European Embedded Control Institute

Graduate School on Control — Spring 2010

# The Behavioral Approach to Modeling and Control

# EXERCISES- second part

## Exercise 26: State and McMillan degree

**We need to introduce the notion of *row-reduced matrix*.**
**Let $r = \begin{bmatrix} r_1 & \dots & r_{\mathtt{w}} \end{bmatrix} \in \mathbb{R}^{1\times \mathtt{w}}[\xi]$; then $\delta$ is the *degree of* $r$ if**

$$\delta = \max\{d \mid d = \deg(r_i), i = 1,\dots,\mathtt{w}\}.$$

**Note that if $\delta = \deg(r)$, then $r(\xi) = \xi^{\delta} r_{\text{hc}} + r'(\xi)$, where $r_{\text{hc}} \in \mathbb{R}^{1\times \mathtt{w}}$ and $\deg(r') < \delta$. We call $r_{\text{hc}}$ the *highest coefficient* of $r$. Given a matrix $R = \text{col}(r_i)_{i=1,\dots,\mathtt{p}}$, we write $r_i(\xi) = \xi^{\delta_i} r_{i,\text{hc}} + r_i'(\xi)$, with $\delta_i = \deg(r_i)$ and $\deg(r') < \delta_i$, $i = 1,\dots,\mathtt{p}$. We call $R_{\text{hc}} := \text{col}(r_{i,\text{hc}})_{i=1,\dots,\mathtt{p}}$ the *highest row coefficient matrix of* $R$. A matrix $R$ is called *row-reduced* if its highest row coefficient matrix has full rank.**
**It can be shown that if $R$ is a polynomial matrix of full row rank, then there exists a unimodular matrix $U$ such that $UR$ is row-reduced. Also, it can be shown that if $R_1$ and $R_2$ are row-reduced, with row-degrees arranged in e.g. ascending order, and if $R_1 = UR_2$ for some unimodular $U$, then the row-degrees of $R_1$ and $R_2$ are the same.**

1. **The matrix**
$$R(\xi) = \begin{bmatrix} \xi+1 & 2\xi+\frac{5}{2} \\ 2\xi^2+\xi+1 & 4\xi^2+3\xi \end{bmatrix}$$
**is not row proper: verify it. Find a unimodular matrix $U$ such that $UR$ is row proper.**

2. **Prove that if $R$ is row-reduced with row degrees $\delta_i$, $i = 1,\dots,\mathtt{p}$, then its maximal degree $\mathtt{p}\times\mathtt{p}$ minor has degree equal to $\sum_{i=1}^{\mathtt{p}} \delta_i$.**

3. **Let $\mathscr{B} = \mathtt{kernel}(R)$, with $R$ row-reduced. Prove that $\mathtt{n}(\mathscr{B}) = \deg(\det(P))$ for every matrix $P$ such that**
$$\begin{bmatrix} P & -Q \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = 0$$
**is an input/output representation of $\mathscr{B}$ with $P^{-1}Q$ proper.**

**4. Assume that $R$ is row-reduced with row degrees $\delta_i$, $i = 1,\dots,\mathtt{p}$, and denote with $\Sigma_R$ the polynomial matrix obtained stacking the results of the shift-and-cut map, i.e. $\Sigma_R := \mathrm{col}\left(\sigma_+^i(R)\right)_{i=1,\dots}$. Prove that the subspace of $\mathbb{R}^{1\times\mathtt{w}}[\xi]$**

$$\Xi_R = \{f \in \mathbb{R}^{1\times\mathtt{w}}[\xi] \mid \exists\ \alpha \in \mathbb{R}^{1\times\bullet} \text{ s.t. } f = \alpha\Sigma_R\}$$

**has dimension equal to $\mathtt{n}(\mathscr{B})$.**

**5. Prove that if $\mathscr{B} = \mathtt{kernel}(R)$ with $R$ a row-reduced matrix, then a minimal state map for $\mathscr{B}$ can be computed selecting the nonzero rows of**

$$\Sigma_R := \mathrm{col}\left(\sigma_+^i(R)\right)_{i=1,\dots}.$$

## Exercise 27: State and state equations

1. **Let $\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{x}}, \mathscr{B}_{\text{full}})$ be a discrete-time latent variable system. Assume that it is *complete*, i.e. that**

$$(w,x) \in \mathscr{B}_{\text{full}} \Longleftrightarrow (w,x)\mid_{[t_0,t_1]} \in \mathscr{B}_{\text{full}}\mid_{[t_0,t_1]} \text{ for all } -\infty < t_0 \le t_1 < \infty.$$

