European Embedded Control Institute

Graduate School on Control — Spring 2010

The Behavioral Approach to Modeling and Control

Lecture III

KERNEL REPRESENTATIONS

Theme

We discuss some issues regarding LTIDSs.

- **The polynomial matrix that defines a LTIDS through a kernel representation is not unique.**
  What is the relation between all the polynomial matrices that lead to the same behavior?
- **What do we mean by an input/output representation?**
  Does every LTIDS admit an input/output interpretation?
- **How does the transfer function fit in?**
- **While a LTIDS is defined in terms of an ODEs involving polynomial matrices, these systems are often represented by rational functions.**
  How are representations with rational symbols defined?

Outline

- **The structure of kernel representations**
- **Minimal kernel representations**
- **Inputs and outputs; the transfer function**
- **Autonomous systems**
- **Rational symbols**

The structure of kernel representations

Non-uniqueness

$$[\![\mathscr{B} \in \mathscr{L}^\bullet]\!] :\Leftrightarrow$$

$$[\![\mathscr{B} = \texttt{kernel}\left(R\left(\frac{d}{dt}\right)\right) \text{ for some } R \in \mathbb{R}[\xi]^{\bullet\times\bullet}]\!].$$

$R\left(\frac{d}{dt}\right)w = 0$ **determines** $\mathscr{B}$**, but** $\mathscr{B}$ **does not determine** R**.**
Obviously, R **and** UR **determine the same behavior if** U **is unimodular.**

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$R\left(\frac{d}{dt}\right)w = 0$ determines $\mathscr{B}$, but $\mathscr{B}$ does not determine R. Obviously, R and UR determine the same behavior if U is unimodular. This leads to the following question

When do

$$R_1\left(\frac{d}{dt}\right)w = 0 \quad \textit{and} \quad R_2\left(\frac{d}{dt}\right)w = 0$$

determine the same system?

The annihilators

The polynomial vector $n \in \mathbb{R}[\xi]^{1\times \mathtt{w}}$ **is said to be an**

$$[\![\textbf{annihilator} \textbf{ of } \mathscr{B} \in \mathscr{L}^{\mathtt{w}}]\!] :\Leftrightarrow [\![n\left(\frac{d}{dt}\right)\mathscr{B} = 0]\!],$$

that is, n **is an annihilator** $:\Leftrightarrow n\left(\frac{d}{dt}\right)w = 0$ **for all** $w \in \mathscr{B}$**.**

Denote the set of annihilators of $\mathscr{B}$ **by** $\mathscr{N}_{\mathscr{B}}$**.**

It is easy to see that $\mathscr{N}_{\mathscr{B}}$ **is an** $\mathbb{R}[\xi]$**-module.**
Obviously, $n_1, n_2 \in \mathscr{N}_{\mathscr{B}}$ **and** $p \in \mathbb{R}[\xi]$ **imply** $n_1 + pn_2 \in \mathscr{N}_{\mathscr{B}}$**.**

Notation

For $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$**, let** $\langle R \rangle$ **denote the** $\mathbb{R}[\xi]$**-submodule of** $\mathbb{R}[\xi]^{1 \times \mathtt{w}}$ **generated by the rows of** R**.**

$\mathscr{M}^{\mathtt{w}}$ **denotes the set of** $\mathbb{R}[\xi]$**-submodules of** $\mathbb{R}[\xi]^{1 \times \mathtt{w}}$**.**

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$\mathscr{M}^{\mathtt{w}}$ **denotes the set of** $\mathbb{R}[\xi]$**-submodules of** $\mathbb{R}[\xi]^{1 \times \mathtt{w}}$**.**
For $M \in \mathscr{M}^{\mathtt{w}}$**, let** $\mathscr{S}_M$ **denote the behavior induced by** M**.**

$$\mathscr{S}_M := \{ w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid m\left(\frac{d}{dt}\right) w = 0 \ \textbf{for all} \ m \in M \}.$$

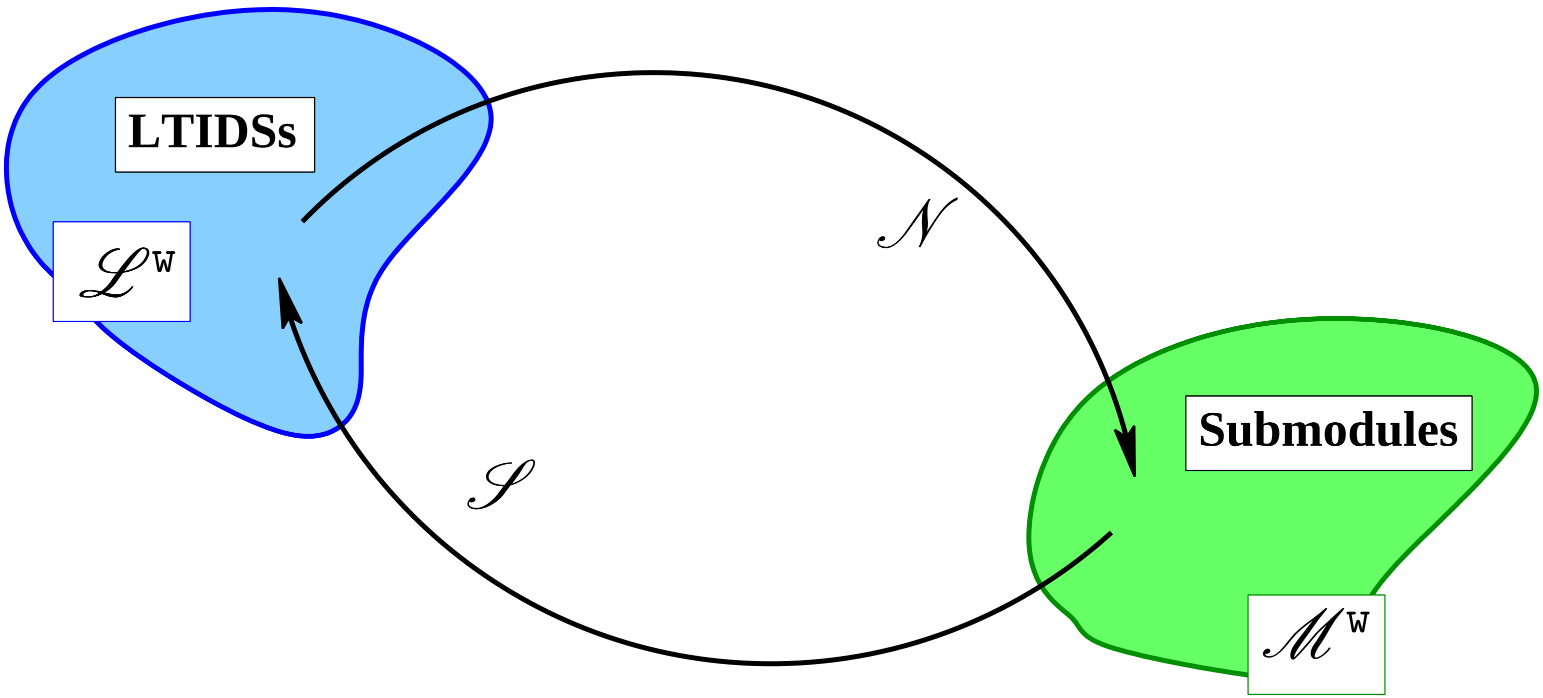
It is easy to see that this behavior belongs to $\mathscr{L}^{\mathtt{w}}$**.**
In fact, if $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$ **is a polynomial matrix whose rows are generators of** M**,** $M = \langle R \rangle$**,**
then $\mathscr{S}_M = \mathtt{kernel}\left(R\left(\frac{d}{dt}\right)\right)$**.**

