European Embedded Control Institute

Graduate School on Control — Spring 2010

The Behavioral Approach to Modeling and Control

Lecture X

ENERGY FLOW in SYSTEMS

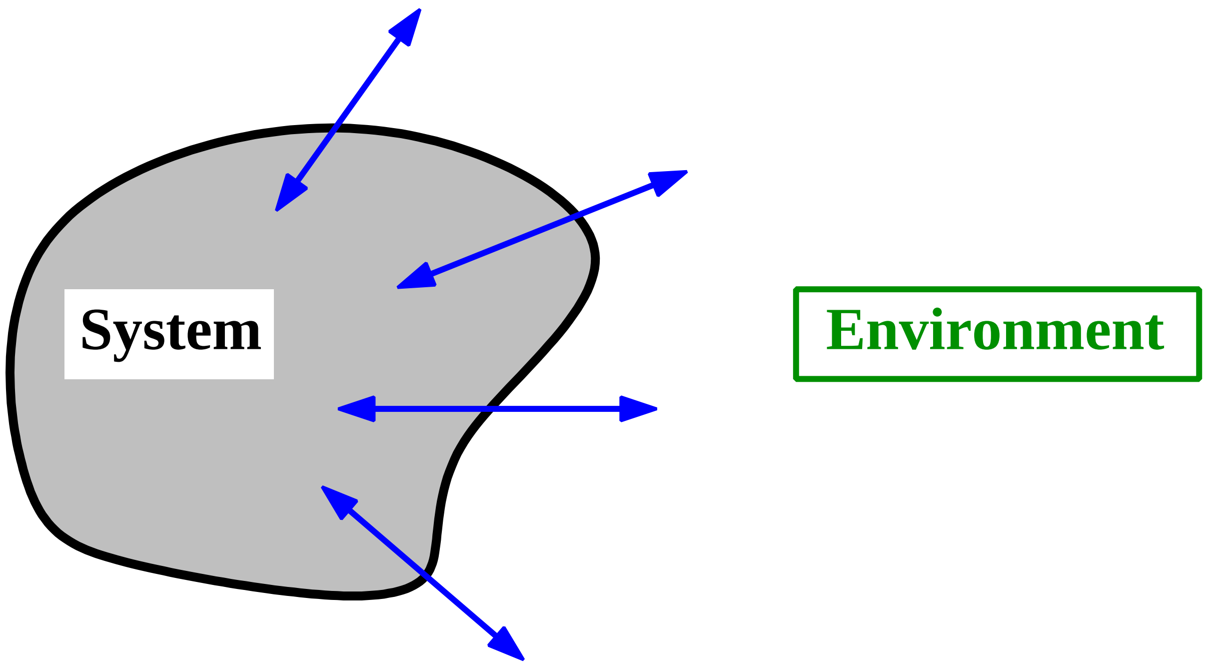
Theme

- How is energy transferred from the environment to a system?
- How is energy transferred between systems?
- Are energy transfer and interconnection related?
- How are passive systems synthesized?

Outline

- Motivation
- KVL, KCL, IUM, and KFL
- Building blocks
- Energy transfer
- Ports
- Circuit synthesis
- The inerter
- Motion energy

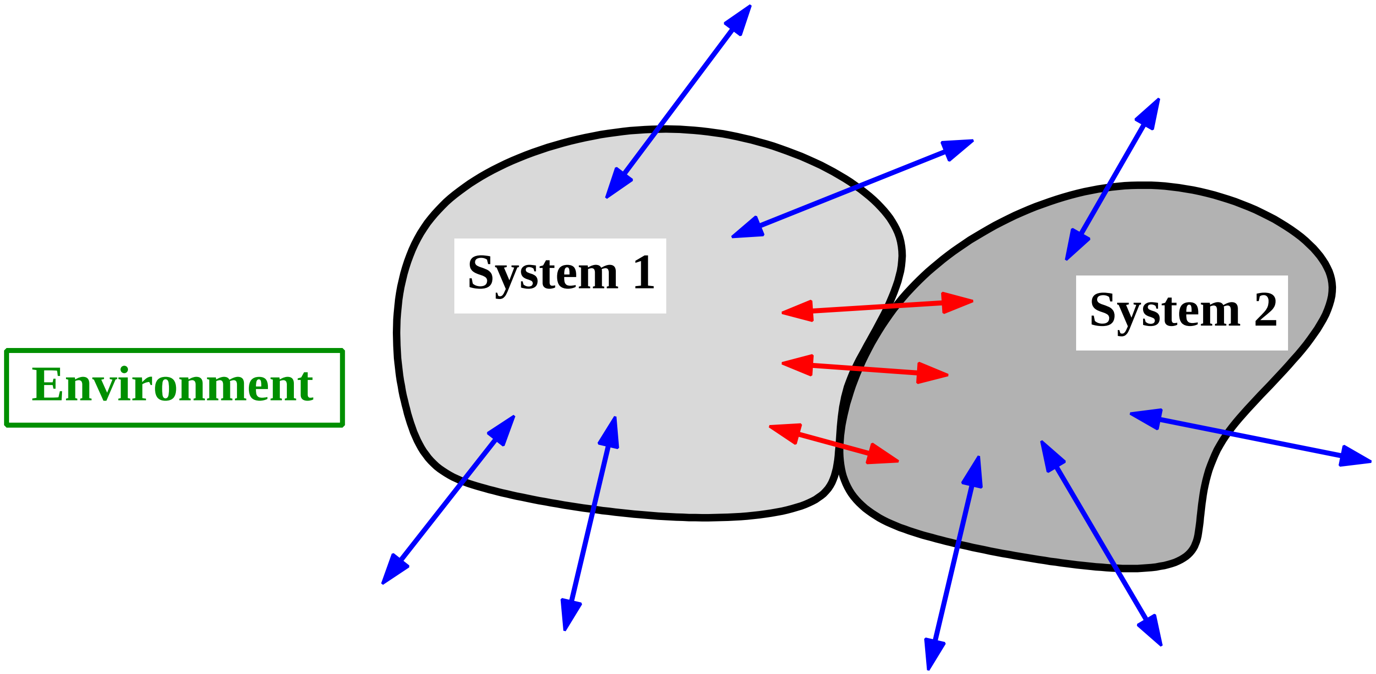
Open systems



Systems are 'open', they interact with their environment.

How is energy transferred from the environment to a system?

Interacting systems



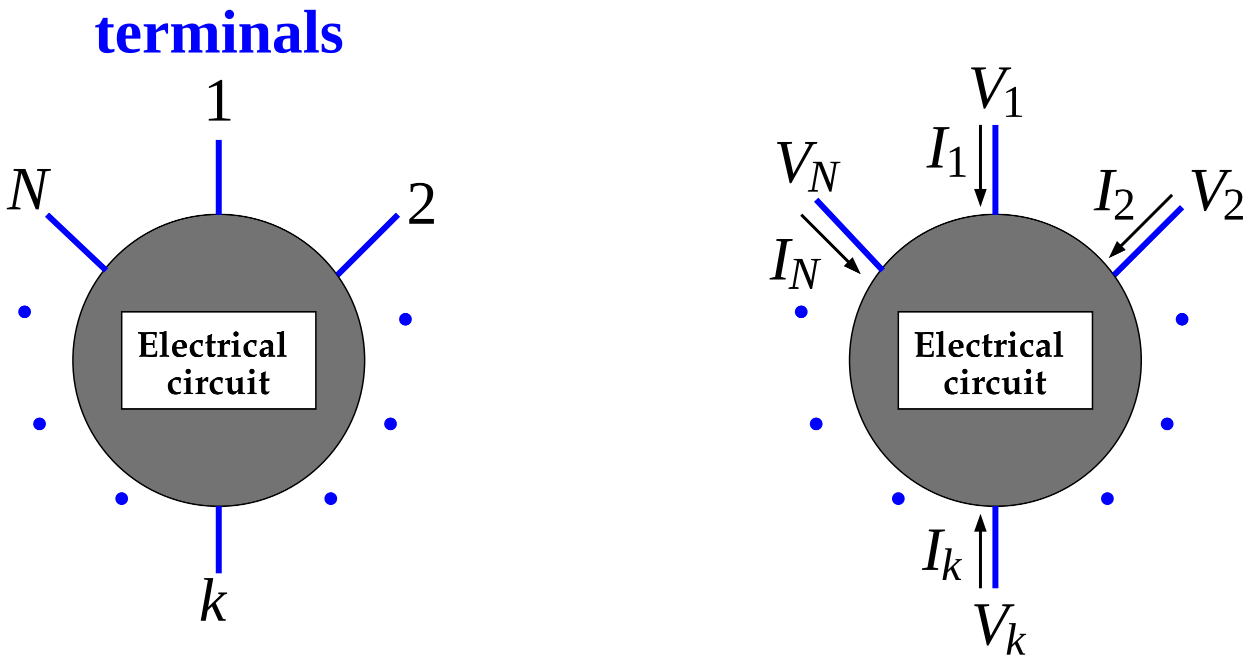
Interconnected systems interact.

How is energy transferred between systems?

Are energy transfer and interconnection related?

Systems with terminals

Electrical circuit



At each terminal:

a potential (!) and a current (counted > 0 into the circuit),

$$\rightsquigarrow \textbf{behavior } \mathscr{B} \subseteq \left(\mathbb{R}^N \times \mathbb{R}^N\right)^{\mathbb{R}}.$$

$(V_1, V_2, \ldots, V_N, I_1, I_2, \ldots, I_N) \in \mathscr{B}$ **means:**

this potential/current trajectory is compatible with the circuit architecture and its element values.

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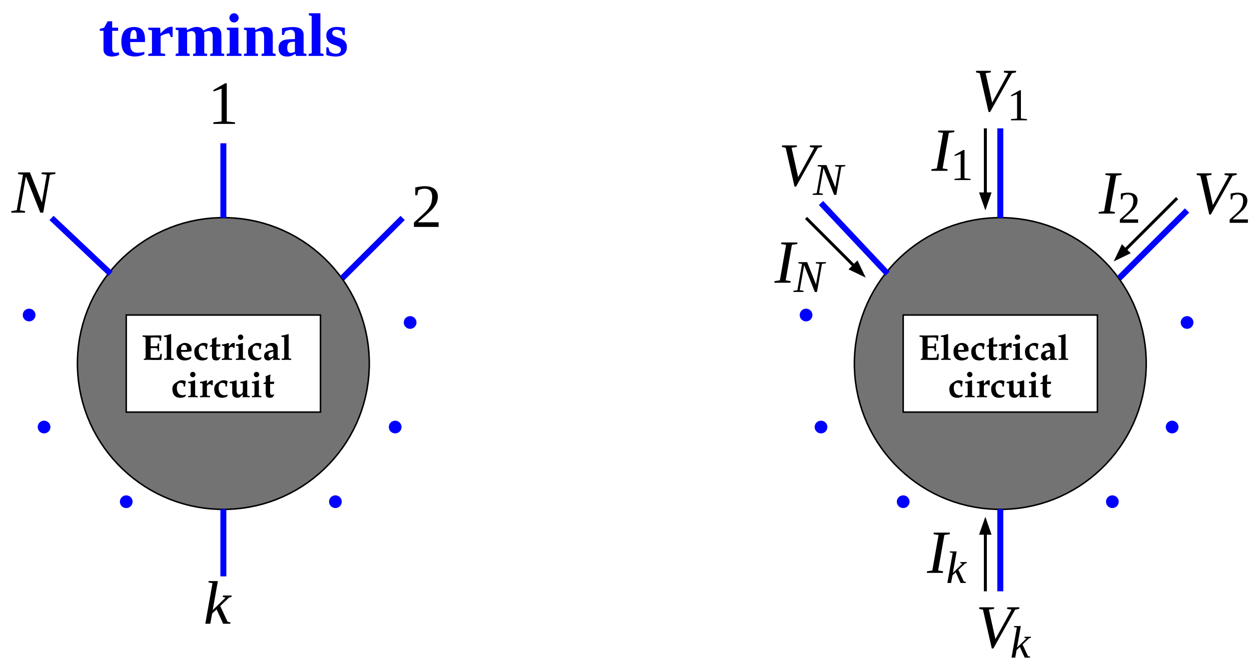
Early sources:

Brockway McMillan

Robert Newcomb

KVL and KCL

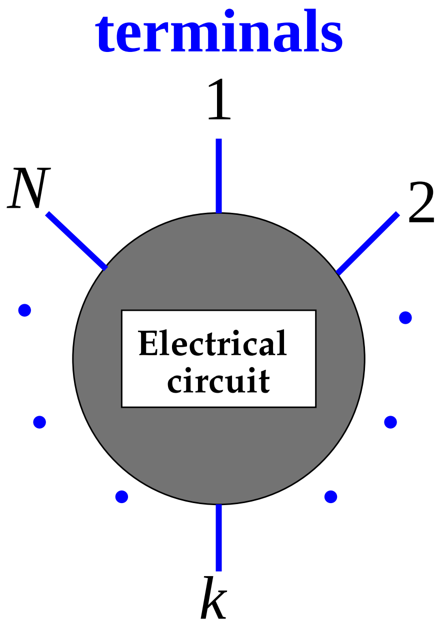


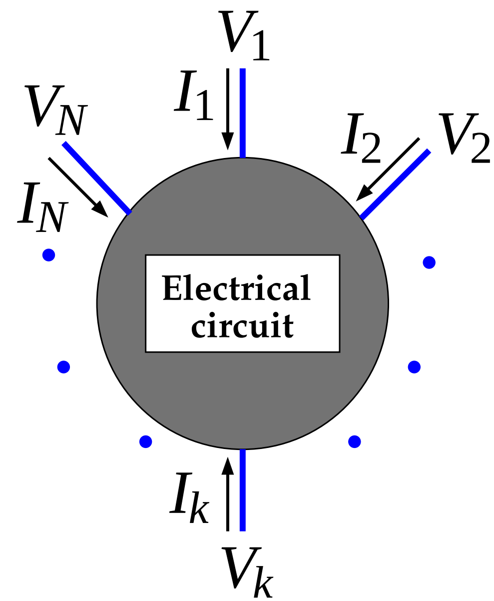
Kirchhoff's voltage law (KVL):

$$[\![(V_1, V_2, \ldots, V_N, I_1, I_2, \ldots, I_N) \in \mathscr{B} \textbf{ and } \alpha : \mathbb{R} \to \mathbb{R}]\!]$$
$$\Rightarrow [\![(V_1 + \alpha, V_2 + \alpha, \ldots, V_N + \alpha, I_1, I_2, \ldots, I_N) \in \mathscr{B}]\!].$$

Equivalently, the behavioral equations contain the V_i's only through the potential differences $V_i - V_j$.

KVL and KCL





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$$\Rightarrow [\![(V_1 + \alpha, V_2 + \alpha, \ldots, V_N + \alpha, I_1, I_2, \ldots, I_N) \in \mathscr{B}]\!].$$

Kirchhoff's current law (KCL):

$$[\![(V_1, V_2, \ldots, V_N, I_1, I_2, \ldots, I_N) \in \mathscr{B}]\!] \Rightarrow [\![I_1 + I_2 + \cdots + I_N = 0]\!].$$

Circuit properties

An N-terminal circuit is said to be

- $[\![$ **linear** $]\!] :\Leftrightarrow [\![\mathscr{B} \subseteq \left(\mathbb{R}^N \times \mathbb{R}^N\right)^{\mathbb{R}}$ **is linear** $]\!]$
- $[\![$ **time-invariant** $]\!] :\Leftrightarrow [\![\sigma^t \mathscr{B} = \mathscr{B}$, **with** σ^t **the** t**-shift** $]\!]$
- $[\![$ **a linear time-invariant differential system (LTIDS)** $]\!]$
  $:\Leftrightarrow [\![\cdots]\!]$

Circuit properties

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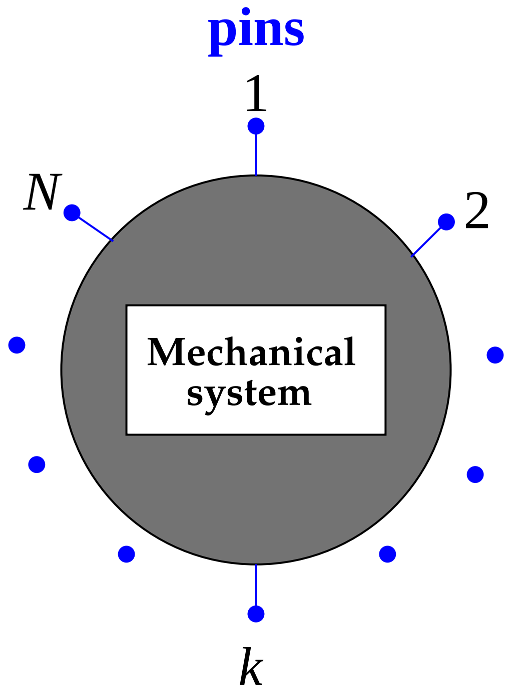
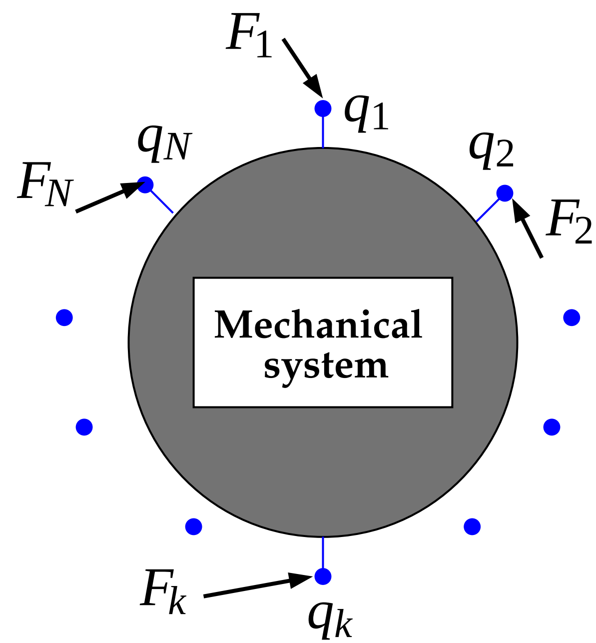
- $[\![$ **a linear time-invariant differential system (LTIDS)** $]\!]$
  $:\Leftrightarrow [\![\cdots]\!]$
- $[\![$ **reciprocal** $]\!]$ $\Leftrightarrow [\![\cdots]\!]$
- $[\![$ **passive** $]\!]$ $:\Leftrightarrow [\![\cdots]\!]$
- $\cdots$

Mechanical device





At each terminal: a position and a force.

$\rightsquigarrow$ **position/force trajectories** $(q, F) \in \mathscr{B} \subseteq ((\mathbb{R}^{\bullet})^{2N})^{\mathbb{R}}$.