**Prove that $\Sigma$ is a state system if and only if there exist $E,F,G \in \mathbb{R}^{\bullet\times\bullet}$ such that $\mathscr{B}_{\text{full}}$ can be described by $E\sigma x + Fx + Gw = 0$.**

**(*Hint:* For the "only if" part, define**

$$\mathscr{V} := \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid \exists (x,w) \in \mathscr{B}_{\text{full}} \text{ s. t. } \begin{bmatrix} x(1) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\}$$

**Prove that $\mathscr{V}$ is a linear space.**

**The "if" part can be proved by induction, using the state property and the completeness of $\mathscr{B}$.)**

2. **Consider the behavior described in kernel form by the equation**

$$p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u$$

**where $p(\xi) = p_0 + \ldots + p_n\xi^n$, $q(\xi) = q_0 + \ldots + q_n\xi^n$. Write the polynomial matrix $X \in \mathbb{R}^{n\times 2}[\xi]$ obtained by applying the shift-and-cut map to the matrix $\begin{bmatrix} p(\xi) & -q(\xi) \end{bmatrix}$. Is $X(\xi)$ obtained in this way a minimal state map? Explain.**

3. **Verify that the matrices $A$, $B$, $C$, and $D$ corresponding to this state map are**

$$A = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & -\frac{p_{n-1}}{p_n} \\ 0 & 1 & 0 & \ldots & 0 & -\frac{p_{n-2}}{p_n} \\ \vdots & \ddots & \ddots & \ddots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -\frac{p_1}{p_n} \\ 0 & 0 & 0 & \ldots & 1 & -\frac{p_0}{p_n} \end{bmatrix} \qquad B = \begin{bmatrix} q_{n-1} - \frac{p_{n-1}q_n}{p_n} \\ q_{n-2} - \frac{p_{n-2}q_n}{p_n} \\ \vdots \\ q_1 - \frac{p_1 q_n}{p_n} \\ q_0 - \frac{p_0 q_n}{p_n} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & \frac{1}{p_n} \end{bmatrix} \qquad D = \frac{q_n}{p_n}$$

**4. Let $\frac{q(\xi)}{p(\xi)} = h_0 + h_1\xi^{-1} + \ldots + h_n\xi^{-n} + \ldots$ be the power series expansion at infinity of the rational function $\frac{q(\xi)}{p(\xi)}$. The numbers $h_i$, $i = 0, \ldots$, are called the *Markov parameters* of the "transfer function" $\frac{q(\xi)}{p(\xi)}$. Define the polynomial matrix**

$$X(\xi) := \begin{bmatrix} 1 & h_0 \\ \xi & h_1 + h_0\xi \\ \vdots & \vdots \\ \xi^{n-1} & h_{n-1} + h_{n-2}\xi + \ldots + h_0\xi^{n-1} \end{bmatrix}$$

**Prove that this matrix induces a state map for the system.**

**5. Find the matrices $A$, $B$, $C$, $D$ corresponding to the state map $X$.**

## Exercise 28: Properties of QDFs

1. **Let $Q_\Phi$ be a QDF associated with the two-variable polynomial matrix $\Phi \in \mathbb{R}^{\bullet\times\bullet}[\zeta,\eta]$. Prove that the derivative of $Q_\Phi$ is associated with the polynomial matrix $(\zeta+\eta)\Phi(\zeta,\eta)$.**

2. **Let $\Phi \in \mathbb{R}[\zeta,\eta]$ (scalar!), and let $\mathscr{B} = \texttt{kernel}\; r\left(\frac{d}{dt}\right)$ . Prove that $Q_\Phi(w) = 0$ for all $w \in \mathscr{B}$ if and only if there exists $f \in \mathbb{R}[\zeta,\eta]$ such that**

$$\Phi(\zeta,\eta) = r(\zeta)f(\zeta,\eta) + f(\eta,\zeta)r(\eta)\,.$$

   **(*Hint*: Assume w.l.o.g. that $r$ is monic. Rewrite every term $\frac{d^k w}{dt^k}\Phi_{k,\ell}\frac{d^\ell w}{dt^\ell}$ of $Q_\Phi(w)$ with $k,\ell \geq \deg(r)$ in terms of derivatives of order less than or equal to $\deg\ (r)-1$. Call the result of these operations $Q_{\Phi'}$. Note that $Q_{\Phi'}(w) = Q_\Phi(w)$ for all $w \in \mathscr{B}$, with only terms involving derivatives of order $\leq \deg\ r-1$. Now you need to prove that $\Phi'(\zeta,\eta) = 0$ (the two-variable zero polynomial); consider what happens at $t = 0$ when $Q_{\Phi'}$ is applied to a $w \in \mathscr{B}$...)**