From behaviors to $\mathbb{R}[\xi]$-modules and back

Summarizing,

$$\mathscr{N} : \mathscr{L}^{\mathtt{w}} \rightarrow \mathscr{M}^{\mathtt{w}}, \qquad \mathscr{B} \overset{\mathscr{N}}{\longmapsto} \mathscr{N}_{\mathscr{B}}.$$

$$\mathscr{S} : \mathscr{M}^{\mathtt{w}} \rightarrow \mathscr{L}^{\mathtt{w}}, \qquad M \overset{\mathscr{S}}{\longmapsto} \mathscr{S}_{M}.$$



The structure theorem

Theorem

1. Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$. Then

$$[\![\mathscr{B} = \mathtt{kernel}\left(R\left(\tfrac{d}{dt}\right)\right)]\!] \Leftrightarrow [\![\mathscr{N}_{\mathscr{B}} = \langle R \rangle]\!].$$

2. Let $\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{L}^{\mathtt{w}}$. Then

$$[\![\mathscr{B}_1 = \mathscr{B}_2]\!] \Leftrightarrow [\![\mathscr{N}_{\mathscr{B}_1} = \mathscr{N}_{\mathscr{B}_2}]\!].$$

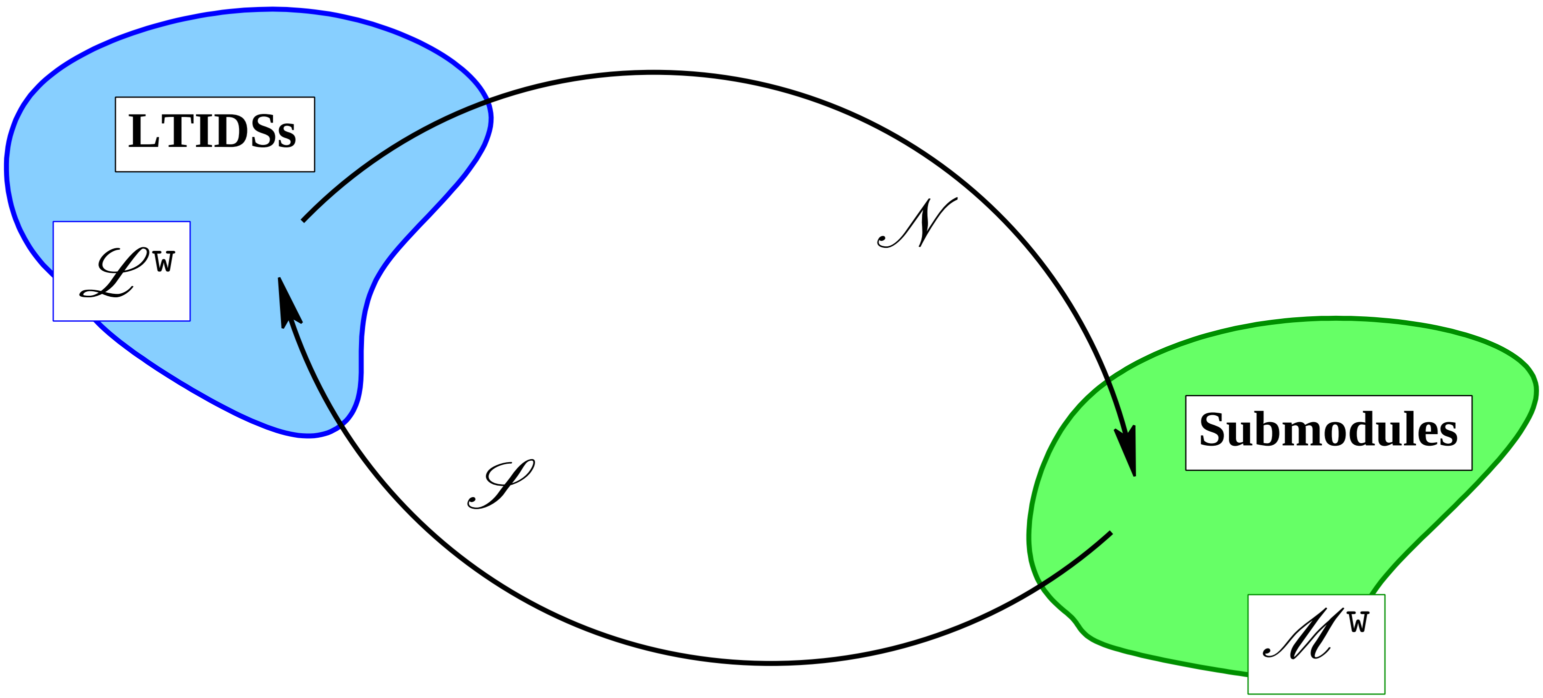
3. The maps $\mathscr{N}$ and $\mathscr{S}$ are each other's inverse, i.e.,

$$\mathscr{S}_{\mathscr{N}_{\mathscr{B}}} = \mathscr{B} \quad \text{and} \quad \mathscr{N}_{\mathscr{S}_M} = M.$$

Hence there exists a one-to-one relation between $\mathscr{L}^{\mathtt{w}}$ and the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{\mathtt{w}}$.