More generally, a position, force, angle, and torque.

Mechanical properties

$\mathscr{B}$ **satisfies invariance under uniform motion (IUM)** $:\Leftrightarrow$
$(q_1, q_2, \ldots, q_N, \quad F_1, F_2, \ldots, F_N) \in \mathscr{B}$ **and**
$v : t \in \mathbb{R} \mapsto (a + bt) \in \mathbb{R}^\bullet$ **imply**

$$(q_1 + v, \, q_2 + v, \ldots, q_N + v, F_1, F_2, \ldots, F_N) \in \mathscr{B}.$$

$\rightsquigarrow$ **other symmetries (rotation, Euclidean group), etc.**

Mechanical properties

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$\rightsquigarrow$ **other symmetries (rotation, Euclidean group), etc.**

$\mathscr{B}$ **satisfies Kirchhoff's force law (KFL)** $:\Leftrightarrow$

$$[\![(q_1, q_2, \ldots, q_N, F_1, F_2, \ldots, F_N) \in \mathscr{B}]\!]$$

$$\Rightarrow [\![\, F_1 + F_2 + \cdots + F_N = 0 \,]\!].$$

KFL is, contrary to IUM, not a universal law.

2-terminal behavior

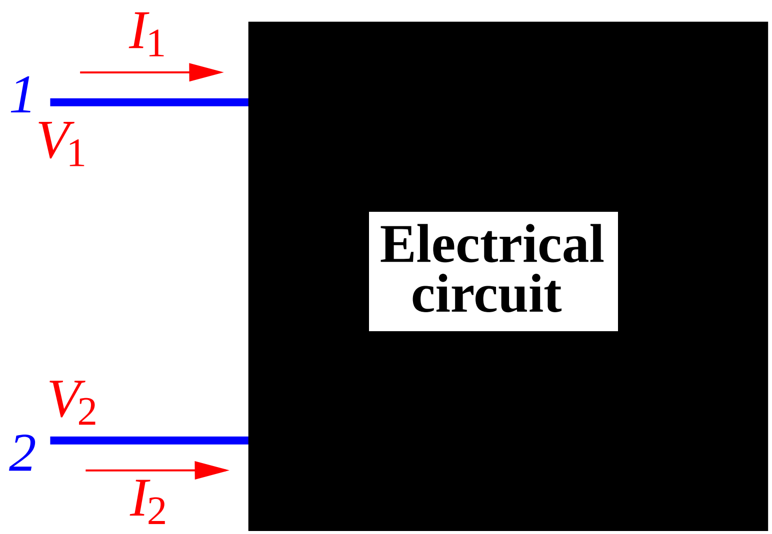
Consider a 2-terminal circuit.

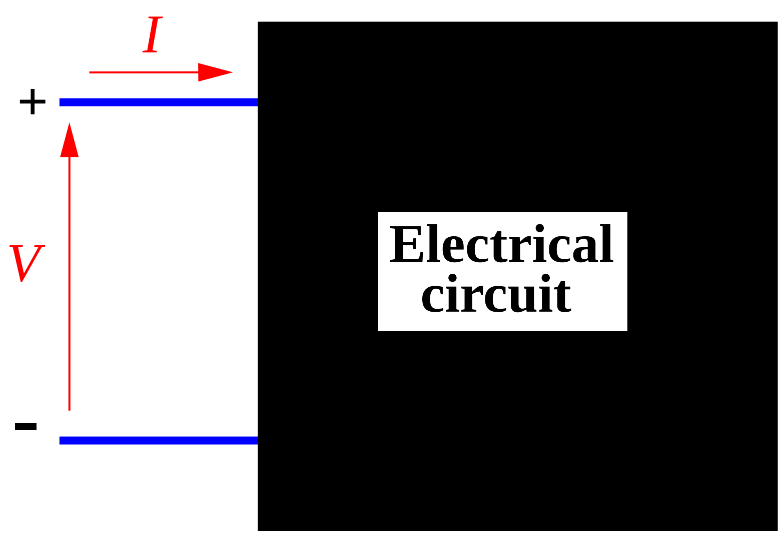
Assume that KVL and KCL hold.

$\rightsquigarrow$ **variables:**

voltage $V = V_1 - V_2$ **across**

current $I = I_1 = -I_2$ **into the circuit along terminal 1.**





Building blocks

2-terminal electrical devices

There are 4 basic variables involved in 2-terminal circuits.

$$V = \textbf{the voltage},$$
$$I = \textbf{the current},$$
$$Q = \textbf{the charge},$$
$$\Phi = \textbf{the flux}.$$

2-terminal electrical devices

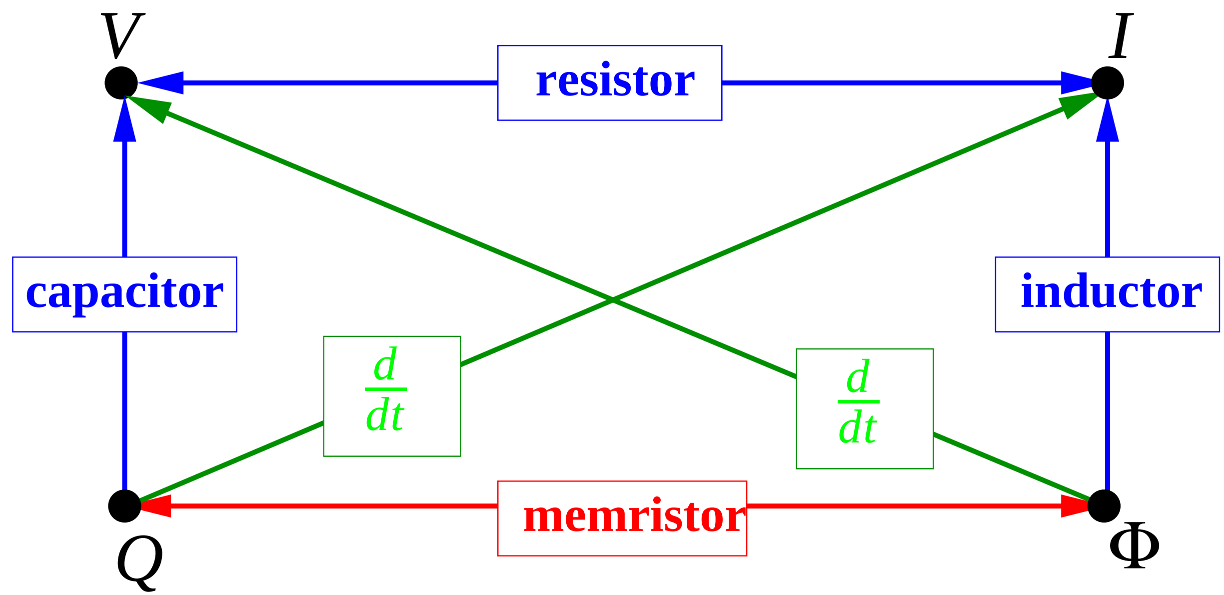
There are 4 basic variables involved in 2-terminal circuits.

V = **the voltage**,

I = **the current**,

Q = **the charge**,

Φ = **the flux**.



These variables are connected by laws and devices.

2-terminal electrical devices

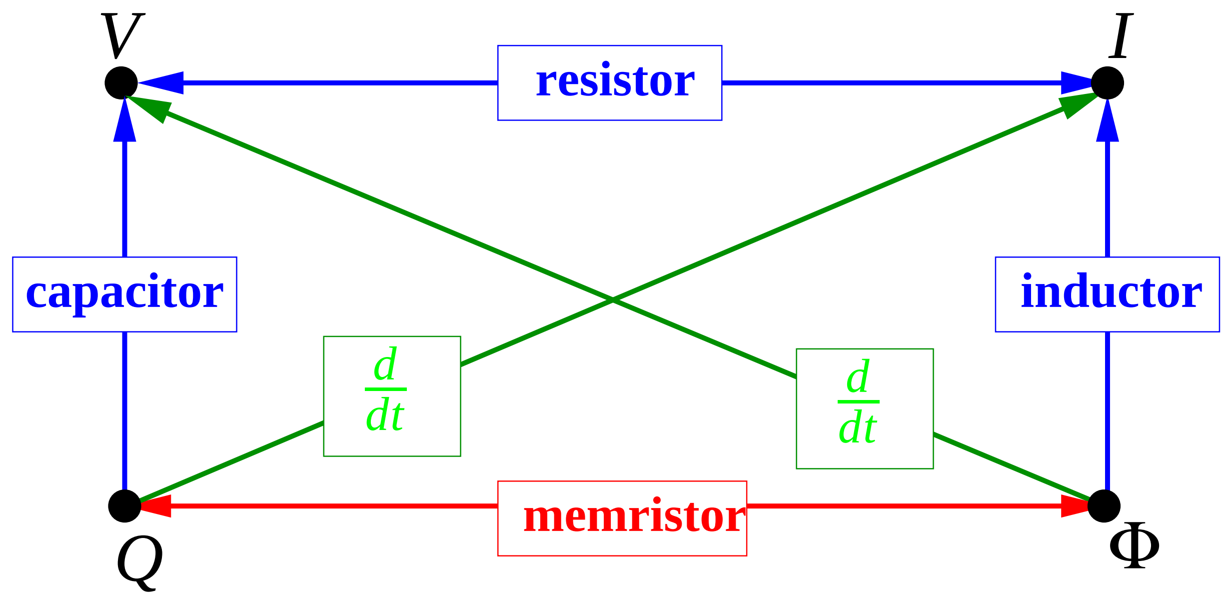
There are 4 basic variables involved in 2-terminal circuits.

V $=$ **the voltage**,

I $=$ **the current**,

Q $=$ **the charge**,

Φ $=$ **the flux**.



The current is the time-derivative of the electrical charge:

$$\frac{d}{dt}Q = I.$$

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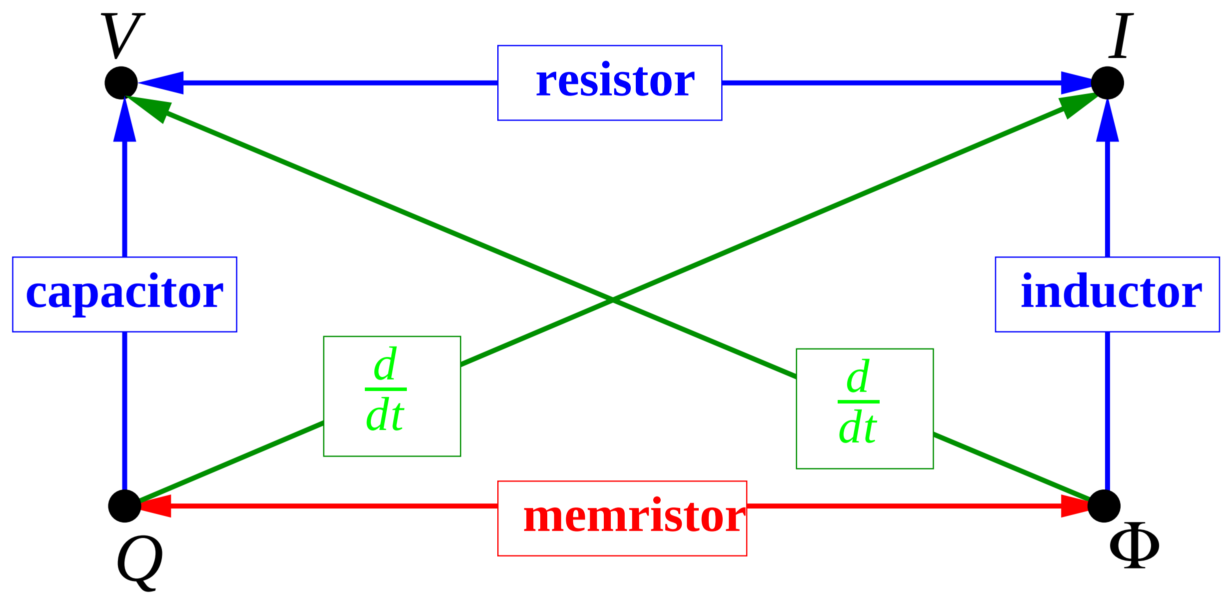

The voltage is the time-derivative of the magnetic flux:

$$\frac{d}{dt}\Phi = V$$

(law of Faraday-Lenz)

Michael Faraday
1791–1867

Heinrich Lenz
1804–1865

2-terminal electrical devices

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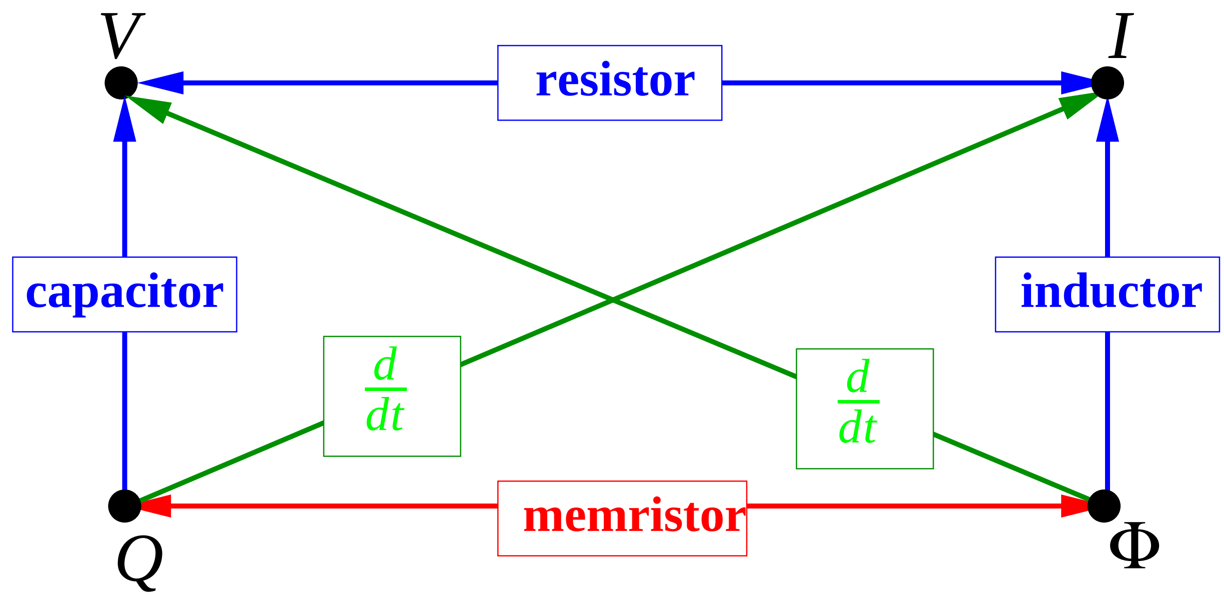
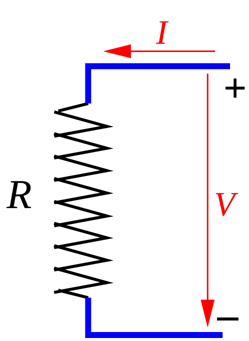
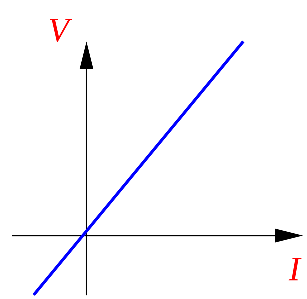
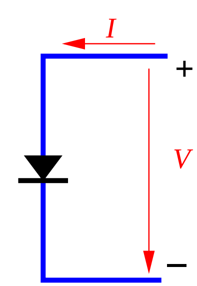
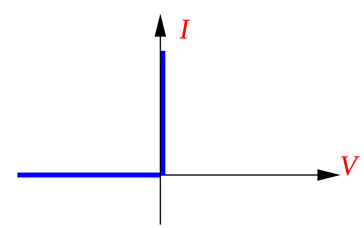


Devices that relate the current and the voltage, I and V, $R(I,V) = 0$, are called resistors. For example,









$$\{V = 0 \wedge I \geq 0\} \vee \{V \geq 0 \wedge I = 0\}$$

$V = RI$ **(Ohmic resistor)** **(Ideal diode)**

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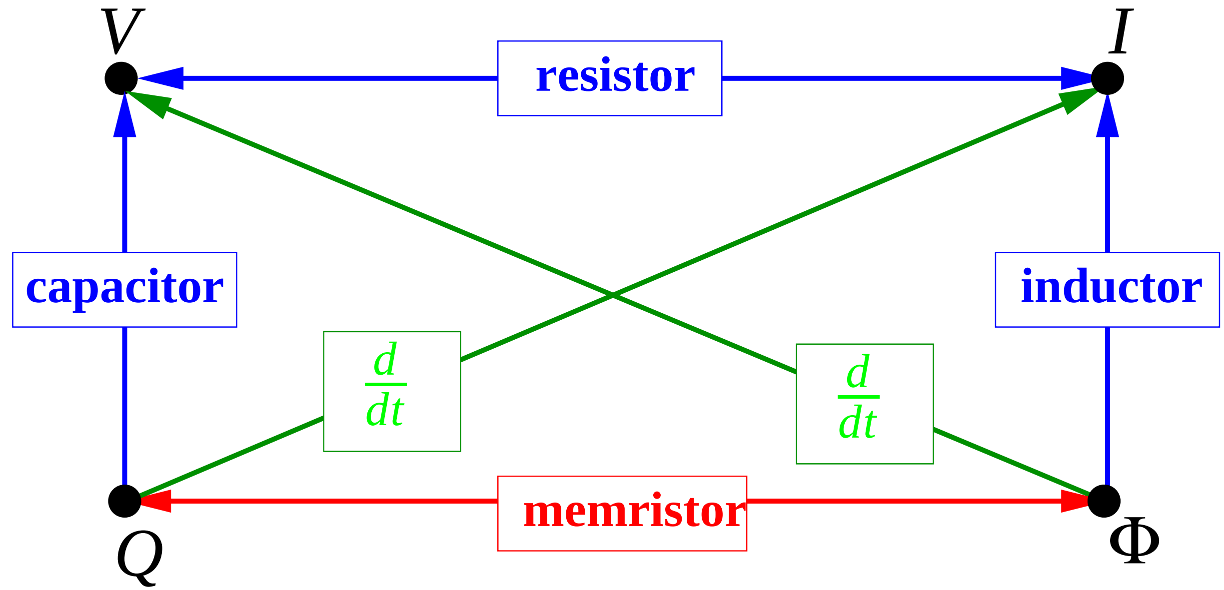
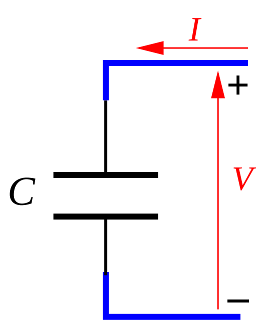