   **This result can easily extended to the multivariable case using the Smith form, obtaining the characterization discussed during the lecture.**

3. **Let $\Phi \in \mathbb{R}[\zeta,\eta]$ (scalar!), and let $\mathscr{B} = \texttt{kernel}\; r\left(\frac{d}{dt}\right)$ . Prove that $Q_\Phi(w) \geq 0$ for all $w \in \mathscr{B}$ if and only if there exist $f \in \mathbb{R}[\xi]$, $g \in \mathbb{R}[\zeta,\eta]$ such that**

$$\Phi(\zeta,\eta) = f(\zeta)f(\eta) + r(\zeta)g(\zeta,\eta) + g(\eta,\zeta)r(\eta)\,. \tag{1}$$

   **(*Hint*: Follow the hint of Question 2. )**

   **This result can easily extended to the multivariable case using the Smith form, obtaining the characterization discussed during the lecture.**

4. **Let $\Phi \in \mathbb{R}[\zeta,\eta]$ (scalar!), and let $\mathscr{B} = \texttt{kernel}\; r\left(\frac{d}{dt}\right)$ . Prove that $Q_\Phi(w) > 0$ for all $w \in \mathscr{B}$ if and only if there exists $f \in \mathbb{R}[\xi]$, $g \in \mathbb{R}[\zeta,\eta]$ such that (1) holds, and moreover $GCD(f,r) = 1$.**

   **This result can easily extended to the multivariable case using the Smith form, obtaining the characterization discussed during the lecture.**

## Exercise 29: QDFs and oscillatory systems

1. A behavior $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ is called *oscillatory* if

$$[w \in \mathscr{B}] \Longrightarrow [w \text{ is bounded on } (-\infty, +\infty)] \ .$$

   Prove that if $\mathscr{B}$ is oscillatory, then it is autonomous.

2. Let $\mathscr{B} = \ker R(\frac{d}{dt})$, with $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$. Prove that $\mathscr{B}$ is oscillatory if and only if every nonzero invariant polynomial of $R$ has distinct and purely imaginary roots.

3. Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$, and let $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$. We call a QDF $Q_\Phi$ a *conserved quantity* for $\mathscr{B}$ if

$$[w \in \mathscr{B}] \Longrightarrow \left[\frac{d}{dt} Q_\Phi(w) = 0\right] \ .$$

   Prove that $Q_\Phi$ is a conserved quantity if and only if there exists $Y \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ such that

$$(\zeta + \eta)\Phi(\zeta, \eta) = R(\zeta)^\top Y(\zeta, \eta) + Y(\eta, \zeta)^\top R(\eta) \ .$$

4. Let $\mathscr{B} \in \mathscr{L}^1$ (scalar system!), and let $\mathscr{B} = \mathtt{kernel}\ r\left(\frac{d}{dt}\right)$. Prove that $Q_\Phi$ is a $r$-canonical conserved quantity for $\mathscr{B}$ if and only if there exists $y \in \mathbb{R}[\xi]$ (univariate!), $\deg\, y < \deg\, r$, such that $(\zeta + \eta)\Phi(\zeta, \eta) = r(\zeta)y(\eta) + y(\zeta)r(\eta)$.

5. Assume now that $\mathscr{B}$ is oscillatory, without characteristic frequencies at zero. Use the result of Question 3 to construct a basis for the space of $r$-canonical conserved quantities for $\mathscr{B}$.

   (*Hint*: Let $\zeta = -\xi$, $\eta = \xi$ in the result of Question 3. Then $r(-\xi)y(\xi) + y(-\xi)r(\xi) = 0$. What does this equation tell about the polynomial $y$?)