From behaviors to $\mathbb{R}[\xi]$-modules and back

$$\mathscr{N} : \mathscr{L}^{\mathtt{w}} \to \mathscr{M}^{\mathtt{w}}, \qquad \mathscr{B} \stackrel{\mathscr{N}}{\longmapsto} \mathscr{N}_{\mathscr{B}}.$$
$$\mathscr{S} : \mathscr{M}^{\mathtt{w}} \to \mathscr{L}^{\mathtt{w}}, \qquad M \stackrel{\mathscr{S}}{\longmapsto} \mathscr{S}_{M}.$$



The above picture illustrates the $1 \leftrightarrow 1$ relation between $\mathscr{L}^{\mathtt{w}}$ and $\mathscr{M}^{\mathtt{w}}$.

Proof in telegram-style

1. **The claim is equivalent to** $\mathscr{N}_{\mathtt{kernel}(R(\frac{d}{dt}))} = \langle R \rangle.$

- **First prove the case** $\mathtt{w} = 1$ **by applying Proposition 1 of the section on differential operators (see Lecture II).**
- **Then show that, without loss of generality, it can be assumed that** R **is in Smith form.**
- **Finally, prove the case that** R **is in Smith form by repeated application of the case** $\mathtt{w} = 1$**.**

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2. $(\Leftarrow)$ **follows from**

$[\![\langle R_1 \rangle = \langle R_2 \rangle]\!] \Leftrightarrow [\![\exists\ F_1, F_2$ **such that** $R_2 = F_1 R_1$ **and** $R_1 = F_2 R_2]\!]$, **which implies** $\mathscr{B}_1 = \mathscr{B}_2$.

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3. **is a consequence of 1.**

Example

Let $\mathtt{w} = 1$**. Let** $\mathscr{B}$ **be described by**

$$r_1\left(\frac{d}{dt}\right)w = 0, r_2\left(\frac{d}{dt}\right)w = 0, \ldots, r_{\mathtt{n}}\left(\frac{d}{dt}\right)w = 0,$$

with $r_1, r_2, \ldots, r_{\mathtt{n}} \in \mathbb{R}[\xi]$**. The annihilators consist of all polynomials that have** $r \in \mathbb{R}[\xi]$**, the greatest common divisor of** $r_1, r_2, \ldots, r_{\mathtt{n}}$**, as a factor. Hence**

$$r\left(\frac{d}{dt}\right)w = 0$$

is also a kernel representation of $\mathscr{B}$**.**

The systems $\mathscr{L}^1$ **and the** $\mathbb{R}[\xi]$**-submodules of** $\mathbb{R}[\xi]$ **stand in** $1 \leftrightarrow 1$ **relation with the monic polynomials in** $\mathbb{R}[\xi]$**.**

Example

Let $\mathtt{w} = 1$. Assume that, instead of taking the $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$-solutions of $R\left(\frac{d}{dt}\right) w = 0$ as the behavior, we take the $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$-solutions *of compact support*. Then there are only two cases: either $\mathscr{B} = \{0\}$, or $\mathscr{B} =$ all $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$-functions of compact support.

Therefore, if we had taken the $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$-solutions of compact support as the definition of the behavior, the $1 \leftrightarrow 1$ relation with the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]$ fails.

This shows that the structure theorem is crucially dependent on the solution concept used. The theory of LTIDSs does not only depend on *algebra*, through submodules and the like, but also on *analysis*, through the sulotion concept of differential equations used.

Inclusion of behaviors

Let $\mathscr{B}_1 = \texttt{kernel}\left(R_1\left(\frac{d}{dt}\right)\right), \mathscr{B}_2 = \texttt{kernel}\left(R_2\left(\frac{d}{dt}\right)\right)$**. Then**

$$[\![\mathscr{B}_1 \subseteq \mathscr{B}_2]\!] \Leftrightarrow [\![\mathscr{N}_{\mathscr{B}_1} = \langle R_1 \rangle \supseteq \langle R_2 \rangle = \mathscr{N}_{\mathscr{B}_2}]\!]$$

$$\Leftrightarrow [\![\ \exists\, F \in \mathbb{R}[\xi]^{\bullet \times \bullet} \textbf{ such that } R_2 = F R_1]\!]$$

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$$\Leftrightarrow [\![\, \exists\, F \in \mathbb{R}[\xi]^{\bullet\times\bullet} \text{ such that } R_2 = FR_1]\!]$$

Therefore,

$$[\![\mathscr{B}_1 = \mathscr{B}_2]\!] \Leftrightarrow [\![\exists\, F_1, F_2 \in \mathbb{R}[\xi]^{\bullet\times\bullet} \text{ such that } R_1 = F_2R_2, R_2 = F_1R_1]\!].$$

In particular,

$$[\![\mathscr{B}_1 = \mathscr{B}_2]\!] \text{ if } [\![\exists\, U \in \mathbb{R}[\xi]^{\bullet\times\bullet} \text{ unimodular such that } R_1 = UR_2]\!].$$

Minimal kernel representations

The representation $R\left(\frac{d}{dt}\right)w = 0$ **of** $\mathscr{B} \in \mathscr{L}^\bullet$ **is said to be a** **minimal** **kernel representation if, among all kernel representations of** $\mathscr{B}$**,** R **has a minimal number of rows.**

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Theorem

Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$**. The following are equivalent.**

1. $R\left(\frac{d}{dt}\right)w = 0$ **is a minimal kernel representation of** $\mathscr{B}$**.**
2. R **has full row rank.**
3. **All minimal kernel representations of** $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **are generated from one minimal kernel representation,** $R\left(\frac{d}{dt}\right)w = 0$**, by the transformation group**

$$R \quad \underset{U \text{ unimodular}}{\overset{U \in \mathbb{R}[\xi]^{\mathtt{w}\times\mathtt{w}}}{\longmapsto}} \quad UR$$

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Proof: Follows immediately from the structure theorem.