Devices that relate the voltage and the electrical charge, V and Q, $C(V,Q)=0$, are called capacitors. For example,



$$Q = CV$$

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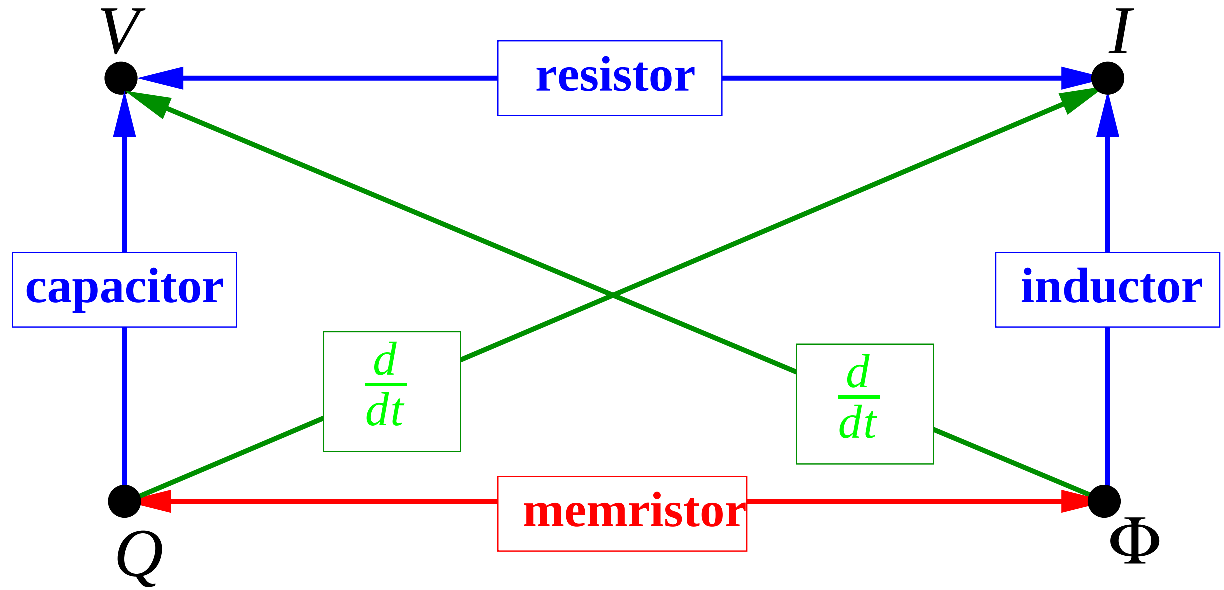
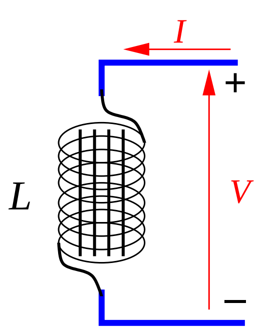


Devices that relate the current and the magnetic flux, I and Φ, $L(I,\Phi)=0$, are called inductors. For example,



$$\Phi = LI$$

2-terminal electrical devices

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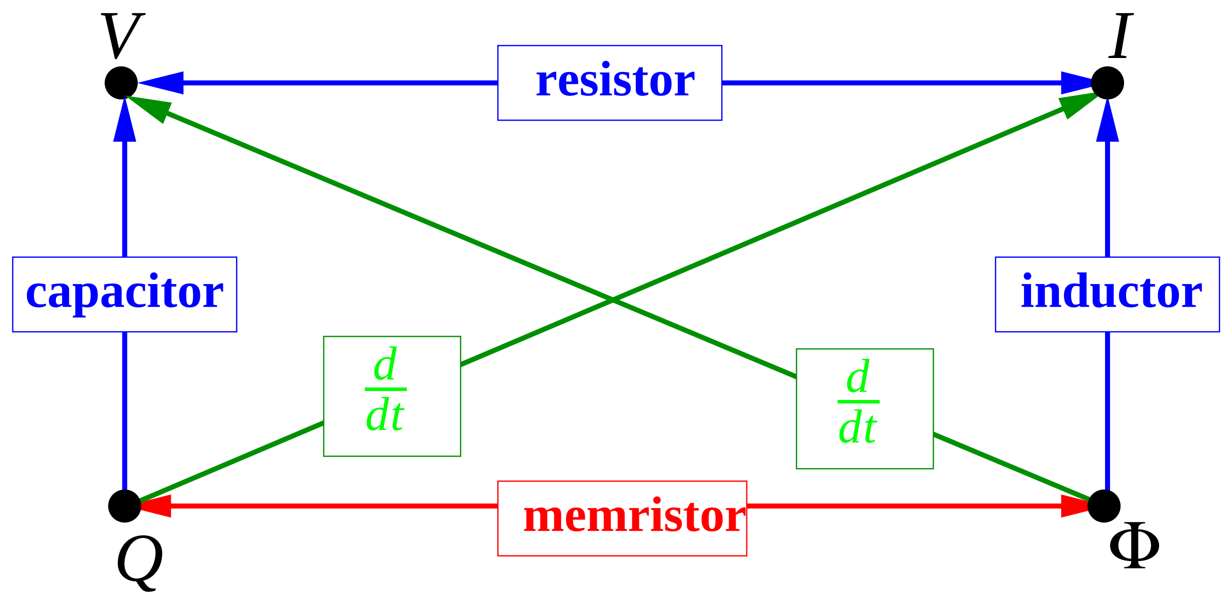
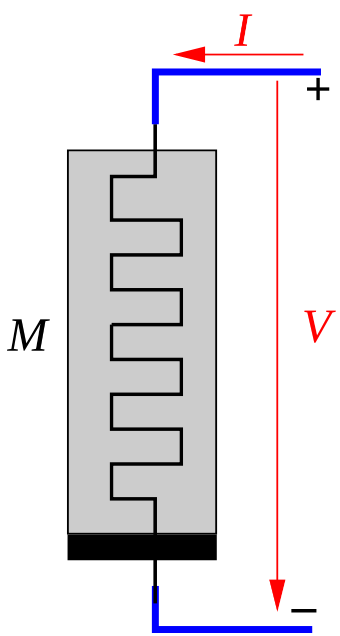


Resistors, capacitors, and inductors are the classical 2-terminal circuit elements.

Are there devices that relate Q and Φ?

The missing element: the memristor

Devices that relate the electrical charge and the magnetic flux, Q and Φ, $M(Q,\Phi) = 0,$ are called memristors.



The missing element: the memristor

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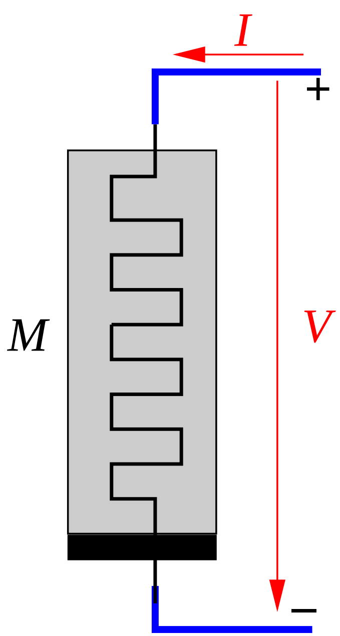

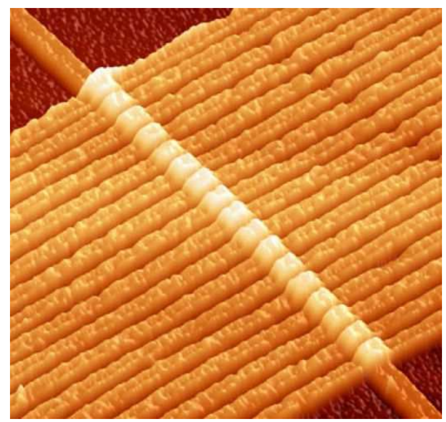

The existence of this device was postulated by Chua in 1971. In 2009, it was manufactured by HP.

Leon Chua (1936–)

The missing element: the memristor

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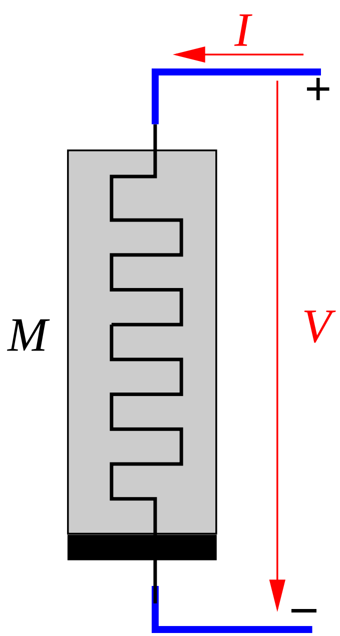


$$\Phi = \widehat{M}(Q) \quad \rightsquigarrow \quad V = R(Q)I, \quad R = \widehat{M}\,',$$

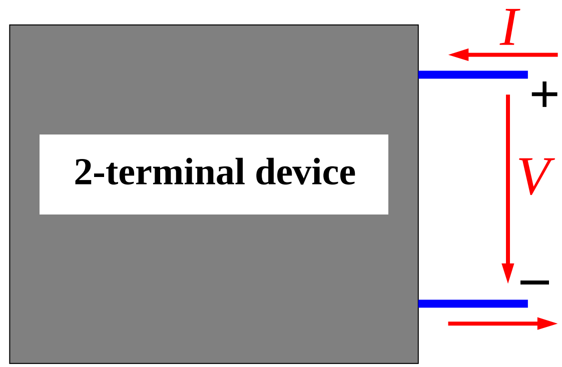
a charge-controlled resistor.

$$Q = \widehat{\widehat{M}}(Q) \quad \rightsquigarrow \quad I = G(Q)V, \quad G = \widehat{\widehat{M}}\,',$$

a flux-controlled resistor.

$'$ **denotes derivative.**

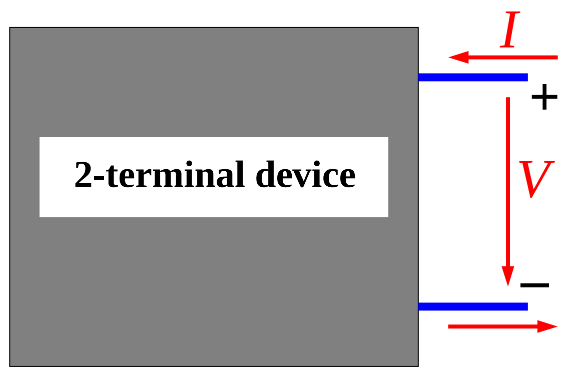
Terminal behavior



resistor	$R(V,I) = 0,$
capacitor	$C(V,Q) = 0, \quad \frac{d}{dt}Q = I,$
inductor	$L(I,\Phi) = 0, \quad \frac{d}{dt}\Phi = V,$
memristor	$M(Q,\Phi) = 0, \quad \frac{d}{dt}Q = I, \quad \frac{d}{dt}\Phi = V.$

Q and Φ are latent variables that cannot be eliminated in the nonlinear case.

Terminal behavior



Linear case :

resistor $V = RI, \quad \textbf{or} \quad I = GI,$

capacitor $C\frac{d}{dt}V = I,$

inductor $L\frac{d}{dt}I = V,$

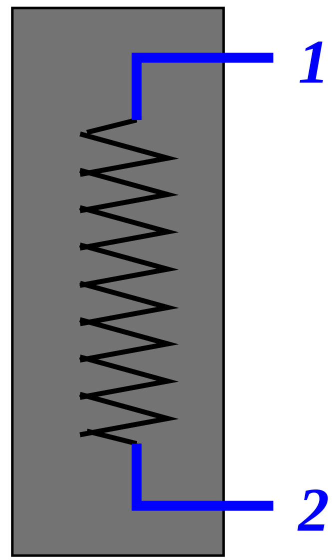
memristor $V = RI, \quad \textbf{or} \quad I = GI.$

Note that a linear memristor is a resistor.
It is a device that is useful only in the nonlinear case.

The classical electrical elements

Linear 2-terminal circuit elements

Resistor



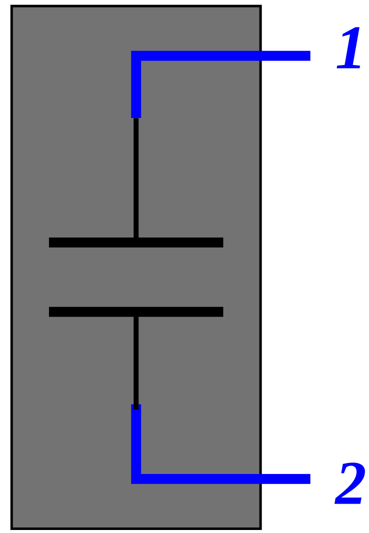
$$V_1 - V_2 = RI_1 \qquad I_1 + I_2 = 0$$

R = 'resistance'

Satisfies KVL and KCL.

Linear 2-terminal circuit elements

Capacitor



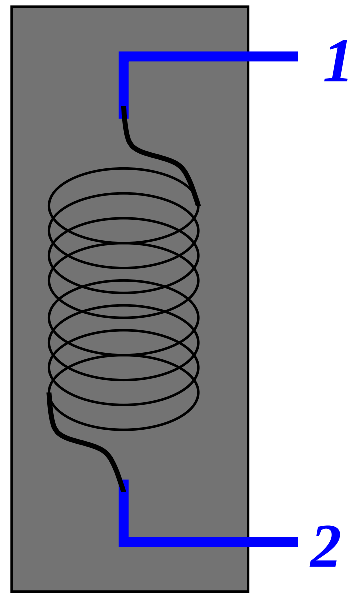
$$C\frac{d}{dt}\left(V_1 - V_2\right) = I_1 \qquad I_1 + I_2 = 0$$

C = ‘capacitance’

Satisfies KVL and KCL.

Linear 2-terminal circuit elements

Inductor



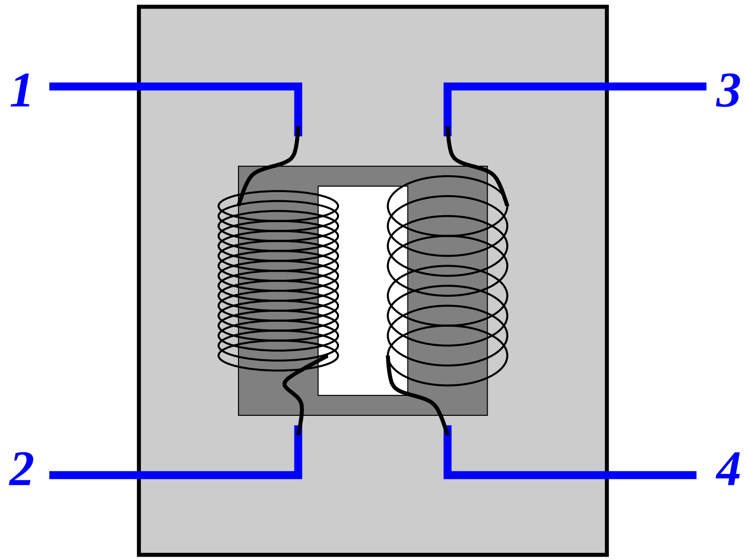
$$L\frac{d}{dt}I_1 = V_1 - V_2 \qquad I_1 + I_2 = 0$$

L = 'inductance'

Satisfies KVL and KCL.

Examples of 4-terminal circuit elements

Transformer



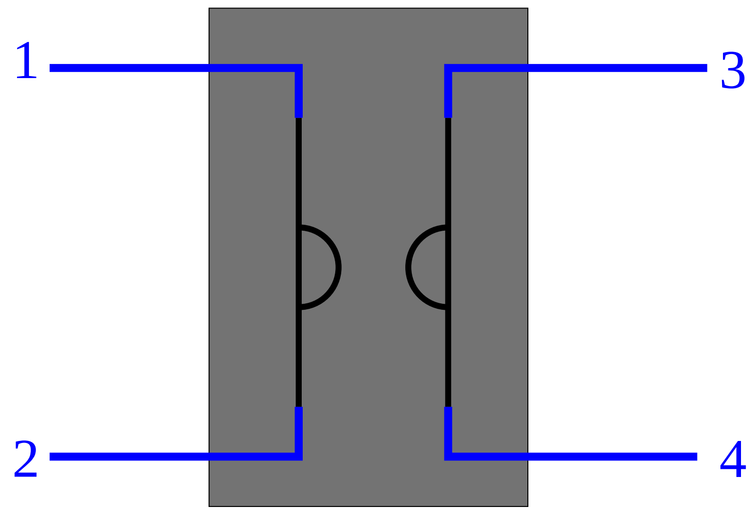
$$V_1 - V_2 = n\left(V_3 - V_4\right), -nI_1 = I_3 \qquad I_1 + I_2 = 0, I_3 + I_4 = 0$$

n = 'turns ratio'

Satisfies KVL and KCL.

Examples of 4-terminal circuit elements

Gyrator



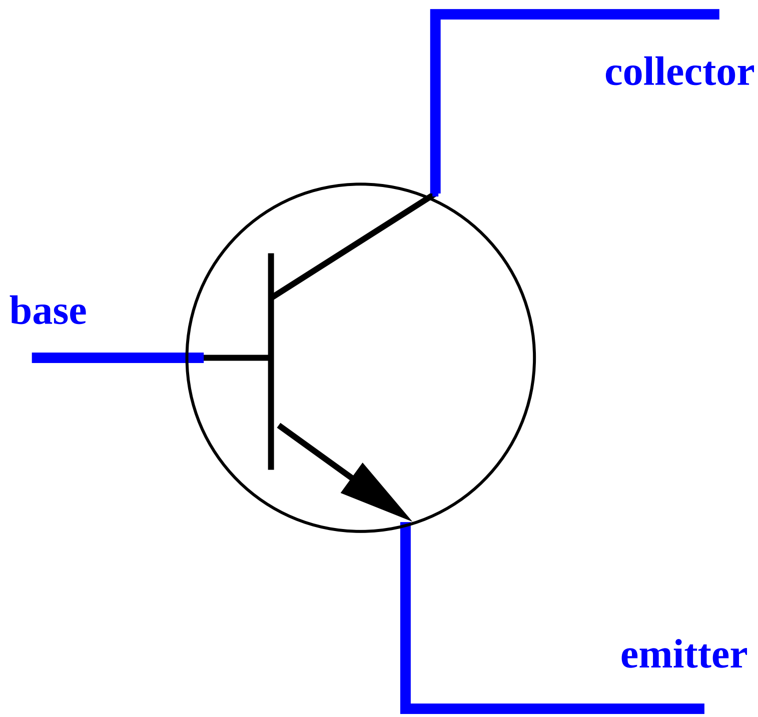
$$V_1 - V_2 = gI_3, V_3 - V_4 = -gI_1 \qquad I_1 + I_2 = 0, I_3 + I_4 = 0$$

g = 'gyrator resistance'

Satisfies KVL and KCL.