## Exercise 30: QDFs and physical systems

**Consider the mechanical system in Figure 1. The equation relating $w$ and $F$**

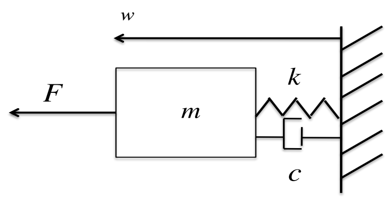


Figure 1: The mechanical system for exercise 22

**is $m\frac{d^2w}{dt^2} + c\frac{d}{dt}w + kw - F = 0$. Assume that all constants have value 1 (in the appropriate physical unit). The system is then described in kernel form by the matrix $R(\xi) = \begin{bmatrix} \xi^2+\xi+1 & -1 \end{bmatrix}$, and in observable image form (verify this!) by $M(\xi) = \begin{bmatrix} 1 \\ \xi^2+\xi+1 \end{bmatrix}$ (you may find working with $M$ easier in the following).**

1. **Using only the calculus of quadratic differential forms (*not* physical insight!), write down the dissipation equality for this system, corresponding to the supply rate $Q_\Phi(w,F) = F\frac{d}{dt}w$.**

2 **Using your physical insight, write an expression for the total energy of the system. Write also the two-variable polynomial matrix corresponding to the total energy.**

3 **Using your physical insight, write an expression for the energy dissipated in the system. Write also the two-variable polynomial matrix corresponding to the dissipated energy.**

4. **Prove that for every trajectory of the system, the derivative of the total energy equals the opposite of the dissipated energy.**

5. **A behavior $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ is called *asymptotically stable* if $\lim_{t\to\infty} w(t) = 0$ for all $w \in \mathscr{B}$. Prove that if $\mathscr{B}$ is asymptotically stable, then it is autonomous.**

**6. Prove the following statement: let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$, and assume that there exists $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that**

**(i)** $Q_\Psi(w) \geq 0$ **for all** $w \in \mathscr{B}$**;**

**(ii) there exists** $D \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ **such that**

$$\frac{d}{dt}Q_\Psi(w) = -\left(D\left(\frac{d}{dt}\right)w\right)^\top D\left(\frac{d}{dt}\right)w$$

**for all** $w \in \mathscr{B}$**, and** $\operatorname{rank}(\operatorname{col}(D(\lambda), R(\lambda))) = \mathtt{w}$**. Then** $\mathscr{B}$ **is asymptoti-**

Figure 2: Alexandr Mikhailovich Lyapunov, 1857-1918

**cally stable.**

(***Hint*: Integrate the relation** $\frac{d}{dt}Q_\Psi(w) = -\left(D\left(\frac{d}{dt}\right)w\right)^\top D\left(\frac{d}{dt}\right)w$ **between** $0$ **and** $T$**.)**

**Relate this result with the behavior $\mathscr{B}$ considered in Questions 1–4, assuming that $F = 0$ in Figure 1.**

## Exercise 31: Dissipativity and the Algebraic Riccati Equation

**Consider the controllable behavior described by**

$$\mathscr{B} = \left\{ (x,u) \in \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{n}+\mathtt{m}}) \mid \frac{d}{dt}x = Ax + Bu \right\} .$$

**It follows from the material illustrated in Lecture 1 of this course that** $\mathscr{B}$ **controllable** $\iff (A,B)$ **controllable. Let now** $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}[\xi]$ **and** $U \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\xi]$ **be such that**

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} X(\frac{d}{dt}) \\ U(\frac{d}{dt}) \end{bmatrix} \ell$$

**is an observable image representation of** $\mathscr{B}$**. It can be shown that this implies** $X(\xi)U(\xi)^{-1} = (\xi I - A)^{-1}B$**. Now assume that** $\mathscr{B}$ **is dissipative with respect to**

$$\Sigma := \begin{bmatrix} Q & S^{\top} \\ S & R \end{bmatrix} \rightsquigarrow x^{\top}Qx + 2x^{\top}S^{\top}u + u^{\top}Ru \,;$$

**then the QDF**

$$\Phi(\zeta,\eta) := \begin{bmatrix} X(\zeta)^{\top} & U(\zeta)^{\top} \end{bmatrix} \begin{bmatrix} Q & S^{\top} \\ S & R \end{bmatrix} \begin{bmatrix} X(\eta) \\ U(\eta) \end{bmatrix}$$