Free variables

Free variables

Let $\mathbb{I} = \{i_1, i_2, \ldots, i_{|\mathbb{I}|}\} \subseteq \{1, 2, \ldots, \mathtt{w}\}$**.**
Define, for $w = (w_1, w_2, \ldots, w_{\mathtt{w}}) \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **and** $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$**,**

$$\Pi_{\mathbb{I}} w := (w_{i_1}, w_{i_2}, \ldots, w_{i_{|\mathbb{I}|}}),$$

$$\Pi_{\mathbb{I}} \mathscr{B} := \{\Pi_{\mathbb{I}} w \mid w \in \mathscr{B}\}.$$

Btw, by the elimination theorem (see Lecture IV),

$$[\![\mathscr{B} \in \mathscr{L}^{\mathtt{w}}]\!] \Rightarrow [\![\Pi_{\mathbb{I}} \mathscr{B} \in \mathscr{L}^{|\mathbb{I}|}]\!].$$

Free variables

The variables $\{w_{i_1}, w_{i_2}, \ldots, w_{i_{|\mathbb{I}|}}\}$ **are said to be free in** $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **if**

$$\Pi_{\mathbb{I}}\mathscr{B} = \mathscr{C}^{\infty}\left(\mathbb{R}, \mathbb{R}^{|\mathbb{I}|}\right),$$

i.e., if $\mathscr{B}$ **does not constrain the variables** $\{w_{i_1}, w_{i_2}, \ldots, w_{i_{|\mathbb{I}|}}\}$**.**

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The variables $\{w_{i_1}, w_{i_2}, \ldots, w_{i_{|\mathbb{I}|}}\}$ **are said to be maximally free in** $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **if**

- $\Pi_{\mathbb{I}}\mathscr{B} = \mathscr{C}^{\infty}\left(\mathbb{R}, \mathbb{R}^{|\mathbb{I}|}\right)$,
- $[\![\mathbb{I}' = \{i'_1, i'_2, \ldots, i'_{|\mathbb{I}'|}\} \subseteq \{1, 2, \ldots, \mathtt{w}\}, \mathbb{I} \subsetneq \mathbb{I}']\!]$
  $\Rightarrow [\![$**the variables** $\{w_{i'_1}, w_{i'_2}, \ldots, w_{i'_{|\mathbb{I}'|}}\}$ **are not free in** $\mathscr{B}]\!]$ **.**

In words, these variables are unconstrained, but adding other variables results in a set of variables that is constrained.

Free variables in LTIDSs

Partition $w = (w_1, w_2), \quad w_1 : \mathbb{R} \longrightarrow \mathbb{R}^{\mathtt{w}_1}, w_2 : \mathbb{R} \longrightarrow \mathbb{R}^{\mathtt{w}_2}$**.**
Let $R_1\left(\frac{d}{dt}\right) w_1 = R_2\left(\frac{d}{dt}\right) w_2$, **be a minimal kernel representation of** $\mathscr{B} \in \mathscr{L}^{\mathtt{w}_1+\mathtt{w}_2}$**.**

Proposition 5:

1. $[\![w_2$ **is free in** $\mathscr{B}]\!] \Leftrightarrow [\![R_1$ **has full row rank**$]\!]$.
2. $[\![w_2$ **is maximally free in** $\mathscr{B}]\!]$
   $\Leftrightarrow [\![R_1$ **is square and** $\mathtt{determinant}(R_1) \neq 0]\!]$.

Note that, by Proposition 4 from the section on differential operators (see Lecture II), 2. is equivalent to:

2'. w_2 **is free and the elements of the form** $(w_1, 0) \in \mathscr{B}$ **form a finite-dimensional subspace.**

Proof in telegram-style

1. $(\Leftarrow)$ $R_1\left(\frac{d}{dt}\right)$ **is surjective, hence** w_2 **is free.**

1. $(\Rightarrow)$ **If** R_1 **does not have full row rank, then after pre-multiplication by a unimodular matrix, the minimal kernel representation looks like**

$$\begin{bmatrix} R_1'\left(\frac{d}{dt}\right) \\ 0_{\mathtt{rank}(R_1)\times \mathtt{w}_1} \end{bmatrix} w_1 = \begin{bmatrix} R_2'\left(\frac{d}{dt}\right) \\ R_2''\left(\frac{d}{dt}\right) \end{bmatrix} w_2,$$

with R_2'' **of full row rank. Therefore** w_2 **satisfies** $R_w''\left(\frac{d}{dt}\right) w_2 = 0$ **and is hence not free.**

Proof in telegram-style

2. $(\Leftarrow)$ w_2 **is free, by 1. Moreover, the elements of the form** $(w_1, 0) \in \mathscr{B}$ **form a finite-dimensional subspace, and therefore there are no additional free variables.**

2. $(\Rightarrow)$ **By 1.** R_1 **has full row rank. If** R_1 **is 'wide' (less rows than columns), then it possible to delete a column from** R_1 **and add it to** R_2**, and preserve the full row rank property. Then by 1.** w_2 **augmented with the variable from** w_1 **corresponding to the deleted column remains free.**

Examples

- **Consider** $$r_1\left(\frac{d}{dt}\right)w_1 = r_2\left(\frac{d}{dt}\right)w_2,$$

  with $r_1, r_2 \in \mathbb{R}[\xi]$**,** $r_1 \neq 0$ **and** $r_2 \neq 0$**, and** $w_1, w_2 : \mathbb{R} \to \mathbb{R}$**.** **Then both** w_1 **and** w_2 **are maximally free.**

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- **Consider**

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}.$$

  u **is free, and since the set of** y**'s corresponding to** $u = 0$ **is finite-dimensional, it is maximally free.**