Example of a 3-terminal circuit element

pnp transistor

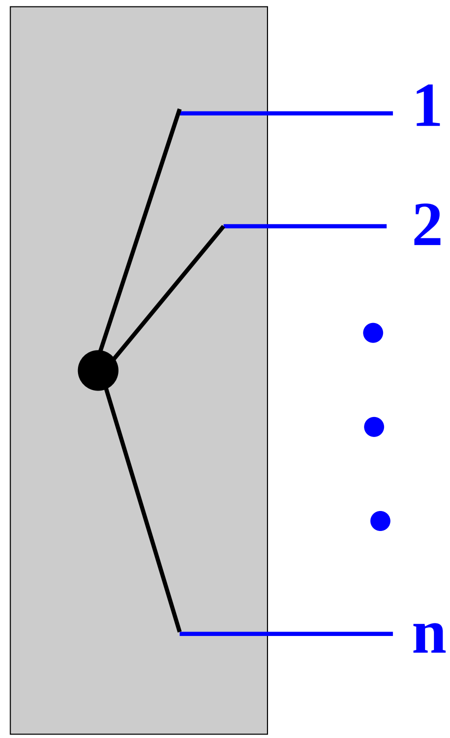


$$I_e = f_e(V_e - V_b, V_c - V_b), I_c = f_c(V_e - V_b, V_c - V_b), \quad I_e + I_c + I_b = 0.$$

Satisfies KVL and KCL.

Example of an n-terminal circuit element

Connector

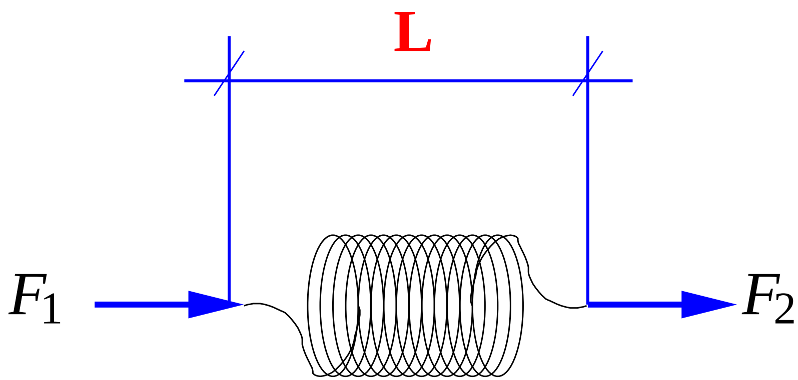


$$V_1 = V_2 = \cdots = V_n, \qquad I_1 + I_2 + \cdots + I_n = 0.$$

Satisfies KVL and KCL.

Linear mechanical building blocks

Spring

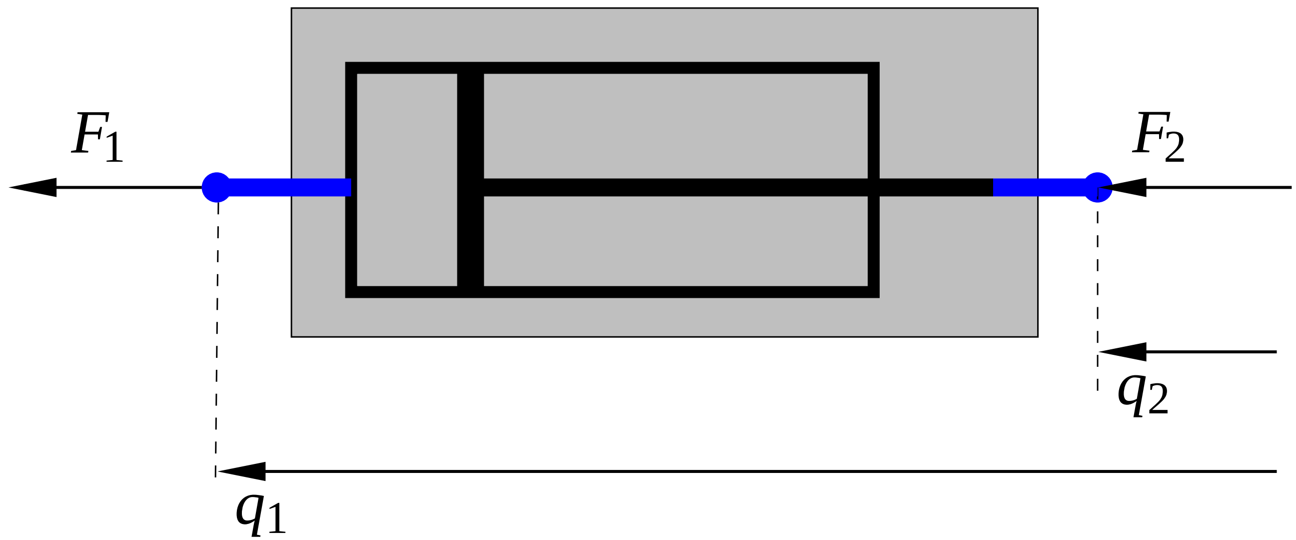


$$F_1 + F_2 = 0, \quad K(q_1 - q_2) = F_1$$

IUM and KFL

Linear mechanical building blocks

Damper

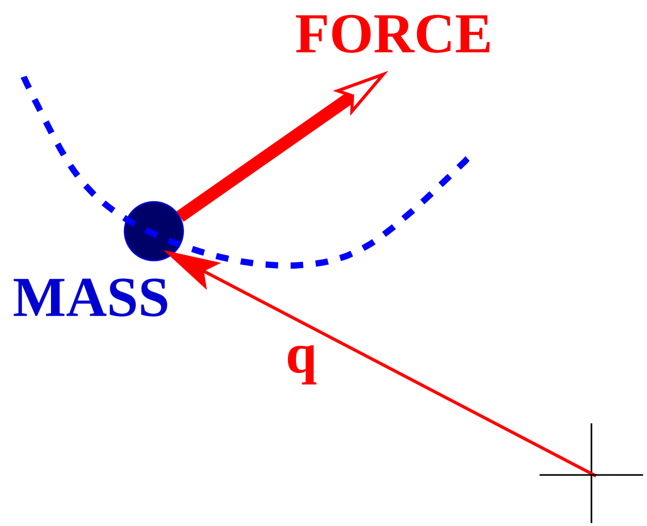


$$F_1 + F_2 = 0, \quad D\frac{d}{dt}(q_1 - q_2) = F_1.$$

IUM and KFL

Linear mechanical building blocks

Mass



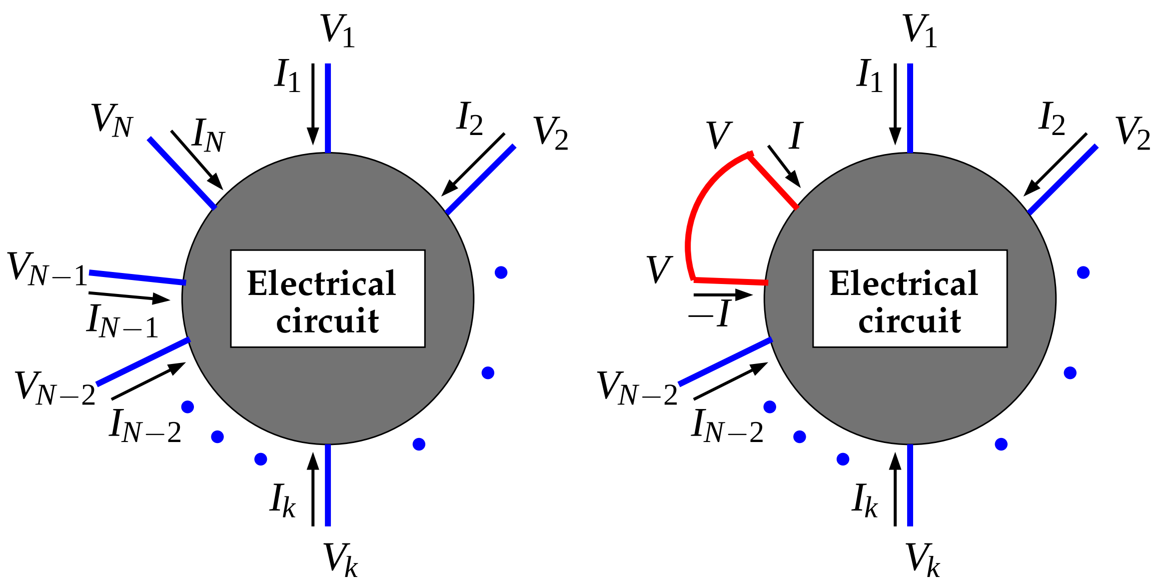
$$M \frac{d^2}{dt^2} q = F.$$

IUM, but not KFL

Interconnection

Connection of circuit terminals

Interconnection = connecting terminals, like soldering wires together.

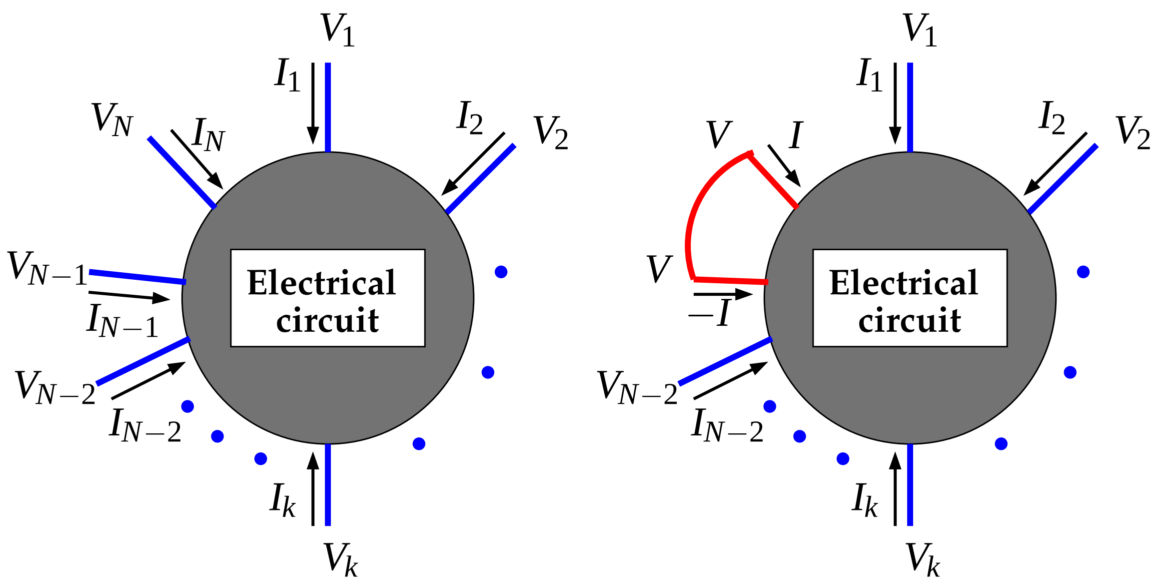


Connecting terminals $N-1$ and N leads to

$$V_{N-1} = V_N, \quad I_{N-1} + I_N = 0.$$

After interconnection the terminals share the variables V_{N-1}, V_N, and I_{N-1}, I_N (up to a sign).

Connection of circuit terminals



Connecting terminals $N-1$ and N leads to

$$V_{N-1} = V_N, \quad I_{N-1} + I_N = 0.$$

The interconnected circuit has $N-2$ terminals. Its behavior =

$$\mathscr{B}' = \{(V_1, I_1, V_2, I_2, \ldots, V_{N-2}, I_{N-2}) : \mathbb{R} \to \mathbb{R}^{2(N-2)} \mid \exists\ V, I$$

$$\text{such that } (V_1, I_1, V_2, I_2, \ldots, V_{N-2}, I_{N-2}, V, I, V, -I) \in \mathscr{B}\}.$$

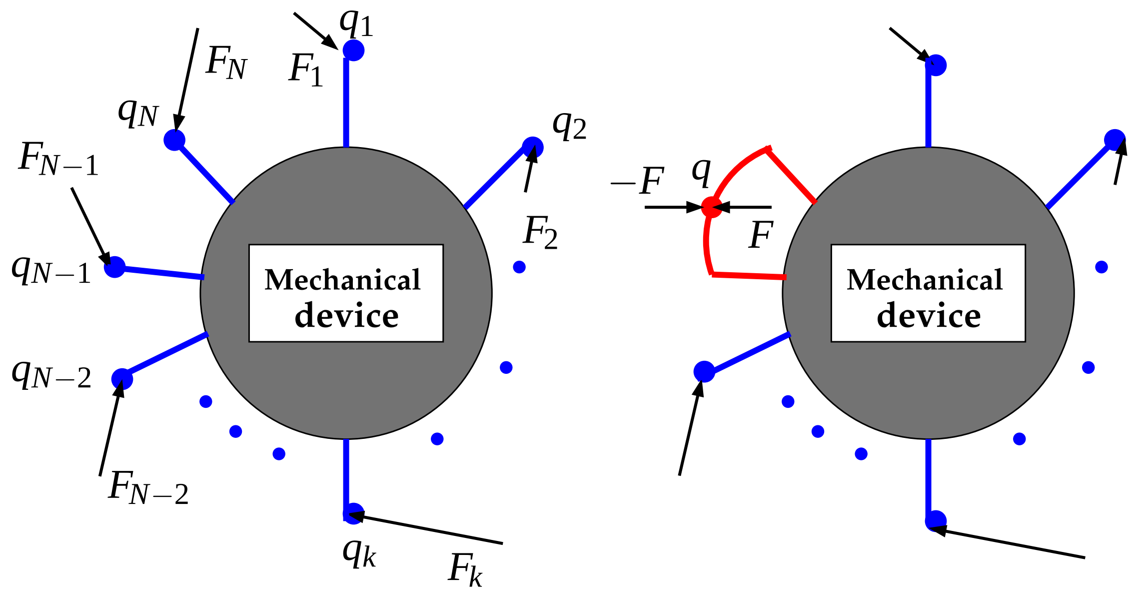
Preservation of properties under interconnection

- $[\![\mathscr{B}$ **satisfies KVL** $]\!] \Rightarrow [\![$**so does** $\mathscr{B}']\!]$
- $[\![\mathscr{B}$ **satisfies KCL** $]\!] \Rightarrow [\![$**so does** $\mathscr{B}']\!]$
- $[\![\mathscr{B}$ **linear** $]\!] \Rightarrow [\![\mathscr{B}'$ **linear** $]\!]$
- $\cdots$

An interconnection of resistors, inductors, capacitors, connectors, transformers, gyrators, transistors, etc. has a terminal behavior that satisfies KVL and KCL.

Connection of mechanical terminals

Interconnection = connecting terminals, like screwing pins together.

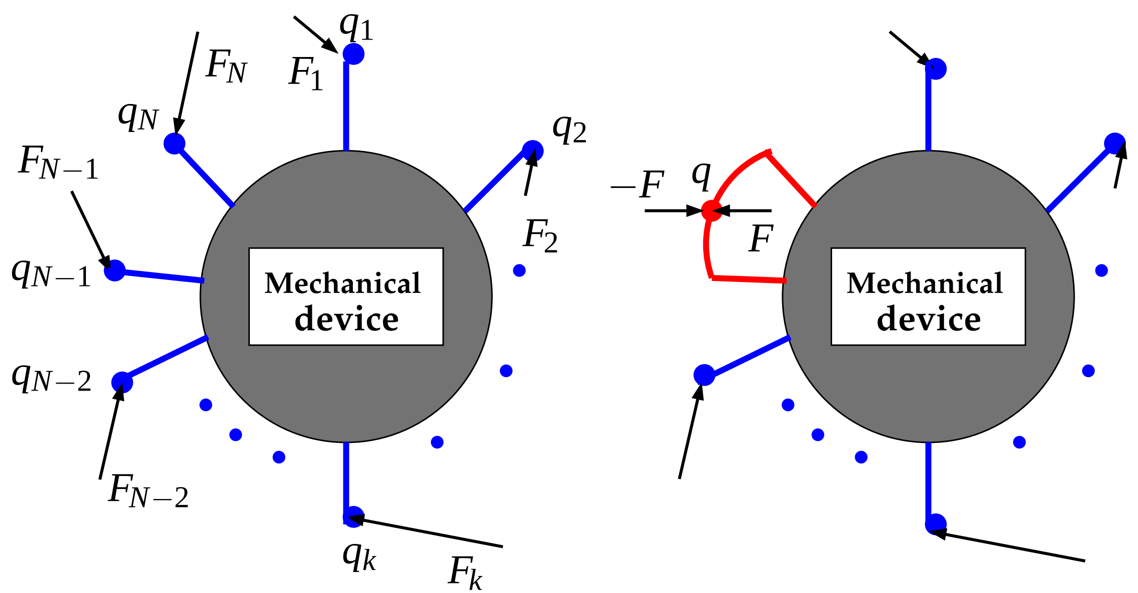


Connecting terminals $N-1$ and N leads to

$$q_{N-1} = q_N, \quad F_{N-1} + F_N = 0.$$

After interconnection the terminals share the variables q_{N-1}, q_N, and F_{N-1}, F_N (up to a sign).

Connection of mechanical terminals



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$$q_{N-1} = q_N, \quad F_{N-1} + F_N = 0.$$

The interconnected circuit has $N-2$ terminals. Its behavior =

$$\mathscr{B}' = \{(q_1, F_1, q_2, F_2, \ldots, q_{N-2}, F_{N-2}) : \mathbb{R} \to \mathbb{R}^{2(N-2)} \mid \exists\ q, F$$

$$\textbf{such that } (q_1, F_1, q_2, F_2, \ldots, q_{N-2}, F_{N-2}, q, F, q, -F) \in \mathscr{B}\}.$$

Preservation of properties under interconnection

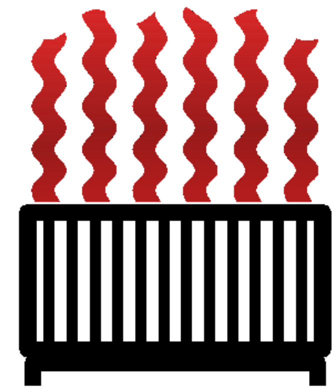
- $[\![\mathscr{B}$ **satisfies IUM** $]\!] \Rightarrow [\![$**so does** $\mathscr{B}']\!]$
- $[\![\mathscr{B}$ **satisfies KVL** $]\!] \Rightarrow [\![$**so does** $\mathscr{B}']\!]$
- $[\![\mathscr{B}$ **linear** $]\!] \Rightarrow [\![\mathscr{B}'$ **linear** $]\!]$
- $\cdots$

An interconnection of springs, dampers, and masses satisfies IUM.
An interconnection of springs and dampers satisfies KFL.