**acting on** $\mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$ **satisfies the dissipation inequality.**

**1. Let** $K = K^{\top} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$**, and consider the QDF associated with the two-variable polynomial matrix** $X(\zeta)^{\top}KX(\eta)$**. Show that**

$$\begin{aligned} (\zeta+\eta)X(\zeta)^{\top}KX(\eta) &= X(\zeta)^{T}A^{T}KX(\eta) + U(\zeta)^{T}B^{T}KX(\eta) \\ &\quad + X(\zeta)^{T}KAX(\eta) + X(\zeta)^{T}KBU(\eta) \end{aligned}$$

**(*Hint:* Use the fact that** $X(\xi)U(\xi)^{-1} = (\xi I - A)^{-1}B$**).**

**2. Consider $\Phi(\zeta,\eta)$ defined above. Show that $K$ is such that $X(\zeta)^\top K X(\eta)$ induces a storage function for $Q_\Phi$, if and only if the *Linear Matrix Inequality***

$$\begin{bmatrix} Q-A^\top K-KA & -KB+S^\top \\ -B^\top K+S & R \end{bmatrix} \geq 0$$

**holds.**

**(*Hint:* Show that the map**

$$\mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{m}}) \to \mathbb{R}^{\mathtt{n}} \times \mathbb{R}^{\mathtt{m}}$$
$$\ell \mapsto \begin{pmatrix} (X(\frac{d}{dt})\ell)(0) \\ (U(\frac{d}{dt})\ell)(0) \end{pmatrix}$$

**is surjective. Then use the result proven in 2.1.)**

**3. Prove that the matrix $\begin{bmatrix} Q-A^T K-KA & -KB+S^T \\ -B^T K+S & R \end{bmatrix}$ has rank $\mathtt{m}$.**

**(*Hint*: Denote with $H \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi]$ a semi-Hurwitz spectral factor of $\Phi(-\xi,\xi)$. Prove that since $H(\xi) = H_0 + H_1\xi + \ldots + H_L\xi^L$ is nonsingular, the coefficient matrix $\tilde{H} := \begin{bmatrix} H_0 & H_1 & \ldots & H_L \end{bmatrix}$ has full row rank.)**

Figure 3: Jacopo Francesco Riccati, 1676-1754

**4. Prove that if $R > 0$ then the algebraic Riccati equation**

$$Q - A^\top K - KA - (-KB + S^\top)R^{-1}(-BK + S) = 0$$

**holds.**

**(*Hint:* Write the Schur complement of $R$ in the matrix of the LMI.)**

## Exercise 32: The MPUM for exponential trajectories

**In Lecture XII we have dealt with discrete-time systems only. In this exercise we extend part of the results to continuous-time.**

**1. Let $v \in \mathbb{R}^{\mathtt{w}}$ and $\lambda \in \mathbb{R}$. Prove that the dimension of**

$$\mathscr{B} = \texttt{kernel}\ \frac{vv^\top}{v^\top v}\frac{d}{dt} - \lambda I$$

**as a subspace of $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ equals one. Write down an expression for the general trajectory in $\mathscr{B}$.**

**2. In Lecture XII we discussed a procedure to recursively construct a representation of the MPUM for discrete-time data. We now discuss an analogous procedure for the case of continuous-time data consisting of a finite set of *vector-exponential trajectories* $w_i$, $i = 1, \ldots, n$:**

$$\{w_i(t) = v_i e^{\lambda_i t} \mid v \in \mathbb{R}^{\mathtt{w}}, \lambda_i \in \mathbb{R}, i = 1, \ldots, n\} \ .$$

**Define a representation for $w_1$ as in Question 1, and call it $R_1$. Now define the first error trajectory as $e_1 := R\left(\frac{d}{dt}\right) w_2$. Prove that $e_1$ is vector-exponential.**

**3. Let $E_1\left(\frac{d}{dt}\right)$ induce a representation for the MPUM for $e_1$. Prove that $E_1 R_1$ induces a kernel representation for the data set $\{w_1, w_2\}$. Infer now a procedure to model recursively the data set $\{w_i\}_{i=1,\ldots,n}$.**

**4. Let $v \in \mathbb{R}^2$, $\lambda \in \mathbb{R}_+$ and $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Assume that $v^\top \Sigma v \neq 0$. Denote with $v^\perp \in \mathbb{R}^2$ any nonzero vector such that $v^\top \Sigma v^\perp = 0$. Define**

$$R(\xi) := (\xi + \overline{\lambda}) I_2 - v \left( \frac{v^\top \Sigma v}{\lambda + \overline{\lambda}} \right)^{-1} v^\top \Sigma \ .$$

