I_1 = I_{e_1} + I_{e_2}, \quad I_{e_2} = I_{e_3}, \quad I_{e_4} = I_{e_4}, \quad I_2 = I_{e_1} + I_{e_2};$$

$$P_1 = P_{e_1}, \quad P_2 = P_{e_4}.$$

The MNA circuit equations can be set up in a straightforward way. The manifest variables are the terminal currents and potentials. MNA illustrates the systematic way in which equations can be set up from first principles, with as choice of latent variables are the vertex potentials and edge currents.

Recapitulation

Summary

- First principles models invariably contain latent variables.
- It may or may not be possible to eliminate the latent variables.

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- The behavior of LTIDSs is closed under projection.
  In LTIDSs latent variables can be completely eliminated.
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Summary

- **First principles models invariably contain latent variables.**
- **It may or may not be possible to eliminate the latent variables.**
- **The behavior of LTIDSs is closed under projection.**
  In LTIDSs latent variables can be completely eliminated.
- **Elimination algorithms involve computing a set of generators of a left syzygy.**
- **The modeling of the terminal behavior of general RLC circuits can be done by MNA.**
- **A crucial step in this modeling procedure is the choice of latent variables.**

End of Lecture IV