Energy transfer

Energy

Energy := a physical quantity transformable into heat.



Energy

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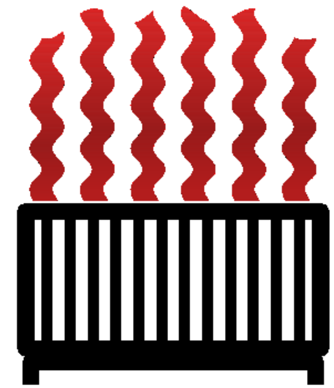
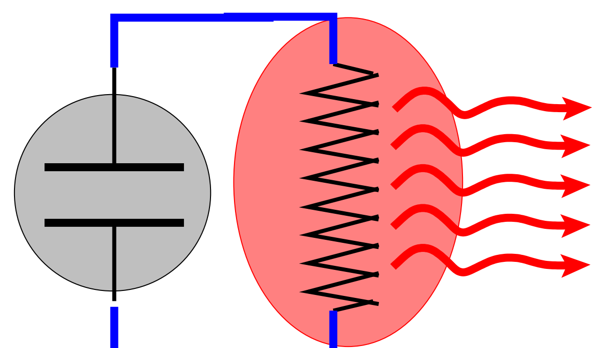


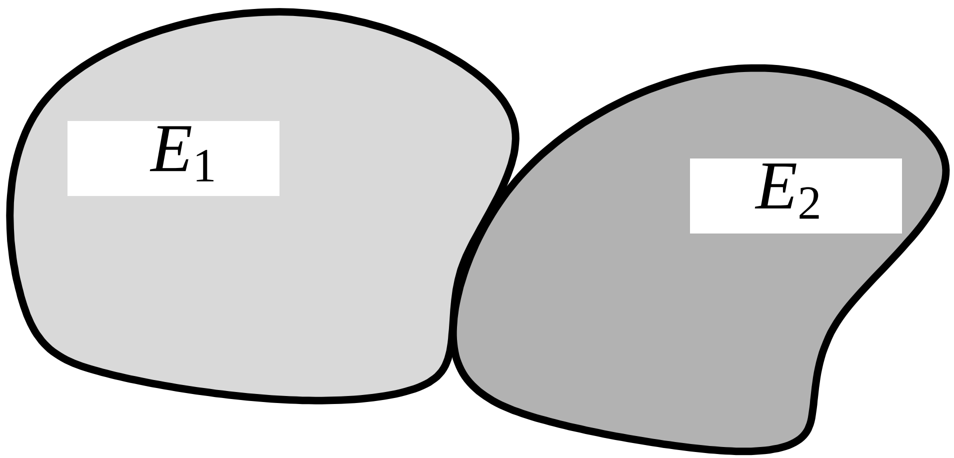
For example capacitor → resistor → heat.

Energy on capacitor = $\frac{1}{2}CV^2$



Energy as an extensive quantity

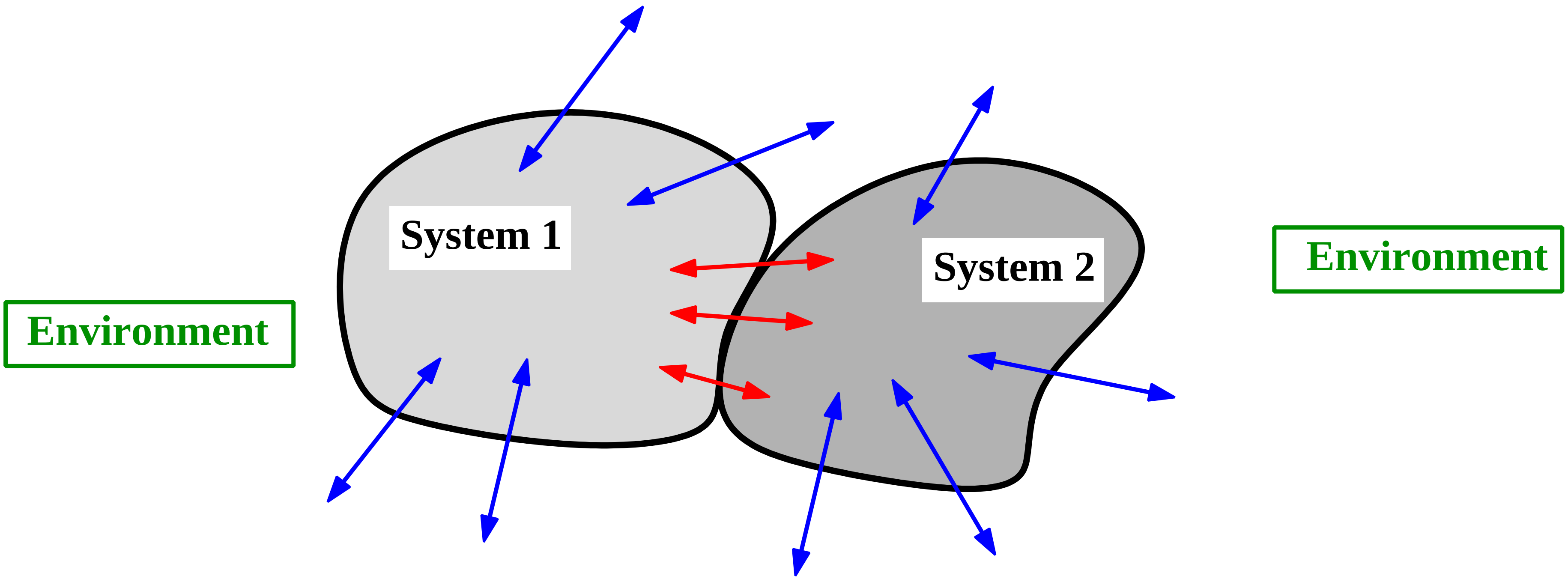
Our intuition has been built to think of energy as an extensive quantity, meaning that it is additive



$$E_{\text{total}} = E_1 + E_2.$$

Energy as an extensive quantity

Our intuition has been built to think of energy as an extensive quantity,

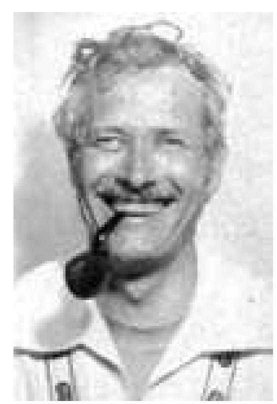


that flows in and out and between systems along the interconnected interfaces (terminals).

Energy as an extensive quantity

Our intuition has been built to think of energy as an extensive quantity, that flows in and out and between systems along the interconnected terminals).

Some methodologies for modeling interconnected systems, as bond-graph modeling and port-Hamiltonian systems, are based on this thinking.



Henry Paynter

Arjan van der Schaft

Energy as an extensive quantity

Our intuition has been built to think of energy as an extensive quantity, that flows in and out and between systems along the interconnected terminals).

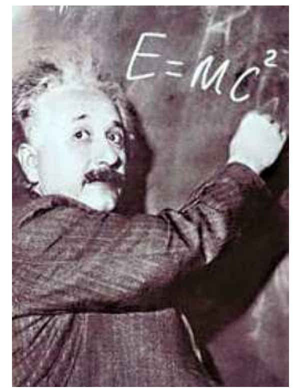
'Power is the universal currency of physical systems'

'In physical systems, the interaction between subsystems is always related to an exchange of energy'

P.J. Gawthrop and G.P. Bevan, *Bond-graph modeling*, IEEE Control Systems Magazine, vol. 27, pp. 2445, 2007.

Energy as an extensive quantity

In many situations, this view is correct. Mass, volume, and energy in the form of heat are extensive quantities.



Energy as an extensive quantity

In many situations, this view is correct. Mass, volume, and energy in the form of heat are extensive quantities.

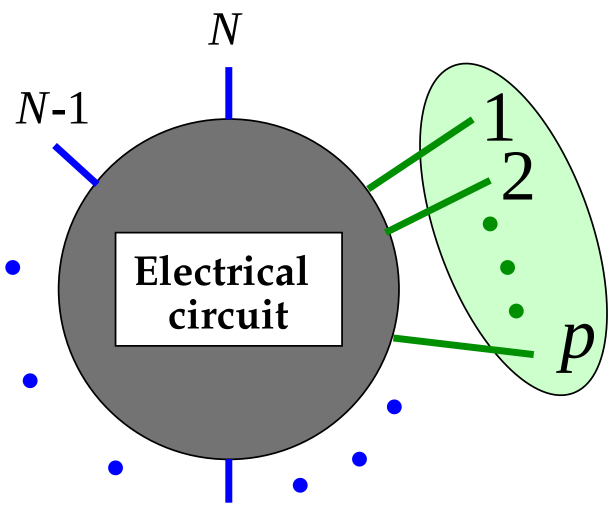
However, energy is more subtle for other forms.

Motion (kinetic) energy is not additive.
Same with energy due to gravitational attraction,
due Coulomb forces, etc.
Heat is a special, extensive, form of energy.

Energy and power are not a 'local' quantities.
They involve 'action at a distance'.

Ports

Ports



Terminals $\{1, 2, \ldots, p\}$ **form a port** $:\Leftrightarrow$

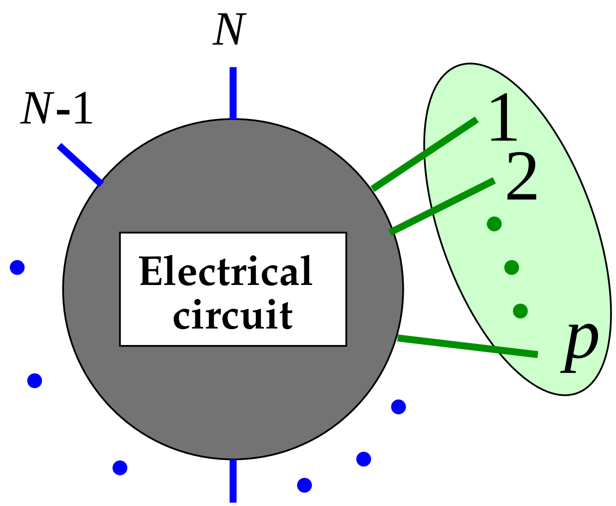
$$(V_1, \ldots, V_p, V_{p+1}, \ldots, V_N, I_1, \ldots, I_p, I_{p+1}, \ldots, I_N) \in \mathscr{B}$$

$$\Rightarrow \quad I_1 + \cdots + I_p = 0.$$

'port KCL'.

(KVL &) KCL $\Rightarrow$ **all terminals together form a port.**

Ports



If terminals $\{1, 2, \ldots, p\}$ **form a port, then**

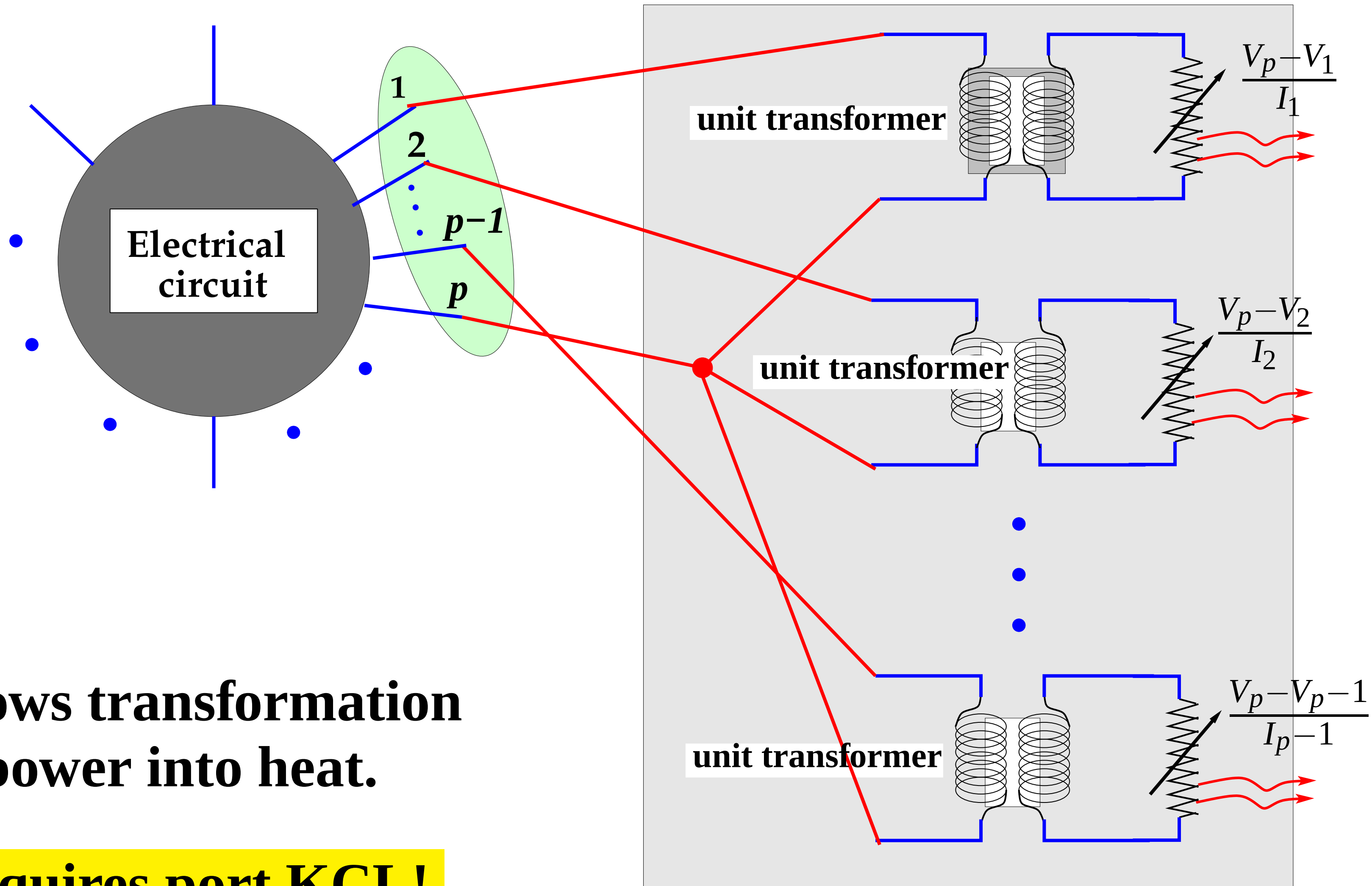
power in along these terminals = $V_1(t)I_1(t) + \cdots + V_p(t)I_p(t)$,

energy in = $\displaystyle\int_{t_1}^{t_2} (V_1(t)I_1(t) + \cdots + V_p(t)I_p(t))\, dt.$

This interpretation in terms of power and energy is not valid unless these terminals form a port !

Dissipation into heat

Justification:



Shows transformation of power into heat.

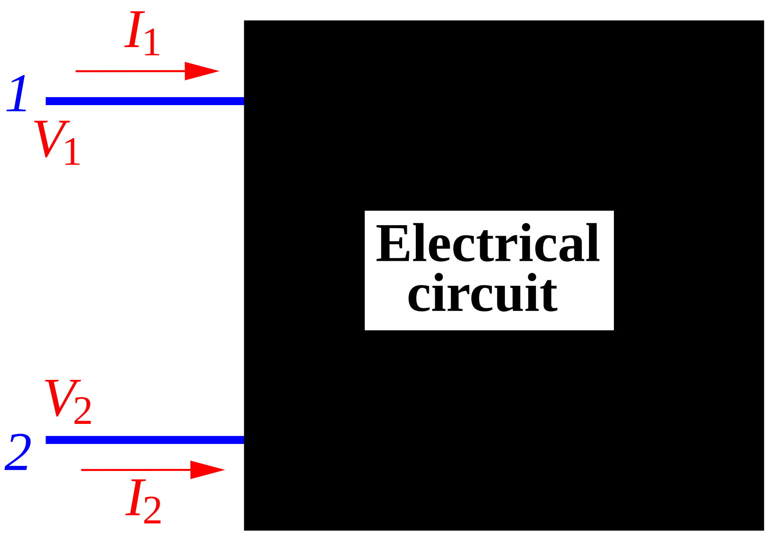
Requires port KCL!

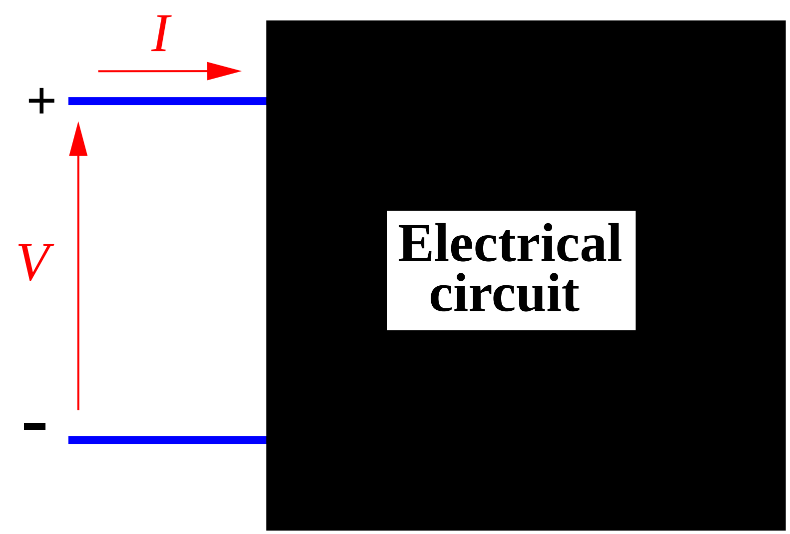
Examples

2-terminal 1-port devices:

resistors, inductors, capacitors, transistors, memristors, gyrators, connectors, etc.

any 2-terminal circuit composed of these.

