European Embedded Control Institute

Graduate School on Control — Spring 2010

The Behavioral Approach to Modeling and Control

Lecture XII

DETERMINISTIC IDENTIFICATION

Theme

- **‘Modeling is forbidding’ $\rightsquigarrow$ Most Powerful Unfalsified Model.**

- **Modeling from data $\equiv$ computing annihilators of Hankel matrix.**

Outline

- **Modeling from data: a language;**
- **The Most Powerful Unfalsified Model;**
- **Modeling discrete-time data;**
- **The Hankel matrix;**
- **Annihilators;**
- **Recursive computation of the MPUM.**

Warning

This lecture deals with exact data, i.e. not corrupted by noise.

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to a kernel representation

$$R_0 w(t) + \cdots + R_L w(t+L) = 0$$

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¡No noise, no stochastics!

Modeling from data: a language

Reminder: events, variables, universum

Physical phenomenon $\rightsquigarrow$ 'outcomes', events

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Example: modeling a resistor

Attributes $\rightsquigarrow$ (voltage, current) $\rightsquigarrow$ $\mathbb{R}^2$

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Physical phenomenon $\rightsquigarrow$ 'outcomes', events

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Example: modeling a gas

Attributes $\rightsquigarrow$ (pressure, temperature,volume) $\rightsquigarrow$ $\mathbb{R}^3_+$

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Physical phenomenon $\rightsquigarrow$ 'outcomes', events

Events are described by variables

Dynamical phenomena: events are maps from time space to variables space

The set of all such maps is the universum $\mathscr{U}$

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Example: modeling a resistor

$$\mathscr{U} = \left\{ (V, I) \in \left(\mathbb{R}^2\right)^{\mathbb{R}} \right\}$$

where $\left(\mathbb{R}^2\right)^{\mathbb{R}} := \{ f : \mathbb{R} \to \mathbb{R}^2 \}$

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Example: modeling a share value

$$\mathscr{U} = \left\{ V \in (\mathbb{R}_+)^{\mathbb{N}} \right\}$$

a discrete-time phenomenon

Models

Every “good” scientific theory is prohibition: it forbids certain things to happen...

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Example: Ohm's resistor

$$\begin{aligned} \mathscr{U} &= \left\{ (V, I) \in \left(\mathbb{R}^2\right)^{\mathbb{R}} \right\} \\ \mathscr{M} &= \{ \mathscr{B} \subset \mathscr{U} \mid \exists\, R \in \mathbb{R}_+ \ \textbf{s.t.}\ (V, I) \in \mathscr{B} \Longrightarrow V = R\, I \} \end{aligned}$$

Models

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A model $\mathscr{B}$ is a subset of $\mathscr{U}$, chosen from a model class $\mathscr{M}$ representing *a priori* knowledge/assumptions

Example: Linear models

$$\begin{aligned} \mathscr{U} &= \mathbb{R}^{\mathtt{w}} \\ \mathscr{M} &= \{\textbf{Linear subspaces of } \mathscr{U}\} \end{aligned}$$

The Most Powerful Unfalsified Model

Modeling from data: the Most Powerful Unfalsified Model

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Fewer possible outcomes, more discriminating model, better!

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- $\mathscr{B} \in \mathscr{M}$ **(i.e. admissible);**

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- $\mathscr{B} \in \mathscr{M}$**;**
- $D \subseteq \mathscr{B}$**;**
- $\mathscr{B}' \in \mathscr{M}, D \subseteq \mathscr{B}' \Longrightarrow \mathscr{B} \subset \mathscr{B}'$ **(i.e. most powerful).**