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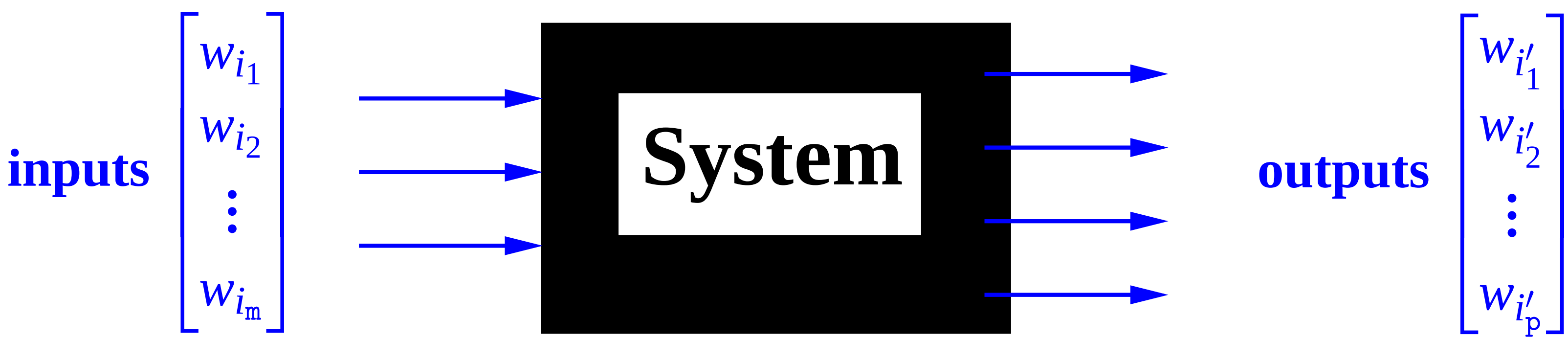
- **Consider** $$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}.$$ u **is free, and since the set of** y**'s corresponding to** $u = 0$ **is finite-dimensional, it is maximally free.**
- **Assume that the DAE** $E\frac{d}{dt}x = Ax + Bu$ **is 'regular' (meaning** E, A **square and** `determinant`$(E\xi - A) \neq 0$**). Then** u **is maximally free.**

Inputs and outputs

Input/output partition

Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ and $w = (u, y)$ with u maximally free in $\mathscr{B}$.
Then u is said to be input and y is said to be output in $\mathscr{B}$.
The corresponding partition $w = (u, y)$ is said to be an input/output partition for $\mathscr{B}$.



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The corresponding partition $w = (u, y)$ **is said to be an input/output partition for** $\mathscr{B}$**.**

It follows from Proposition 5 that $w = (u, y)$ **is an input/output partition if and only if** $\mathscr{B}$ **has a minimal kernel representation**

$$P\left(\frac{d}{dt}\right) y = Q\left(\frac{d}{dt}\right) u,$$

with P **square and** $\mathtt{determinant}(P) \neq 0.$

Input/output partition

Theorem

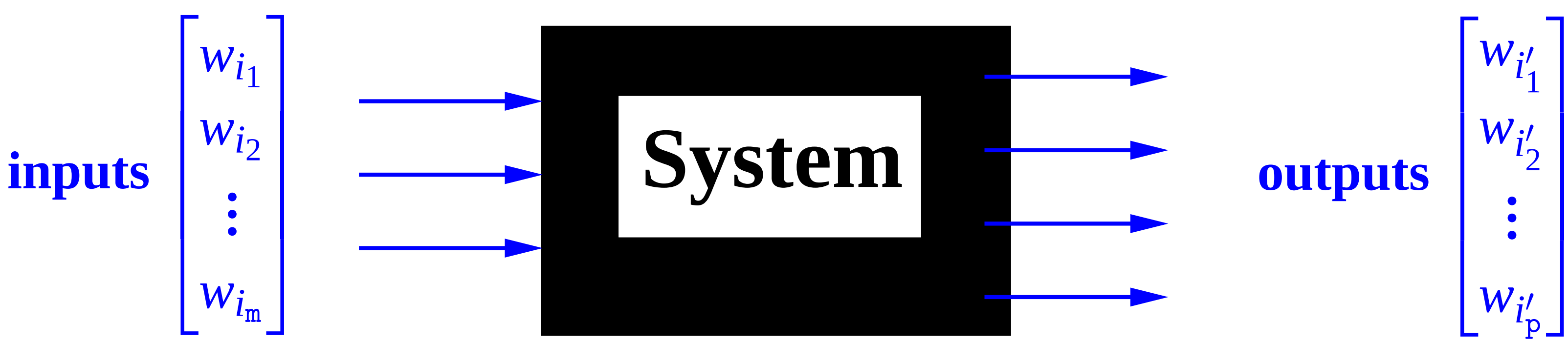
Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$. There exists a partition of the index set $\{1, 2, \ldots, \mathtt{w}\}$ into two parts,

$$\{i_1, i_2, \ldots, i_{\mathtt{m}}\} \text{ and } \{i'_1, i'_2, \ldots, i'_{\mathtt{p}}\}$$

such that

$$u = (w_{i_1}, w_{i_2}, \ldots, w_{i_{\mathtt{m}}}), \quad y = (w_{i'_1}, w_{i'_2}, \ldots, w_{i'_{\mathtt{p}}})$$

is an input/output partition for $\mathscr{B}$.



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such that

$$u = (w_{i_1}, w_{i_2}, \ldots, w_{i_{\mathtt{m}}}), \quad y = (w_{i'_1}, w_{i'_2}, \ldots, w_{i'_{\mathtt{p}}})$$

is an input/output partition for $\mathscr{B}$.

Proof: Let $R\left(\frac{d}{dt}\right) w = 0$ be a minimal kernel representation of $\mathscr{B}$. Choose $\{i'_1, i'_2, \ldots, i'_{\mathtt{p}}\}$ such that the columns $\{i'_1, i'_2, \ldots, i'_{\mathtt{p}}\}$ of R form a square and nonsingular matrix.

Input/output partition

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such that

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is an input/output partition for $\mathscr{B}$.

It follows from the construction used in this proof that an input/output partition for $\mathscr{B}$ is in general not unique. However, the *number* of input and output components is uniquely determined by $\mathscr{B}$.