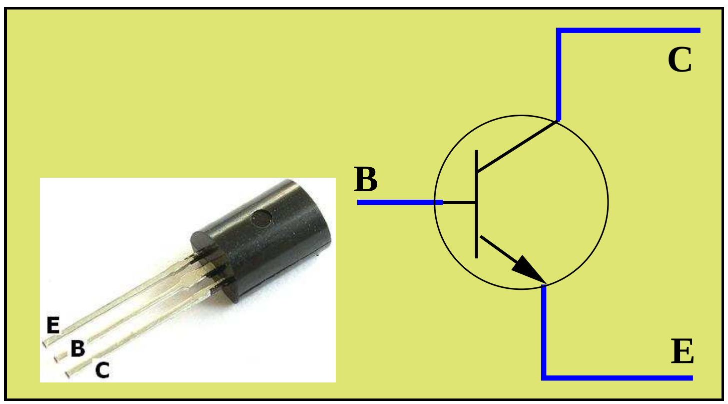


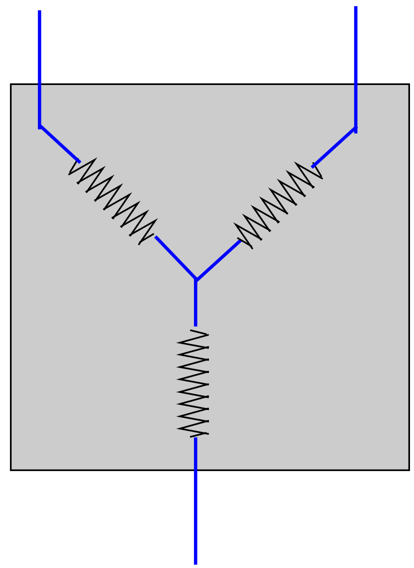


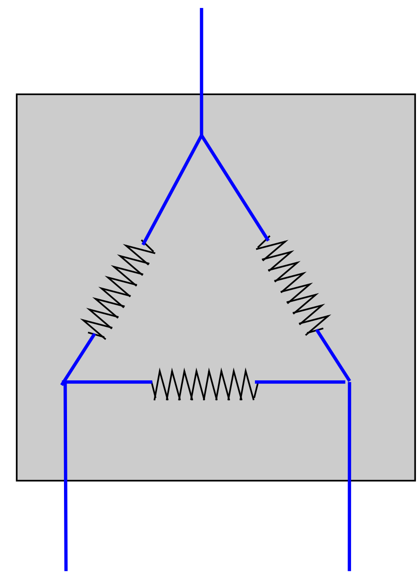
Examples

3-terminal 1-port devices:

transistors, Y's, Δ's.



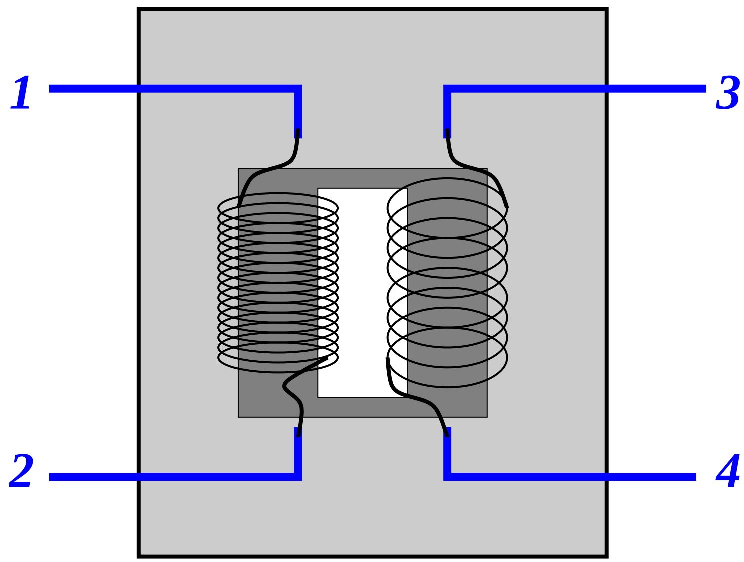




Examples

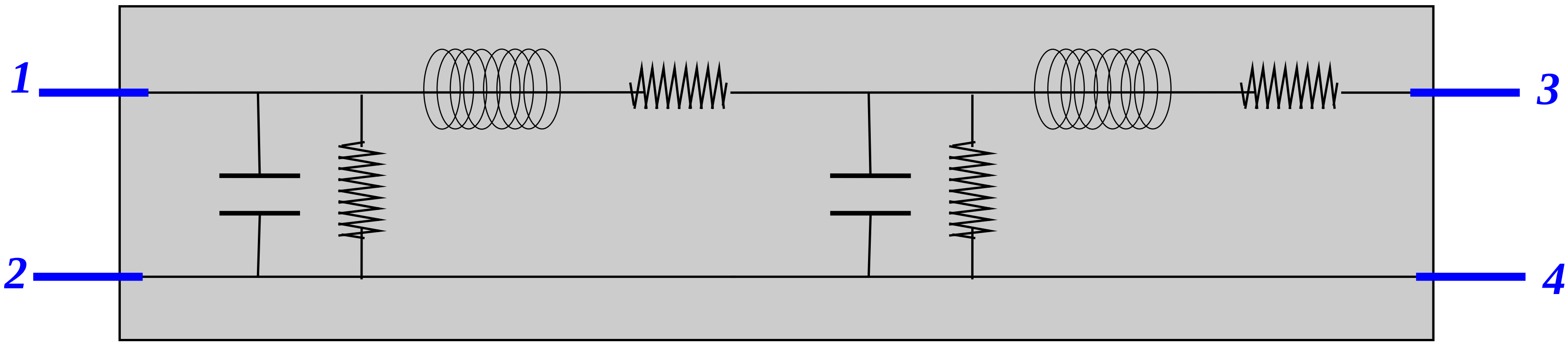
4-terminal 2-port devices:

Transformers, gyrators.



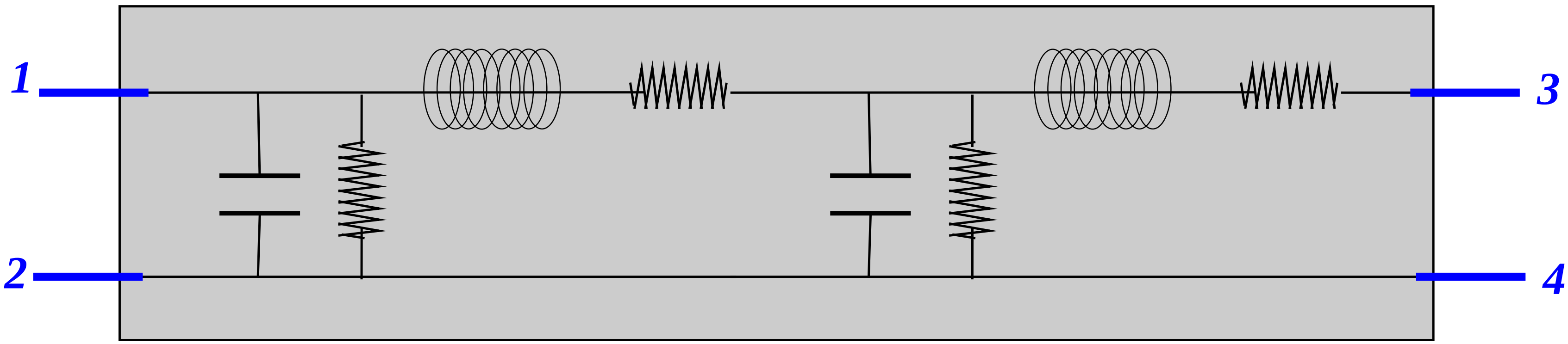
$$V_1 - V_2 = n\,(V_3 - V_4)\,, -nI_1 = I_3 \qquad I_1 + I_2 = 0, I_3 + I_4 = 0$$

Examples



Terminals $\{1, 2, 3, 4\}$ form a port. But $\{1, 2\}$ and $\{3, 4\}$ do not.

Examples



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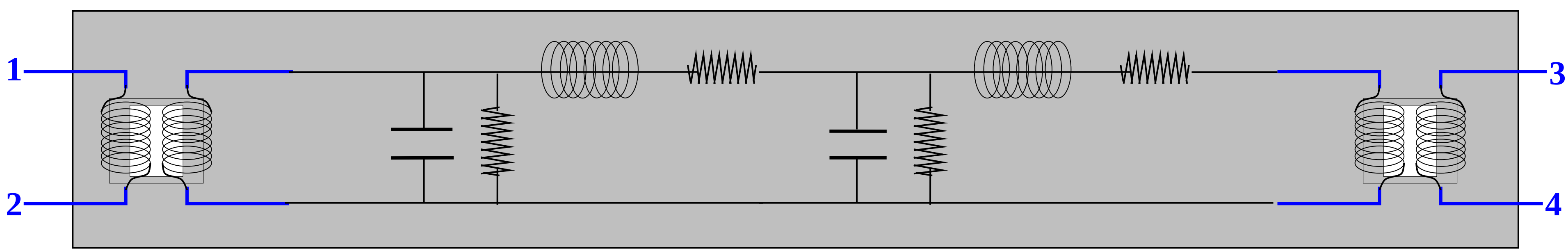
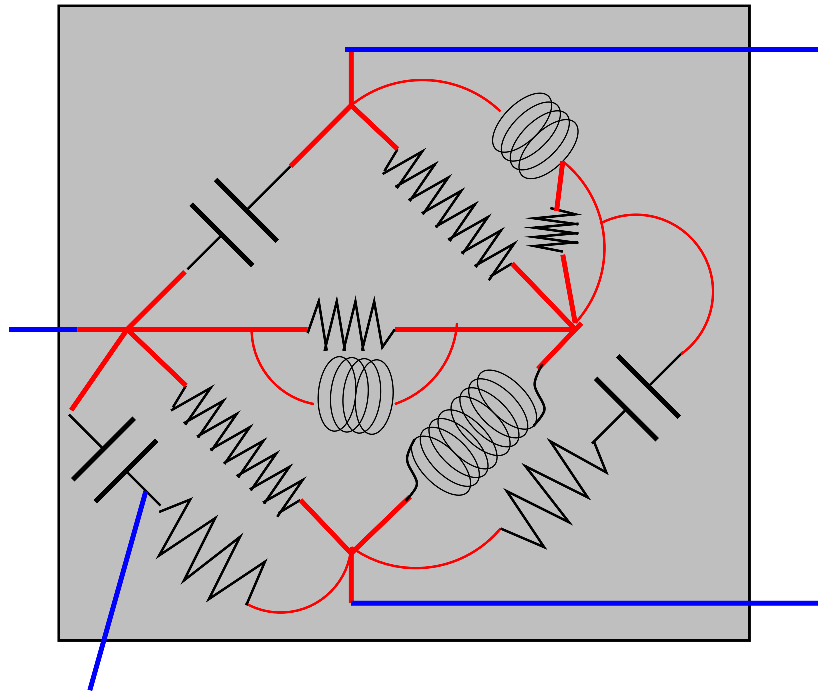


Terminals $\{1,2\}$ and $\{3,4\}$ form a port.

Are ports common?



<u>Theorem</u>: Consider an electrical circuit consisting of an interconnection of (linear passive) R's, L's, and C's. If every pair of terminals of the circuit graph is connected, then

the only port is the one that consists of all the terminals.

Are ports common?

Corollary: Consider an electrical circuit consisting of an interconnection of (linear passive) 2-terminal 1-port impedances. If every pair of terminals of the circuit graph is connected, then

the only port is the one that consists of all the terminals.

Follows from the theorem, combined with Bott-Duffin (every positive real impedance can be viewed as an RLC circuit).In order to have non-trivial ports, we need

2-port building blocks like transformers in the circuit.

Independence

$$(V_1, \ldots, V_p, V_{p+1}, \ldots, V_N, I_1, \ldots, I_p, I_{p+1}, \ldots, I_N) \in \mathscr{B}, \alpha : \mathbb{R} \to \mathbb{R}$$

$$\Rightarrow \quad (V_1 + \alpha, \ldots, V_p + \alpha, V_{p+1}, \ldots, V_N, I_1, \ldots, I_p, I_{p+1}, \ldots, I_N) \in \mathscr{B}.$$

'port KVL'

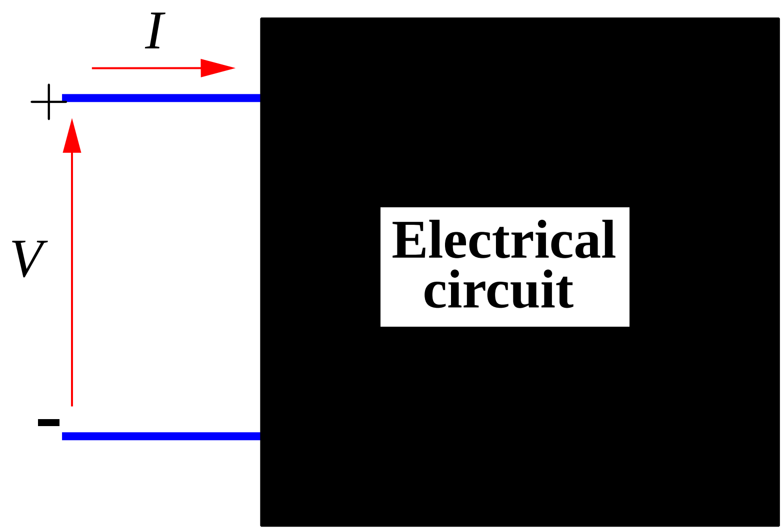
For linear passive circuits, there holds

port KVL $\Leftrightarrow$ **port KCL**.

For energy: port KCL $\quad I_1 + I_2 + \cdots + I_p = 0.$

Electrical circuit synthesis

Synthesis question

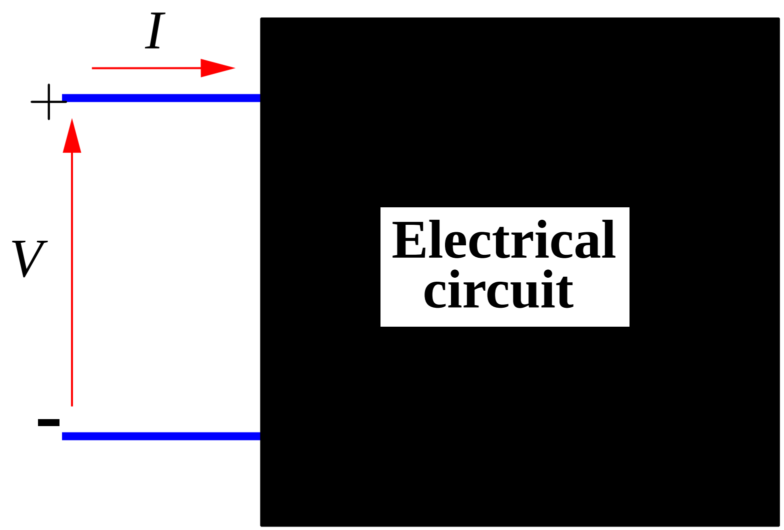


Assume that the circuit consists of an interconnection certain building blocks, say positive R's, L's, C's, T's, G's, etc., or combinations of these,

which external behaviors can occur ?

This was the prime theoretical electrical engineering question until 1960.

Synthesis question



LTIDS case $\rightsquigarrow$ relation between V and I

$$d\left(\frac{d}{dt}\right)V = n\left(\frac{d}{dt}\right)I \qquad n, d \in \mathbb{R}[\xi].$$

Which polynomial pairs (n, d) can occur?

Positive realness

Introduction the 'impedance' $Z := \frac{n}{d}$.

Positive realness

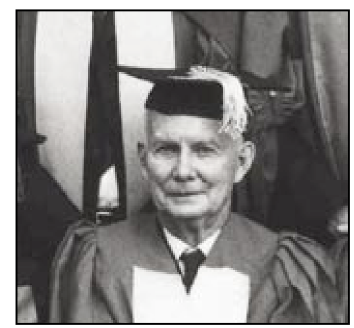
<u>Theorem</u>: The following are equivalent

- Z **is realizable using (positive, linear) R, L, & C's** ***and transformers.***

- Z **is 'positive real',**
  i.e., $[\![\mathtt{Real}(\lambda) > 0]\!] \Rightarrow [\![\mathtt{Real}(Z(\lambda)) > 0]\!]$.

- $\int_{-\infty}^{0} V(t)I(t)\,dt \geq 0 \quad \forall$ **compactly supported** $(V,I) \in \mathscr{B}$,

- $\ldots$



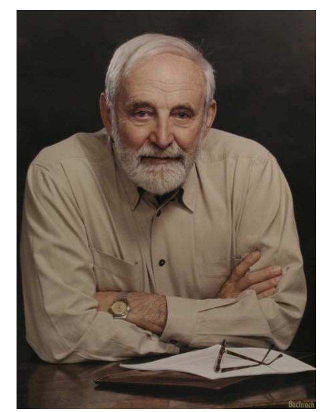
Otto Brune
1901-1982

Positive realness

In 1949 Raoul Bott and Richard Duffin in a joint paper dramatically improved Brune's 1931 result.