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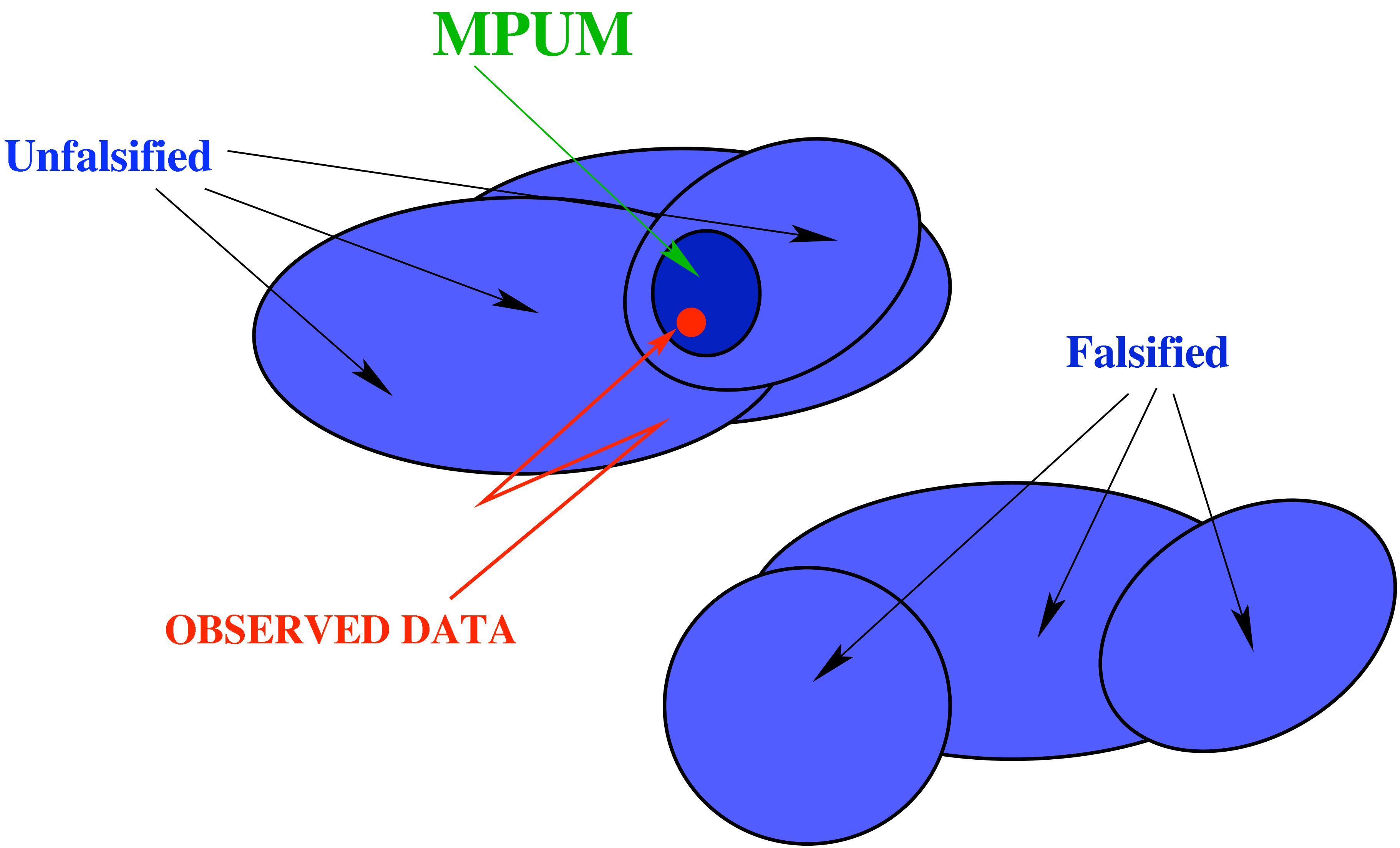
Existence? Uniqueness? Representations? Algorithms?