Integer invariants

$\mathtt{w} : \mathscr{L}^\bullet \to \mathbb{N}, \quad \mathtt{w}(\mathscr{B}) :=$ **the number of variables in** $\mathscr{B}$,

$\mathtt{m} : \mathscr{L}^\bullet \to \mathbb{N}, \quad \mathtt{m}(\mathscr{B}) :=$ **the number of input components in** $\mathscr{B}$,

$\mathtt{p} : \mathscr{L}^\bullet \to \mathbb{N}, \quad \mathtt{p}(\mathscr{B}) :=$ **the number of output components in** $\mathscr{B}$.

Of course $\mathtt{m} + \mathtt{p} = \mathtt{w}$**.**

Note the following formulas for $\mathtt{p}$**:**

$$\mathtt{p}(\mathscr{B}) = \mathtt{dimension}\,(\mathscr{N}_{\mathscr{B}})\,,$$

and, with $R\left(\frac{d}{dt}\right) w = 0$ **a minimal kernel representation of** $\mathscr{B}$**,**

$$\mathtt{p}(\mathscr{B}) = \mathtt{rowdimension}\,(R)\,.$$

The transfer function

The transfer function

Let $w = (u, y)$ **be an input/output partition of** $\mathscr{B} \in \mathscr{L}^{\mathtt{m}(\mathscr{B})+\mathtt{p}(\mathscr{B})}$**, with minimal kernel representation**

$$P\left(\frac{d}{dt}\right) y = Q\left(\frac{d}{dt}\right) u.$$

The $\mathtt{m}(\mathscr{B}) \times \mathtt{p}(\mathscr{B})$ **matrix of real rational functions**

$$\boxed{G = P^{-1} Q}$$

is called the transfer function corresponding to this input/output partition.

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is called the transfer function corresponding to this input/output partition.

Note that for each $\lambda \in \mathbb{C}$, not a pole of G, and for each $\mathrm{u}_\lambda \in \mathbb{C}^{\mathtt{m}(\mathscr{B})}$, the exponential trajectory

$$t \mapsto \left(\mathrm{u}_\lambda e^{\lambda t}, \mathrm{y}_\lambda e^{\lambda t}\right), \ \textbf{with } \mathrm{y}_\lambda = G(\lambda)\mathrm{u}_\lambda,$$

belongs to $\mathscr{B}$ (complexified).

It is most insightful to think of the transfer function in terms of this formula for the exponential responses. It avoids irrelevant considerations of domains of convergence encountered in Laplace transforms.

Proper transfer functions

The real rational function $f = \frac{n}{d} \in \mathbb{R}(\xi), n, d \in \mathbb{R}[\xi]$ **is said to be** $[\![$ **proper** $]\!]$ $:\Leftrightarrow$ $[\![\texttt{degree}(d) \geq \texttt{degree}(n)]\!]$,

and $[\![$ **strictly proper** $]\!]$ $:\Leftrightarrow$ $[\![\texttt{degree}(d) > \texttt{degree}(n)]\!]$.

A matrix of real rational functions is said to be proper, or strictly proper $:\Leftrightarrow$ **each element is proper, or strictly proper.**

Proper transfer functions

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Theorem

Let $\mathscr{B} \in \mathscr{L}^{\texttt{w}}$**. There exists a partition of the index set** $\{1, 2, \ldots, \texttt{w}\}$ **into 2 parts,** $\{i_1, i_2, \ldots, i_{\texttt{m}(\mathscr{B})}\}$ **and** $\{i'_1, i'_2, \ldots, i'_{\texttt{p}(\mathscr{B})}\}$ **such that the input/output partition**

$$u = (w_{i_1}, w_{i_2}, \ldots, w_{i_{\texttt{m}(\mathscr{B})}}), \quad y = (w_{i'_1}, w_{i'_2}, \ldots, w_{i'_{\texttt{p}(\mathscr{B})}})$$

for $\mathscr{B}$ **has a proper transfer function.**

Proper transfer functions

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<u>Proof</u>: When selecting $\mathtt{p}(\mathscr{B})$ columns of R corresponding to a minimal kernel representation $R\left(\frac{d}{dt}\right)w = 0$ of $\mathscr{B}$, choose the columns $\{i'_1, i'_2, \ldots, i'_{\mathtt{p}(\mathscr{B})}\}$ such that the determinant of the matrix formed by these columns has largest degree among all $\mathtt{p}(\mathscr{B}) \times \mathtt{p}(\mathscr{B})$ submatrices of R.

Significance of a proper transfer functions

- **For continuous-time system the significance of a proper transfer function lies in the fact that**
  the output is at least as smooth as the input.
  Unfortunately, this cannot be illustrated in our $\mathscr{C}^{\infty}$-setting. However, if the behavior is defined as a set of distributions, properness comes down to the implication

$$[\![(u,y) \in \mathscr{B}, u \in \mathscr{C}^{\mathtt{k}}(\mathbb{R}, \mathbb{R}^{\mathtt{m}(\mathscr{B})})]\!] \Rightarrow [\![y \in \mathscr{C}^{\mathtt{k}}(\mathbb{R}, \mathbb{R}^{\mathtt{p}(\mathscr{B})})]\!].$$

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- **For discrete-time systems, properness implies that**
  the output does not anticipate the input.
  This is made precise in Exercise ????.

Examples

- **Consider**

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}.$$

In order to compute the transfer function from u to y, it is easiest to proceed via the exponential response. This yields

$$G(\xi) = D + C(I\xi - A)^{-1}B$$

for the transfer function. This matrix of rational functions is proper, hence y is at least as smooth as u.

Examples

- **Note that our definitions consider a differentiator**

$$u \mapsto y = \frac{d}{dt}u$$

as a valid input/output system.