**Verify that $\texttt{kernel}\ R\left(\frac{d}{dt}\right)$ is the MPUM for the data set $\{ve^{\lambda t}, v^\perp e^{-\overline{\lambda} t}\}$.**

**5. Prove that there exist $r_i \in \mathbb{R}[\xi]$, $i = 1, 2$, such that**

$$R(\xi) = \begin{bmatrix} r_2(-\xi) & r_1(-\xi) \\ r_1(\xi) & r_2(\xi) \end{bmatrix} ,$$

**where $R$ is the matrix introduced in Question 4.**

## Exercise 33: From data to state model

1. **Let $w \in \mathscr{B}$, with $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$. Split the Hankel matrix of the data in 'past' (blue) and 'future' (pink):**

$$\begin{bmatrix} \mathscr{H}_- \\ \mathscr{H}_+ \end{bmatrix} = \begin{bmatrix} w(0) & w(1) & \cdots & w(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ w(\Delta) & w(\Delta+1) & \cdots & w(t+\Delta-1) & \cdots \\ w(\Delta+1) & w(\Delta+2) & \cdots & w(t+\Delta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ w(2\Delta) & w(2\Delta+1) & \cdots & w(t+2\Delta-1) & \cdots \end{bmatrix}$$

**where $\Delta$ is a 'large' integer. Prove that the intersection of the row spaces of past and future induces a state sequence:**

**row span**$(\mathscr{H}_-) \cap$ **row span**$(\mathscr{H}_+) = \begin{bmatrix} x(\Delta+1) & x(\Delta+2) & \cdots & x(t+\Delta) & \cdots \end{bmatrix}$ .

**(*Hint*: Let $R(\xi) = R_0 + R_1\xi + \cdots + R_L\xi^L$ induce a kernel representation of $\mathscr{B}$. Since $\Delta$ is 'large', we can assume $L < \Delta$. Observe that**

$$\begin{bmatrix} R_0 & R_1 & \cdots & R_{L-1} & R_L \end{bmatrix} \begin{bmatrix} w(\Delta-L+1) & w(\Delta-L+2) & \cdots \\ \vdots & \vdots & \vdots \\ w(\Delta) & w(\Delta+1) & \cdots \\ w(\Delta+1) & w(\Delta+2) & \cdots \end{bmatrix} = 0\,.$$

**'Shifting' we obtain also**

$$\begin{bmatrix} R_0 & R_1 & \cdots & R_{L-2} & R_{L-1} & R_L \end{bmatrix} \begin{bmatrix} w(\Delta-L+2) & w(\Delta-L+3) & \cdots \\ \vdots & \vdots & \vdots \\ w(\Delta) & w(\Delta+1) & \cdots \\ w(\Delta+1) & w(\Delta+2) & \cdots \\ w(\Delta+2) & w(\Delta+3) & \cdots \end{bmatrix} = 0\,.$$

**Now use the notion of state map. )**

**The idea of intersecting past and future of the data lies at the foundation of the *subspace approach* to system identification.**

2. **Assume that a state sequence has been computed, for example following the procedure sketched in the hint for Question 1. How would you compute matrices $E$, $F$, and $G$ corresponding to a state representation of the data-generating behavior $\mathscr{B}$? If a partition of $w$ in input and output variables is known, how would you compute the matrices $A$, $B$, $C$, $D$ corresponding to an input-state-output representation of $\mathscr{B}$?**

3. **Specialize the results of Question 1 to the case in which $\mathtt{w} = 1$ and the data consists of the linear combination of two (scalar) geometric series: $w(k) = \alpha_1\lambda_1^k + \alpha_2\lambda_2^k$, with $\alpha_1, \alpha_2 \neq 0$ and $\lambda_1 \neq \lambda_2$. If one follows the procedure sketched in the hint, what is the state representation corresponding to this data?**

4. **Assume that $\mathscr{B}$ is the behavior of a discrete-time SISO system, and that the data collected is of the form $w(k) = v_1\lambda_1^k + v_2\lambda_2^k$ for some $v_i \in \mathbb{R}^2$, $\lambda_i \in \mathbb{R}$, $i = 1,2$ with $\lambda_1 \neq \lambda_2$. What does the result of intersecting the past and the future data matrices look like? What is a state representation corresponding to this data?**