Theorem: The following are equivalent

- Z **is realizable using (positive, linear) R, L, & C's** ***without transformers.***
- Z **is** ***positive real,***
- $\ldots$



Raoul Bott
1923-2005

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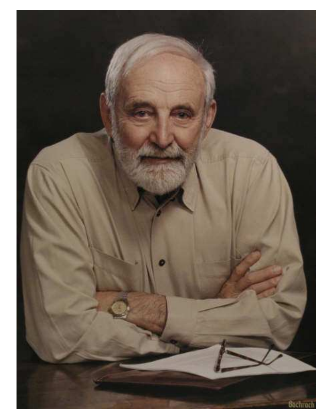
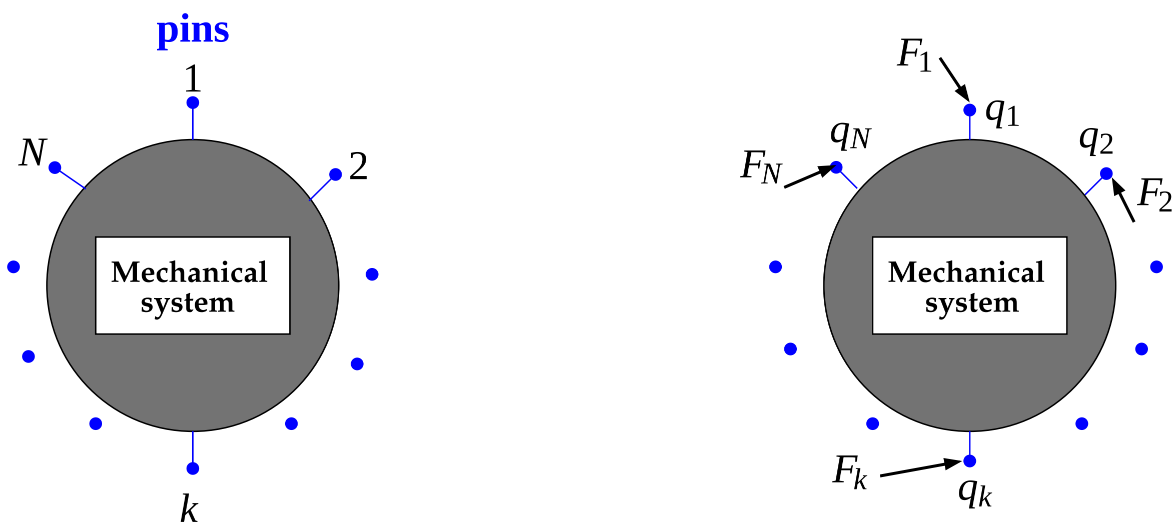


Raoul Bott
1923-2005

Caveat: the n and d obtained in the Bott-Duffin synthesis are NOT coprime!
$\rightsquigarrow$ **uncontrollable (V, I)-behavior.**
$\rightsquigarrow$ **correct impedance, perhaps incorrect ODE.**

Mechanical ports

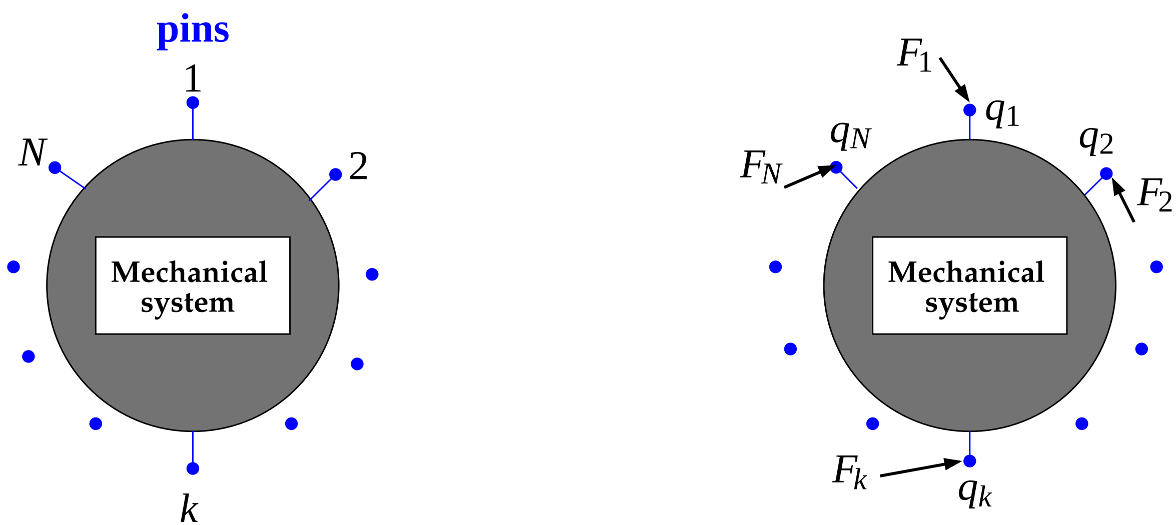
The behavior



At each terminal: a position and a force.

$\rightsquigarrow$ **position/force trajectories** $(q, F) \in \mathscr{B} \subseteq ((\mathbb{R}^\bullet)^{2N})^{\mathbb{R}}$.

The behavior

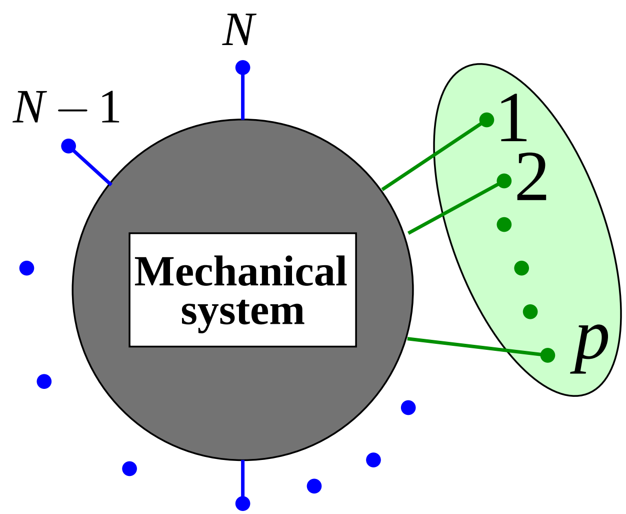


At each terminal: a position and a force.

$\rightsquigarrow$ **position/force trajectories** $(q, F) \in \mathscr{B} \subseteq ((\mathbb{R}^\bullet)^{2N})^{\mathbb{R}}$**.**

What is the analogue of a port?

Port KFL



Terminals $\{1, 2, \ldots, p\}$ **form a (mechanical) port** $:\Leftrightarrow$

$$(q_1, \ldots, q_p, q_{p+1}, \ldots, q_N, F_1, \ldots, F_p, F_{p+1}, \ldots, F_N) \in \mathscr{B},$$

$$\Rightarrow \quad F_1 + F_2 + \cdots + F_p = 0.$$ ***'port KFL'***

Power and energy

If terminals $\{1, 2, \ldots, p\}$ form a port, then

$$\textbf{power in} \;=\; F_1(t)^\top \tfrac{d}{dt} q_1(t) + \cdots + F_p(t)^\top \tfrac{d}{dt} q_p(t),$$

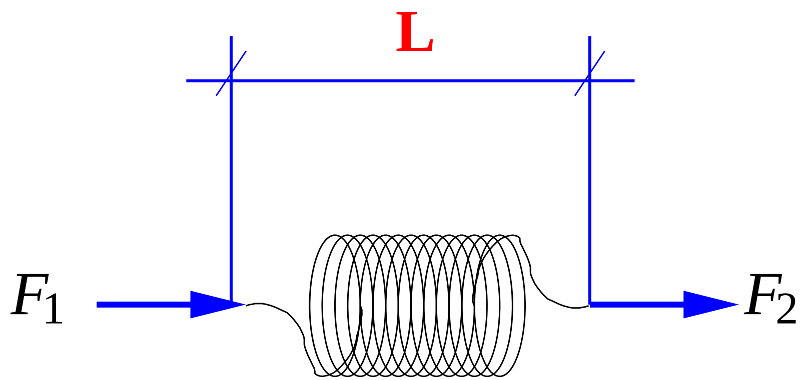
and

$$\textbf{energy in} \;=\; \int_{t_1}^{t_2} \left(F_1(t)^\top \frac{d}{dt} q_1(t) + \cdots + F_p(t)^\top \frac{d}{dt} q_p(t) \right) dt.$$

This interpretation in terms of power and energy is not valid unless these terminals form a port !

Examples

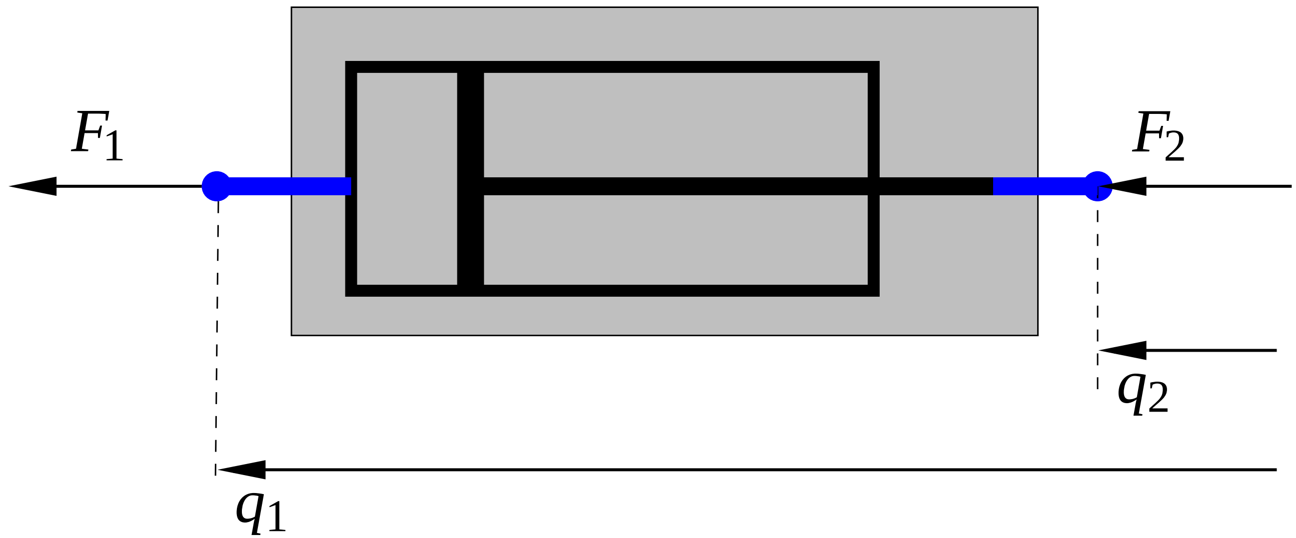
Spring



$$F_1 + F_2 = 0, \quad K(q_1 - q_2) = F_1$$

IUM and KFL

Damper



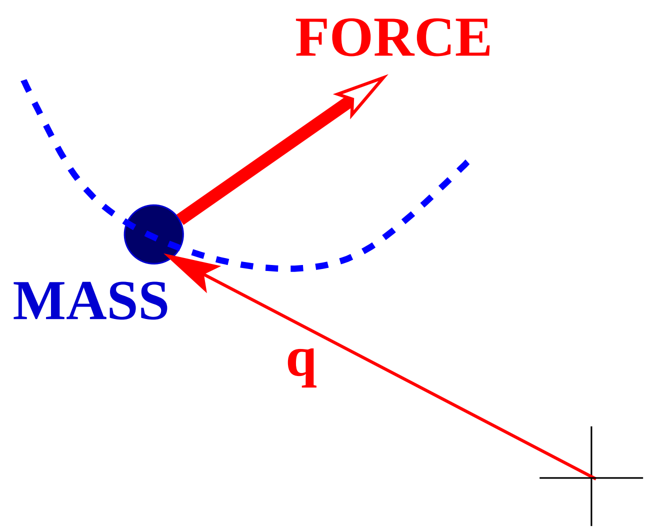
$$F_1 + F_2 = 0, \quad D\frac{d}{dt}(q_1 - q_2) = F_1.$$

IUM and KFL

Springs and dampers, and the interconnection of springs and dampers form ports.

Examples

Mass



$$M\frac{d^2}{dt^2}q = F.$$

IUM but not KFL

Not a port!!!

Consequences

We discuss 2 consequences of the fact that a mass is not a port.

1. **The inerter**
2. **Kinetic energy**

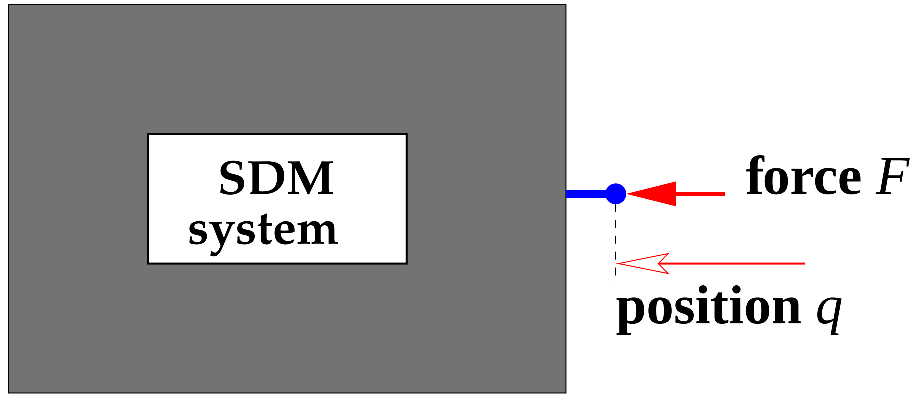
Mechanical synthesis

Electrical and mechanical synthesis

What mechanical impedances are realizable using passive mechanical devices (dampers, springs, and masses)?

Is it possible to use RLC synthesis to obtain mechanical synthesis?

Electrical and mechanical synthesis



Relationship between F and q

$$d\left(\frac{d}{dt}\right)q = n\left(\frac{d}{dt}\right)F \qquad n, d \;\; \textbf{real polynomials}.$$

$$Z(\xi) = \xi \frac{n(\xi)}{d(\xi)} \quad \textbf{positive real ???}$$

Naive! The mass is NOT the mechanical analogue of a capacitor.

Electrical-mechanical analogies

voltage $V \leftrightarrow v$ **velocity**

current $I \leftrightarrow F$ **force**

Resistor $\frac{1}{R}(V_1 - V_2) = I_1,\ \ I_1 + I_2 = 0$	**Damper** $D(v_1 - v_2) = F_1,\ \ F_1 + F_2 = 0$
Inductor $\frac{1}{L}(V_1 - V_2) = \frac{d}{dt} I_1,\ \ I_1 + I_2 = 0$	**Spring** $K(v_1 - v_2) = \frac{d}{dt} F_1,\ \ F_1 + F_2 = 0$
Capacitor $C \frac{d}{dt}(V_1 - V_2) = I_1,\ \ I_1 + I_2 = 0$	**Mass** $M \frac{d}{dt} v = F$

Electrical-mechanical analogies

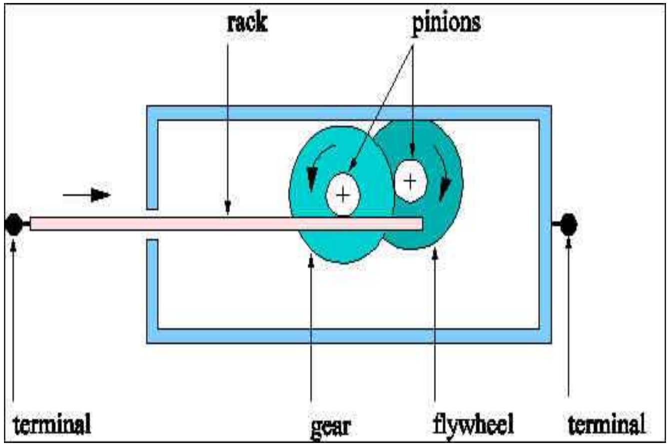
voltage $V \leftrightarrow v$ **velocity** **current** $I \leftrightarrow F$ **force**

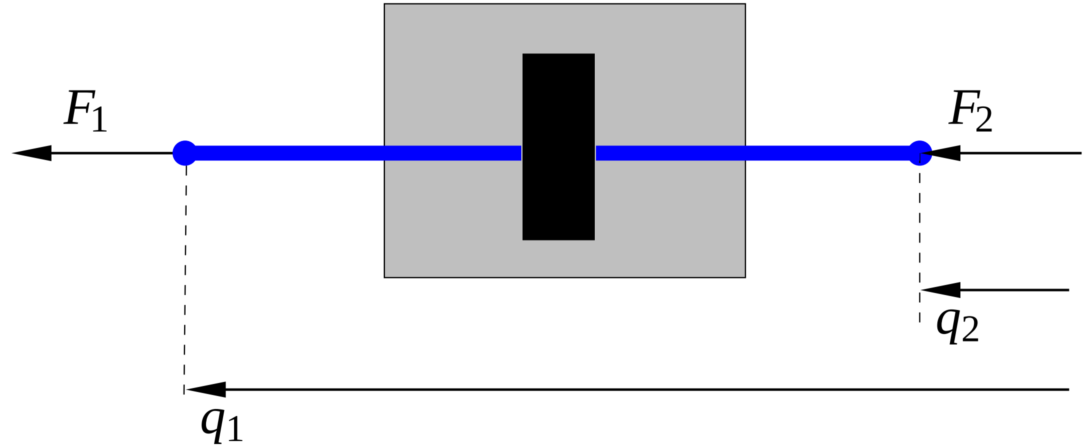
Resistor $\frac{1}{R}(V_1 - V_2) = I_1, \;\; I_1 + I_2 = 0$	**Damper** $D(v_1 - v_2) = F_1, \;\; F_1 + F_2 = 0$
Inductor $\frac{1}{L}(V_1 - V_2) = \frac{d}{dt}I_1, \;\; I_1 + I_2 = 0$	**Spring** $K(v_1 - v_2) = \frac{d}{dt}F_1, \;\; F_1 + F_2 = 0$
Capacitor $C\frac{d}{dt}(V_1 - V_2) = I_1, \;\; I_1 + I_2 = 0$	**Mass** $M\frac{d}{dt}v = F$

The electrical analogue of a mass is a 'grounded' capacitor.

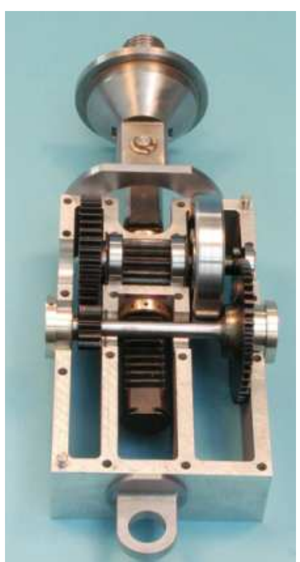
Electrical synthesis $\not\Rightarrow$ **mechanical synthesis.**

The inerter





$$B\frac{d^2}{dt^2}(q_1 - q_2) = F_1, \quad F_1 + F_2 = 0$$ **IUM and KFL**



Malcolm Smith

Electrical-mechanical analogies

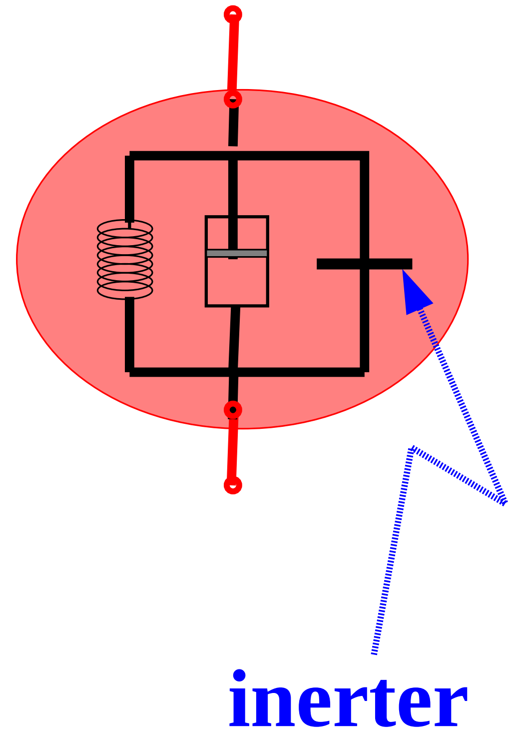
voltage $V \leftrightarrow v$ **velocity**   **current** $I \leftrightarrow F$ **force**

Resistor	Damper
$\frac{1}{R}(V_1 - V_2) = I_1, \;\; I_1 + I_2 = 0$	$D(v_1 - v_2) = F_1, \;\; F_1 + F_2 = 0$
Inductor	**Spring**
$\frac{1}{L}(V_1 - V_2) = \frac{d}{dt} I_1, \;\; I_1 + I_2 = 0$	$K(v_1 - v_2) = \frac{d}{dt} F_1, \;\; F_1 + F_2 = 0$
Capacitor	**Inerter**
$C\frac{d}{dt}(V_1 - V_2) = I_1, \;\; I_1 + I_2 = 0$	$B\frac{d}{dt}(v_1 - v_2) = F_1, \;\; F_1 + F_2 = 0$

electrical RLC synthesis $\Leftrightarrow$ **mechanical SDI synthesis**

Springs, dampers, inerters, and their interconnections form ports!