Its transfer function $G(\xi) = \xi$ **is not proper, and indeed,** y **need not be as smooth as** u**.**

The opposite input/output partition leads to an integrator, an input/output with a strictly proper transfer function $G(\xi) = \dfrac{1}{\xi}$**.**

Recapitulating

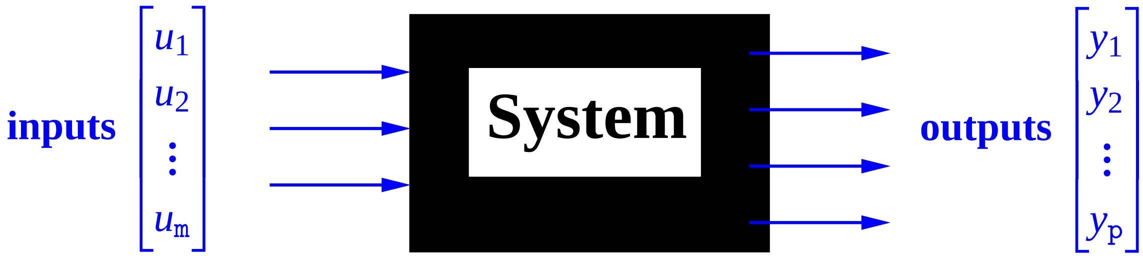
The components of the signal $w = (w_1, w_2, \ldots, w_{\mathtt{w}})$ of a LTIDS $\mathscr{B}$ allow a componentwise partition into inputs and outputs. If this partition is well-chosen, we may even obtain a proper transfer function.

To obtain a strictly proper transfer function, we need in general change the basis in the signal space first, and subsequently choose a componentwise partition into inputs and outputs.

Rational symbols

Laplace transforms

In system theory, it is customary to think of dynamical models in terms of inputs and outputs, viz.



In the LTI case, this leads to transfer functions,
$y = G(s)u$**, with** G **a matrix of rational functions.**

Usually, transfer functions are interpreted in terms of Laplace transforms, with growth conditions and domains of convergence, and such largely irrelevant mathematical traps.

We now learn to interpret '$y = G(s)u$**' in terms of differential equations.**

Factorizations of rational matrices

$M \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ **is *left prime* (over** $\mathbb{R}[\xi]$**)** $:\Leftrightarrow$

$$[\![M = FM', \text{ with } F, M' \in \mathbb{R}[\xi]^{\bullet\times\bullet}]\!] \Rightarrow [\![F \text{ is unimodular}]\!].$$

Equivalently, $M(\lambda)$ **must have full row rank for all** $\lambda \in \mathbb{C}$**.**

It follows from the Smith form that every $M \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ **of full row rank can be written as** $M = FM'$ **with** $F \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ **square and nonsingular, and** $M' \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ **left prime.**

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A *left coprime* polynomial factorization of $M \in \mathbb{R}(\xi)^{\bullet\times\bullet}$ **over** $\mathbb{R}[\xi]$ **is a pair** (P, Q)**, with** $P, Q \in \mathbb{R}[\xi]^{\bullet\times\bullet}$, P **square and nonsingular,** $M = P^{-1}Q$**, and** $\left[\begin{array}{c|c} P & Q \end{array}\right]$ **left prime.**

It is easily seen that every $M \in \mathbb{R}(\xi)^{\bullet\times\bullet}$ **admits a left coprime polynomial factorization. In the scalar case this simply means writing** M **as a ratio of two coprime polynomials.**

ODEs with rational symbols

Defining what a solution is of $R\left(\frac{d}{dt}\right)w = 0$ poses no difficulties worth mentioning when R is a polynomial matrix.

But, *what do we mean by a solution when R is a matrix of rational functions?*

ODEs with rational symbols

Let $F \in \mathbb{R}(\xi)^{\bullet\times\bullet}$**, and consider the 'differential equation'**

$$F\left(\frac{d}{dt}\right)w = 0.$$

$w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **satisfies this differential equation**

$$:\Leftrightarrow \quad Q\left(\frac{d}{dt}\right)w = 0,$$

where $F = P^{-1}Q$ **is a left coprime polynomial factorization.**

This definition of a solution is independent of the particular left coprime polynomial factorization of F **that is taken.**

ODEs with rational symbols

***By definition*, therefore, the behavior defined by $F\left(\frac{d}{dt}\right)w = 0$ is equal to that of $Q\left(\frac{d}{dt}\right)w = 0$.**

F is called the ‘symbol’ associated with this representation.

ODEs with rational symbols

By definition, **therefore, the behavior defined by** $F\left(\frac{d}{dt}\right)w = 0$ **is equal to that of** $Q\left(\frac{d}{dt}\right)w = 0$**.**

F **is called the ‘symbol’ associated with this representation.**

The use of rational symbols in addition to the polynomial symbols has proven to be very valuable.
In one of the exercises, we define norm-preserving representations. These require rational symbols.

Rationalization and justification

Equations with rational symbols

Let $F \in \mathbb{R}(\xi)^{\bullet\times\bullet}$ **and consider the equation**

$$v = F\left(\frac{d}{dt}\right)w.$$

Write this equation as

$$\left[\ F\left(\frac{d}{dt}\right)\ \middle|\ -I_{\mathtt{p}\times\mathtt{p}}\ \right]\begin{bmatrix} w \\ v \end{bmatrix} = 0.$$

A left coprime factorization $F = P^{-1}Q$ **of** F **over** $\mathbb{R}[\xi]$ **leads to a left coprime factorization** $P^{-1}\left[\ P\ \middle|\ -Q\ \right]$ **of** $\left[\ F\ \middle|\ -I_{\mathtt{p}\times\mathtt{p}}\ \right]$ **over** $\mathbb{R}[\xi]$

Hence (v, w) **satisfies** $v = F\left(\frac{d}{dt}\right)w \Leftrightarrow P\left(\frac{d}{dt}\right)v = Q\left(\frac{d}{dt}\right)w.$

Equations with rational symbols

Let $F \in \mathbb{R}(\xi)^{\bullet \times \bullet}$ **and consider the equation**

$$v = F\left(\frac{d}{dt}\right)w.$$

This shows that $F\left(\frac{d}{dt}\right)$ **is in general not a map, since there are many** v**'s that satisfy** $P\left(\frac{d}{dt}\right)v = Q\left(\frac{d}{dt}\right)w$ **for a given** w**.**

Instead, $F\left(\frac{d}{dt}\right)$ **is a point-to-set map, in the sense that it associates a set of** v**'s to one single** w**.**

Behaviors defined by point-to-set maps

For a given point-to-set map $g : \mathscr{U} \to \mathscr{E}$**, with** $0 \in \mathscr{E}$**,**

$$\mathscr{B} = \{u \in \mathscr{U} \mid 0 \in g(u)\}$$

is obviously a behavior.