Graphically



The intersection property

***Example*: Consider**

$$\mathscr{U} = \mathbb{R}^n$$
$$\mathscr{M} = \textbf{Linear subspaces of } \mathbb{R}^n$$

Given measurements

$$D = \{w_1, \cdots, w_k\}$$

MPUM is

$$\textbf{span}\ \{w_i \mid i = 1, \cdots, k\}$$

the **intersection** **of all subspaces containing ($\equiv$ unfalsified by) data.**

The intersection property

Theorem: Assume that $\mathscr{M}$ satisfies

- **The intersection property i.e.**

$$\mathscr{M}' \subset \mathscr{M} \Longrightarrow \left(\bigcap_{\mathscr{B} \in \mathscr{M}'} \mathscr{B} \right) \in \mathscr{M}$$

- **For each $D \in 2^{\mathscr{U}}$ there exists $\mathscr{B} \in \mathscr{M}$ such that $D \subseteq \mathscr{B}$.**

Then for each D there exists a unique MPUM $\mathscr{B}^*$, namely

$$\mathscr{B}^* := \bigcap_{\mathscr{B} \in \mathscr{M},\ D \subseteq \mathscr{B}} \mathscr{B}$$

Examples

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The following are instances in which the intersection property holds:

- $\mathscr{M} = 2^{\mathscr{U}}$, **whatever** $\mathscr{U}$ **is;**
- $\mathscr{U} = \mathbb{R}^{\mathtt{n}}$, $\mathscr{M} = \{V \mid V$ **is linear subspace of** $\mathscr{U}\}$;
- $\mathscr{U}$ **topological vector space, and model class is** $\mathscr{M} = \{V \mid V$ **is closed linear subspace of** $\mathscr{U}\}$.

Dynamical modeling from data

Completeness

A discrete-time behavior $\mathscr{B}$ is complete if and only if

$$w \in \mathscr{B} \iff w|_{\mathbb{Z}\cap[t_0,t_1]} \in \mathscr{B}|_{\mathbb{Z}\cap[t_0,t_1]} \textbf{ for all } -\infty < t_0 \leq t_1 < \infty$$

where

$$\mathscr{B}|_{\mathbb{Z}\cap[t_0,t_1]} := \{v \in \mathbb{R}^{(t_1-t_0+1)\mathtt{w}} \mid \exists w \in \mathscr{B} \textbf{ s.t. } w|_{\mathbb{Z}\cap[t_0,t_1]} = v\}$$

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What happens at $\pm\infty$ is of no consequence.

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$\mathscr{B}$ is locally specified:

$\mathscr{B}$ L-complete $\iff$ $\mathscr{B}$ described by system of difference equations of order L

Completeness

Theorem: A discrete-time behavior $\mathscr{B}$ is linear and complete $\iff$ it is a linear subspace of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ closed in the topology of pointwise convergence.

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Theorem: A discrete-time behavior $\mathscr{B}$ is linear, time-invariant, and complete

$\iff$ it is a linear, shift-invariant subspace of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ closed in the topology of pointwise convergence;

$\iff$ there exists $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that $\mathscr{B} = \mathtt{kernel}\, R(\sigma)$.

Time-series modeling

Problem: given $\mathtt{w}$-dimensional time series

$$w := \{w(0), w(1), \cdots\}$$

find LTI complete behavior $\mathscr{B}$ containing w.

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Universum $\mathscr{U} = (\mathbb{R}^{\mathtt{w}})^{\mathbb{R}}$. Model class $\mathscr{M} = \mathscr{L}^{\mathtt{w}}$.

$\mathscr{L}^{\mathtt{w}}$ satisfies the intersection property: MPUM exists.

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Any unfalsified model is shift-invariant: must contain

$$\begin{aligned} w &= \{w(0), w(1), \cdots\} \\ \sigma w &= \{w(1), w(2), \cdots\} \\ \sigma^2 w &= \{w(2), w(3), \cdots\} \\ &\vdots \end{aligned}$$

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Intersection of all linear unfalsified models yields

$$\mathscr{B}^* = \left(\textbf{span}\ \{w, \sigma w, \sigma^2 w, \cdots\}\right)^{\textbf{closure}}$$

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¿What about representations?