The inerter in Formula 1



Kimi Räikkönen wins the 2005 Grand Prix in Spain with McLaren's 'J-damper', i.e., Smith's inerter.

grandprix.com

AUGUST 21, 2008

Ingenuity still brings success in Formula 1

ShareThis

For years engineers have complained that the rules of Formula 1 mean that there is little room left for innovation but Cambridge University's engineering department has just revealed that this is not the case at all.

Professor Malcolm Smith, a fellow of Gonville and Caius College, created an innovative suspension system in the late 1990s and this was patented by the university. The first details were published in 2002 in the obscure Institute of Electrical and Electronics Engineers's publication called Transactions on Automatic Control. This was spotted by the boffins at McLaren and an exclusive deal was negotiated to allow the team to use the technology in F1. The new system was first used at the Spanish GP in 2005 and Kimi Raikkonen won.

The team used the name "J Damper" to describe the unit - in an effort to confuse the opposition - but it has now been revealed that it is actually called "an inerter". This is grandprix.com

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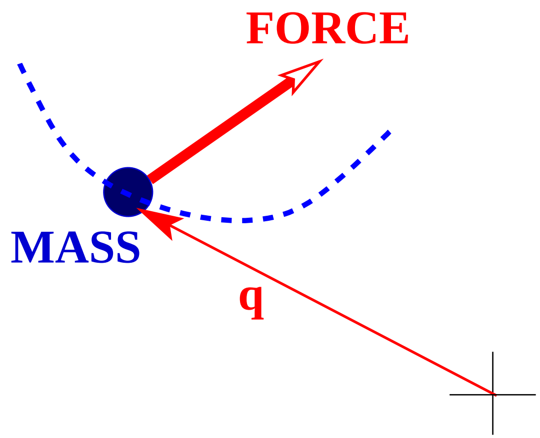
MOTION ENERGY

Back to the mass

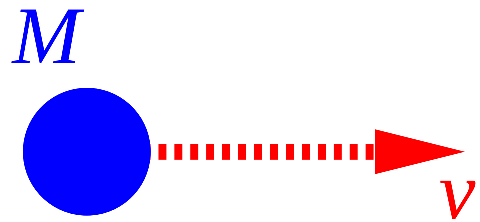


$$M\frac{d^2}{dt^2}q = F \quad \Rightarrow \quad \boxed{\frac{d}{dt}\frac{1}{2}M||\frac{d}{dt}q||^2 = F^\top \frac{d}{dt}q}$$

Since $F^\top v$ is not power,

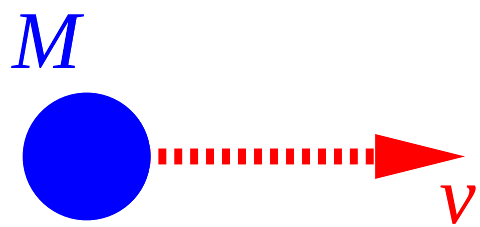
is $\frac{1}{2}M||\frac{d}{dt}q||^2$ not stored (kinetic, motion) energy ???

Kinetic energy and invariance under uniform motions



What is the kinetic energy?

Kinetic energy and invariance under uniform motions



What is the kinetic energy?

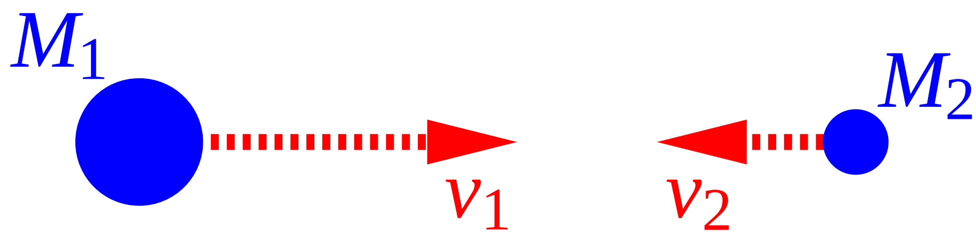
$$\mathcal{E}_{\textbf{kinetic}} = \frac{1}{2}\, M\, ||v||^2$$

Willem ’s Gravesande
1688–1742

Émilie du Châtelet
1706–1749

This expression is not invariant under uniform motion.

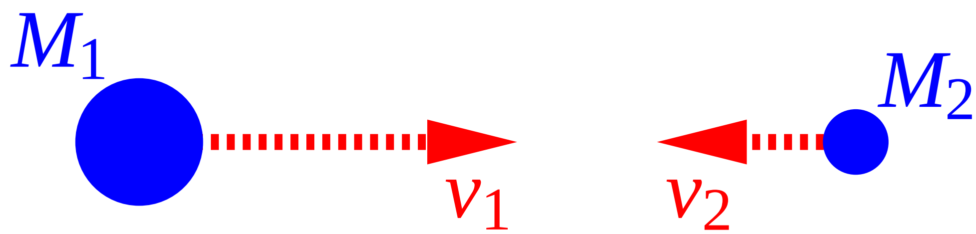
Motion energy



What is the motion energy?

What quantity is transformable into heat?

Motion energy



What is the motion energy?

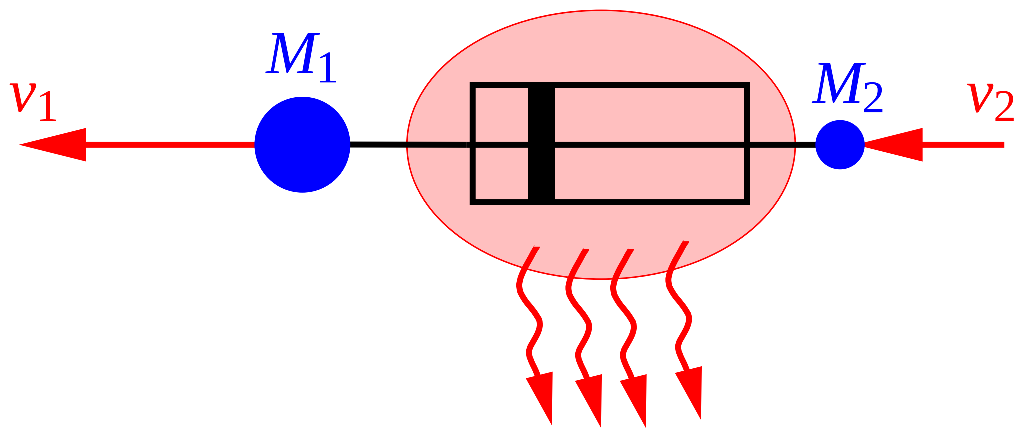
What quantity is transformable into heat?

$$\mathscr{E}_{\mathbf{motion}} = \frac{1}{2}\,\frac{M_1\,M_2}{M_1 + M_2}\,||v_1 - v_2||^2$$

Invariant under uniform motion.

Dissipation into heat

Can be justified by mounting a damper or a spring between the masses.

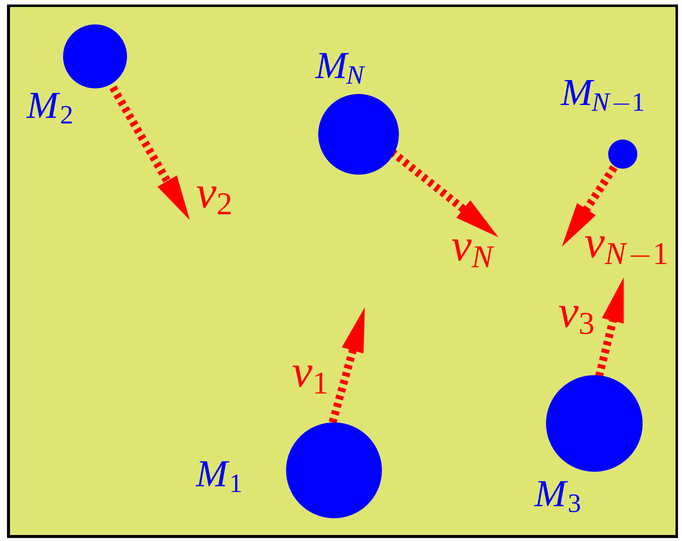


$$\frac{1}{2}\,\frac{M_1\,M_2}{M_1+M_2}\,||v_1-v_2||^2$$

is the heat dissipated in the damper.

Motion energy

Generalization to N masses.

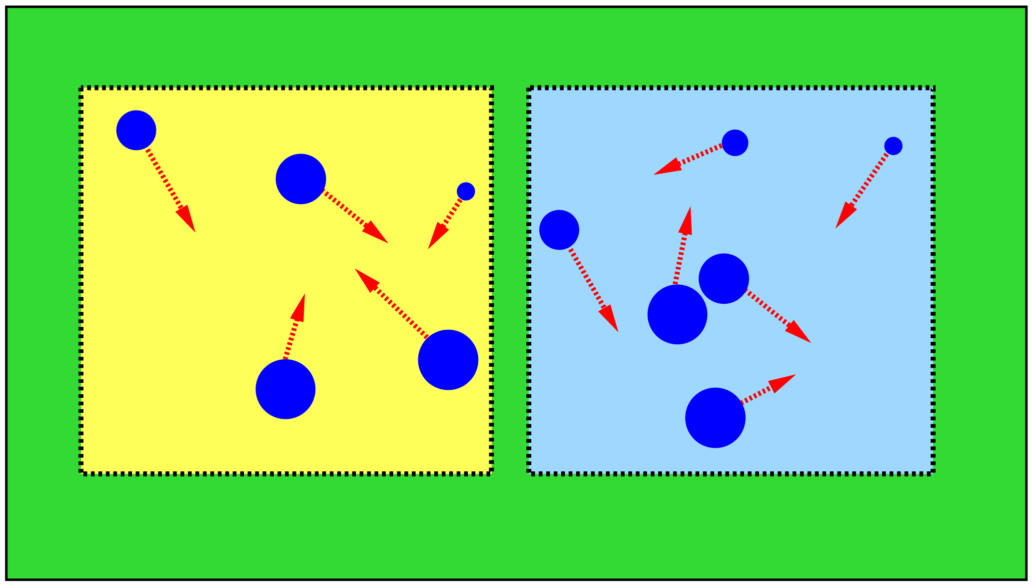


$$\mathscr{E}_{\textbf{motion}} = \frac{1}{4} \sum_{i,j \in \{1,2,\ldots,N\}} \frac{M_i\, M_j}{M_1 + M_2 + \cdots + M_N} ||v_i - v_j||^2.$$

$$\textbf{KFL} \;\Rightarrow\; \frac{d}{dt}\mathscr{E}_{\textbf{motion}} = \sum_{i \in \{1,2,\ldots,N\}} F_i^{\top} v_i.$$

Motion energy

Motion energy is not an extensive quantity, it is not additive.



Total motion energy $\neq$ sum of the parts.

Motion energy

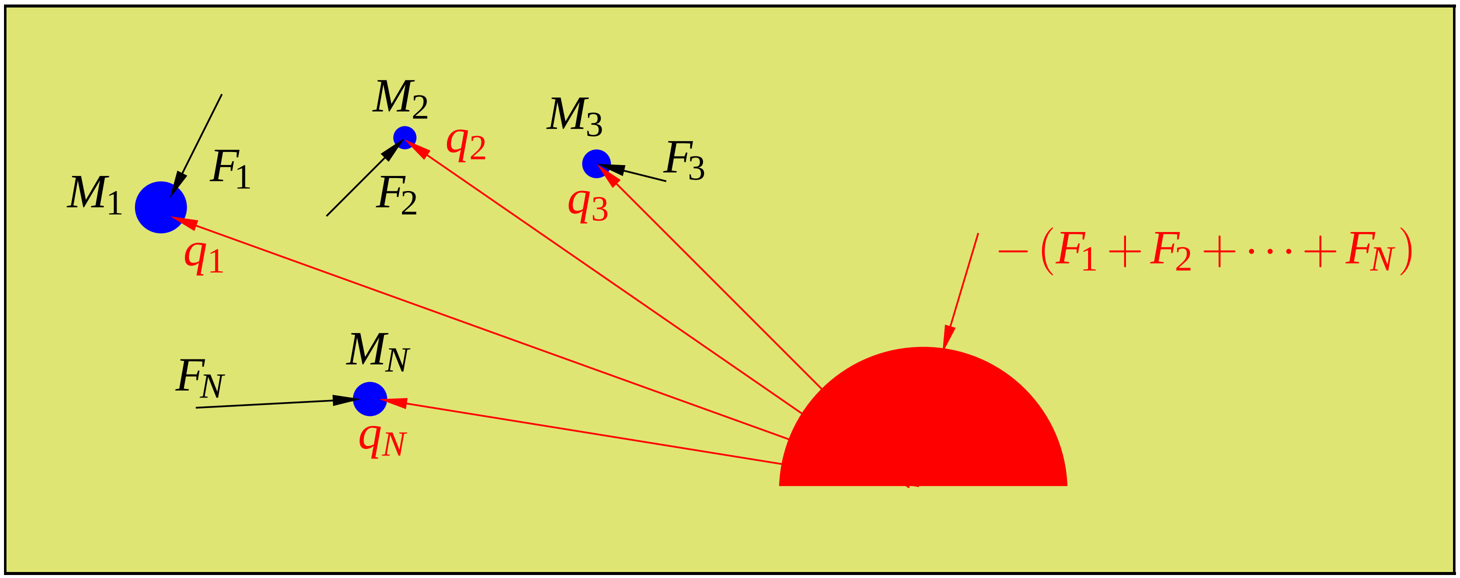
$$\mathscr{E}_{\mathbf{motion}} = \boxed{\frac{1}{4} \sum_{i,j \in \{1,2,\ldots,N\}} \frac{M_i\, M_j}{M_1 + M_2 + \cdots + M_N} ||v_i - v_j||^2.}$$

Distinct from the classical expression of the kinetic energy,

$$\mathscr{E}_{\mathbf{kinetic}} = \boxed{\frac{1}{2} \sum_{i \in \{1,2,\ldots,N\}} M_i\, ||v_i||^2.}$$

Motion energy

Reconciliation: $M_{N+1} = \infty, F_{N+1} = -(F_1 + F_2 + \cdots + F_N)$,



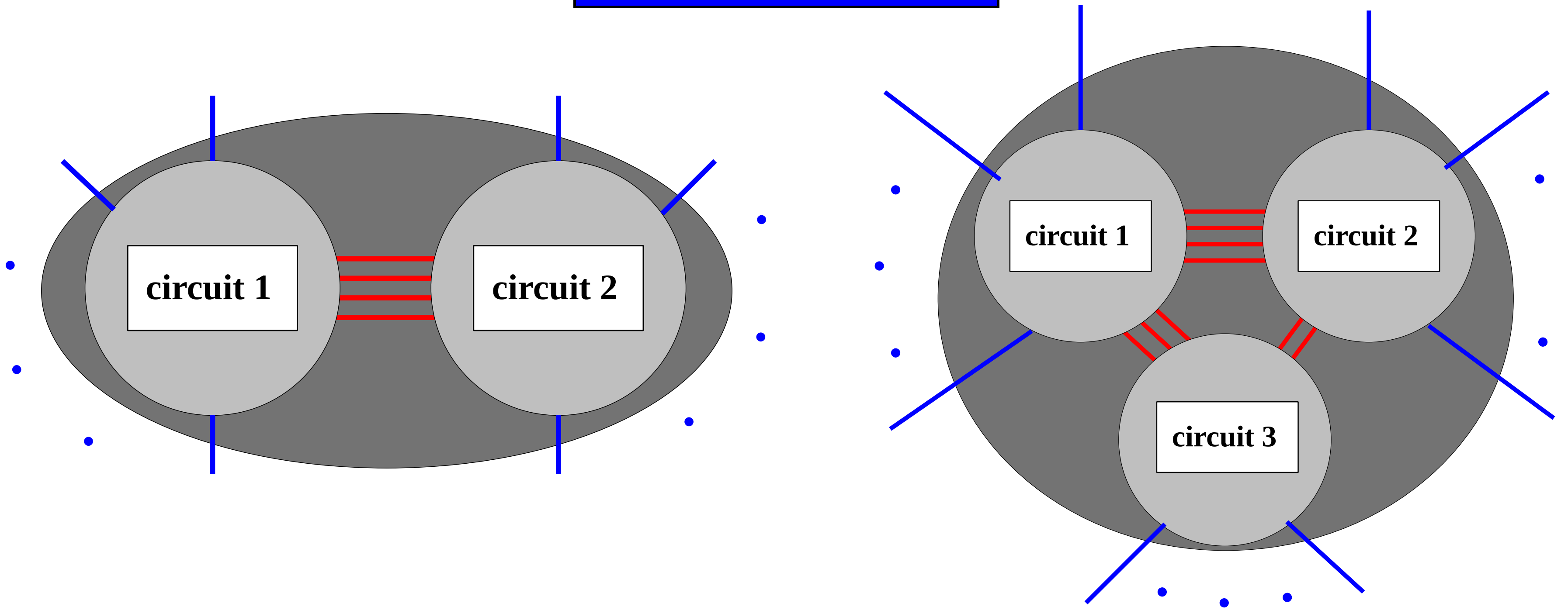
measure velocities w.r.t. this infinite mass ('ground'), then

$$\frac{1}{4} \sum_{i,j \in \{1,2,\ldots,N,N+1\}} \frac{M_i\, M_j}{M_1 + M_2 + \cdots + M_N + M_{N+1}} ||v_i - v_j||^2$$

$$\longrightarrow \quad \frac{1}{2} \sum_{i \in \{1,2,\ldots,N\}} M_i\, ||v_i||^2.$$

PORTS and TERMINALS

Energy transfer



One cannot speak about

"the energy transferred from circuit 1 to circuit 2"
or *"from the environment to circuit 1"*,

unless the relevant terminals form a port.

Analogously for mechanical systems, etc.

Recapitulation

Summary

- **Energy transfer happens via ports,**
  hence it involves action at a distance.
- **Interconnection is ‘local’,**
  power and energy transfer involve ‘action at a distance’.

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  $\Rightarrow$ **the inerter.**
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Summary

- **Energy transfer happens via ports,**
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- **Interconnection is 'local',**
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- **The mass is not the mechanical analogue of the capacitor.**
  $\Rightarrow$ **the inerter.**
  $\Rightarrow$ **a new expression for motion energy.**
- **Terminals are for interconnection,**
  ports are for energy transfer.

End of Lecture X