This can be seen as the model defined by
$0 \in g(u),$ **or, informally, by the 'equation'** $g(u) = 0.$

This is a generalized type of kernel representation.

Set-theoretic inverse

If the map $f : A \to B$ is a bijection, then f^{-1} is a well-defined map from B to A.

However, f^{-1} can also be given a meaning when f is not a bijection, as follows

$$f^{-1}(b) := \{a \in A \mid f(a) = b\}.$$

Hence f^{-1} is a point-to-set map.

If f is not surjective, then $f^{-1}(a)$ may be empty,
if f is not injective, then $f^{-1}(a)$ may be not be a singleton.

We now apply these ideas to rational symbols.

Rational symbols

The set theoretic inverse $P\left(\frac{d}{dt}\right)^{-1}$ **in**

$$P\left(\frac{d}{dt}\right)^{-1} Q\left(\frac{d}{dt}\right) w = 0 \quad \cong \quad 0 \in P\left(\frac{d}{dt}\right)^{-1} Q\left(\frac{d}{dt}\right) w$$

leads to

$$Q\left(\frac{d}{dt}\right) w = 0.$$

Therefore our definition of $F\left(\frac{d}{dt}\right) w = 0$ **as** $Q\left(\frac{d}{dt}\right) w = 0$ **merely implements the set-theoretic inverse idea applied to** $0 \in P\left(\frac{d}{dt}\right)^{-1} Q\left(\frac{d}{dt}\right) w$**.**

Justification

Another justification comes from state space models. Assume F proper. Let $\frac{d}{dt}x = Ax + Bw, v = Cx + Dw$ be a controllable system with transfer function F, i.e., $F(\xi) = C(I\xi - A)^{-1}B + D$. Consider the output nulling inputs

$$\frac{d}{dt}x = Ax + Bw, 0 = Cx + Dw.$$

These w's are exactly those that satisfy $F\left(\frac{d}{dt}\right)w = 0$.

For F not proper, take $F(\xi) = C(I\xi - A)^{-1}B + D(\xi)$ with D polynomial, and

$$\frac{d}{dt}x = Ax + Bw, 0 = Cx + D(\frac{d}{dt})w.$$

Again, these w's are exactly those that satisfy $F\left(\frac{d}{dt}\right)w = 0$.

The transfer function

Rational transfer functions

Viewing the input/output system

$$y = G\left(\frac{d}{dt}\right)u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix},$$

with $G \in \mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{m}}$**, as a system defined in terms of a rational symbol, yields a rigorous definition of this LTIDS. Write it as**

$$\left[\; G\left(\frac{d}{dt}\right) \;\middle|\; -I_{\mathtt{p}\times\mathtt{p}} \;\right] \begin{bmatrix} u \\ y \end{bmatrix} = 0.$$

A left coprime factorization $G = P^{-1}Q$ **over** $\mathbb{R}[\xi]$ **leads to a left coprime factorization** $P^{-1}\left[\; P \;\middle|\; -Q \;\right]$ **of** $\left[\; G \;\middle|\; -I_{\mathtt{p}\times\mathtt{p}} \;\right]$ **over** $\mathbb{R}[\xi]$**, and**

$$P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}.$$

Rational transfer functions

Viewing the input/output system

$$y = G\left(\frac{d}{dt}\right)u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix},$$

with $G \in \mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{m}}$, as a system defined in terms of a rational symbol, yields a rigorous definition of this LTIDS.

Obviously, u is input, y is output, and G the transfer function, in accordance with the nomenclature introduced previously.

Rational transfer functions

Viewing the input/output system

$$y = G\left(\frac{d}{dt}\right)u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix},$$

with $G \in \mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{m}}$, as a system defined in terms of a rational symbol, yields a rigorous definition of this LTIDS.

This leads to a definition of its behavior and of the input/output pairs that is completely independent of Laplace transforms and its mathematical finesses and traps.

Caveats

- **The notation**

$$0 \in F\left(\frac{d}{dt}\right)w$$

is more accurate than

$$F\left(\frac{d}{dt}\right)w = 0$$

(which we use), and

$$y \in G\left(\frac{d}{dt}\right)u$$

is more accurate than the more commonly used

$$y = G\left(\frac{d}{dt}\right)u.$$

Caveats

- $F\left(\frac{d}{dt}\right)$ **is not a map! It associates with an input $u \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^\bullet)$ many (a finite-dimensional linear variety) outputs $y \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^\bullet)$ such that $y = F\left(\frac{d}{dt}\right)u$.**
  $F\left(\frac{d}{dt}\right)$ **is a one-to-many map.**

- **The operators $F_1\left(\frac{d}{dt}\right)$ and $F_2\left(\frac{d}{dt}\right)$ for $F_1, F_2 \in \mathbb{R}(\xi)$ need not commute.**

Recapitulation

Summary

- **There exists a one-to-one relation between the LTIDSs in $\mathscr{L}^{\mathtt{w}}$ and the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{1\times\mathtt{w}}$.**

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- **The variables of a LTIDS allow a componentwise partition in inputs and outputs.**
- **There exists a partition with a proper transfer function.**

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- **There exists a one-to-one relation between the LTIDSs in $\mathscr{L}^{\mathtt{w}}$ and the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{1\times\mathtt{w}}$.**
- **The variables of a LTIDS allow a componentwise partition in inputs and outputs.**
- **There exists a partition with a proper transfer function.**
- **A LTIDS is autonomous if and only if its behavior is finite-dimensional.**

Summary

- There exists a one-to-one relation between the LTIDSs in $\mathscr{L}^{\mathtt{w}}$ and the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{1\times\mathtt{w}}$.
- The variables of a LTIDS allow a componentwise partition in inputs and outputs.
- There exists a partition with a proper transfer function.
- A LTIDS is autonomous if and only if its behavior is finite-dimensional.
- LTIDSs also allow representations with rational symbols.

End of Lecture III