The Hankel matrix

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MPUM is subspace spanned by rows of

$$\mathscr{H}(w) := \begin{bmatrix} w(0) & w(1) & \cdots & w(t'') & \cdots \\ w(1) & w(2) & \cdots & w(t''+1) & \cdots \\ w(2) & w(3) & \cdots & w(t''+2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w(t') & w(t'+1) & \cdots & w(t'+t''-1) & \cdots \\ w(t'+1) & w(t'+2) & \cdots & w(t'+t'') & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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Constant along the block-antidiagonal: Hankel structure

The left kernel of $\mathscr{H}(w)$

Let $R(\xi) = R_0 + R_1\xi + \cdots + R_L\xi^L \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$**.**

Then $R(\sigma)w = 0 \rightsquigarrow$

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Infinite dimensional problem? Not quite!

Annihilators

Module structure of annihilators

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Associate polynomials with left kernel vectors:

$$\begin{bmatrix} r_0 & r_1 & \cdots & r_L & 0 & \cdots \end{bmatrix} \rightsquigarrow r(\xi) := r_0 + r_1\xi + \cdots + r_L\xi^L$$

Then $r(\xi), \xi r(\xi), \cdots$ **also represent left annihilators of** $\mathscr{H}(w)$

Module structure of annihilators

Denote the set of annihilators of $\mathscr{H}(w)$ with

$$\mathscr{N}(\mathscr{H}(w)) := \{r_0 + r_1\xi + \cdots + r_n\xi^n \in \mathbb{R}^{1\times \mathtt{w}}[\xi] \mid \begin{bmatrix} r_0 & r_1 & \cdots & r_n & 0 & \cdots \end{bmatrix} \in \textbf{ left kernel } \mathscr{H}(w)\}$$

Then $\mathscr{N}(\mathscr{H}(w))$ is a submodule of $\mathbb{R}^{1\times \mathtt{w}}[\xi]$, and consequently it is finitely generated: there exist basis elements $a_1(\xi), \cdots, a_p(\xi) \in \mathbb{R}^{1\times \mathtt{w}}[\xi]$ such that for every $b \in \mathscr{N}(\mathscr{H}(w))$

$$\exists\, g_1(\xi), \cdots, g_p(\xi) \in \mathbb{R}[\xi] \textbf{ s.t. } b(\xi) = \sum_{i=1}^{p} g_i(\xi) a_i(\xi)$$

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Not quite "finite-dimensional", but "almost".

Recursive computation of the MPUM

Recursive computation of kernel representation of MPUM

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Problem: find basis for the submodule $\mathscr{N}\left(\mathscr{H}\left(w\right)\right)$

Recursive computation of kernel representation of MPUM

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Basic idea: compute annihilators one by one, at each step using the previous annihilators in order to get a new one.

Recursive computation of kernel representation of MPUM

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Basic idea: compute annihilators one by one, at each step using the previous annihilators in order to get a new one.

Basic technique: unimodular completion of a polynomial matrix

Unimodular completion of a polynomial matrix

$R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ **is** **left-prime** **if** $R(\lambda)$ **has full row rank** $\forall\ \lambda \in \mathbb{C}$**.**

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- **Unimodular completion:** $\exists\, E \in \mathbb{R}^{(\mathtt{w}-\mathtt{p}) \times \mathtt{w}}[\xi]$ **such that**

$$\begin{bmatrix} R \\ E \end{bmatrix}$$

is unimodular.

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Special case $\mathtt{w} = 2$ **leads to Bézout equation**

$$\det\left(\begin{bmatrix} r_1(\xi) & r_2(\xi) \\ e_1(\xi) & e_2(\xi) \end{bmatrix}\right) = r_1(\xi)e_2(\xi) - r_2(\xi)e_1(\xi) = 1$$

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Equivalent with:

- $R = FR' \implies F$ **is unimodular.**
- **Unimodular completion**: $\exists\, E \in \mathbb{R}^{(\mathtt{w}-\mathtt{p}) \times \mathtt{w}}[\xi]$ **such that**

$$\begin{bmatrix} R \\ E \end{bmatrix}$$

is unimodular.

Completion is not unique. Algorithms to compute one available.

Unimodular completion of a polynomial matrix

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Equivalent with:

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- **Behavioral interpretation: If** $\mathscr{B} := \mathbf{kernel}\, R(\sigma)$ **is controllable, then there exists** $\mathscr{B}' := \mathbf{kernel}\, E(\sigma)$ **such that**

$$\mathscr{B} \oplus \mathscr{B}' = (\mathbb{R}^{\mathtt{w}})^{\mathbb{N}}$$

Recursive computation of an MPUM representation

Let $r(\xi) = r_0 + r_1\xi + \cdots + r_L\xi^n \in \mathscr{N}(\mathscr{H}(w))$**, i.e.**

$$\begin{bmatrix} r_0 & r_1 & \cdots & r_L \end{bmatrix} \begin{bmatrix} w(0) & w(1) & \cdots & w(t'') & \cdots \\ w(1) & w(2) & \cdots & w(t''+1) & \cdots \\ w(2) & w(3) & \cdots & w(t''+2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w(L) & w(L+1) & \cdots & w(L+t'') & \cdots \end{bmatrix} = 0$$

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Recursive computation of an MPUM representation

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Compute annihilator $r'(\xi)$ **for the error:**

$$\begin{bmatrix} r'_0 & r'_1 & \cdots & r'_{L'} \end{bmatrix} \begin{bmatrix} e(0) & e(1) & \cdots & e(t'') & \cdots \\ e(1) & e(2) & \cdots & e(t''+1) & \cdots \\ e(2) & e(3) & \cdots & e(t''+2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e(L') & e(L'+1) & \cdots & e(L'+t'') & \cdots \end{bmatrix} = 0$$

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Define error $e := E_r(\sigma)w$**, a** $(\mathtt{w}-1)$**-dimensional time-series.**

Compute annihilator $r'(\xi)$ **for the error. Now**

$$r'(\sigma)E_r(\sigma)w = r'(\sigma)\,(E_r(\sigma)w) = r'(\sigma)e = 0$$

i.e. $r'(\xi)E_r(\xi)$ **is annihilator of** w**.**

Recursive computation of an MPUM representation

Let $r(\xi) = r_0 + r_1\xi + \cdots + r_L\xi^n \in \mathscr{N}(\mathscr{H}(w))$**.**

Compute a unimodular completion E_r **of** r**.**

Define error $e := E_r(\sigma)w$**, a** $(\mathtt{w}-1)$**-dimensional time-series.**

Compute annihilator $r'(\xi)$ **for the error:** $r'(\xi)E_r(\xi)$ **is 'new' annihilator of** w**.**

Compute unimodular completion $E_{r'} \in \mathbb{R}^{(\mathtt{w}-2)\times\mathtt{w}}[\xi]$ **of** $r'(\xi)E_r(\xi)$**; define error** $e' := E_{r'}(\sigma)e$**; find annihilator** r''**.**

Recursive computation of an MPUM representation

Let $r(\xi) = r_0 + r_1\xi + \cdots + r_L\xi^n \in \mathscr{N}(\mathscr{H}(w))$**.**

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$r''(\xi)E_{r'}(\xi)E_r(\xi)$ **is 'new' annihilator of** w**.**

Recursive computation of an MPUM representation

Let $r(\xi) = r_0 + r_1\xi + \cdots + r_L\xi^n \in \mathcal{N}(\mathcal{H}(w))$**.**

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$r''(\xi)E_{r'}(\xi)E_r(\xi)$ **is 'new' annihilator of** w**.**

Continue until error is zero.

Summary

- **A language for modeling**

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- **The most powerful unfalsified model**

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- A language for modeling from data
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- The Hankel matrix is key
- Recursive computation of the MPUM