Lecture 7b

Friday 06-02-2008 **11.00-12.30**

System Identification for Deterministic Systems

Lecturer: Paolo Rapisarda

Outline

- **Modeling from data: a language;**
- **The Most Powerful Unfalsified Model;**
- **Modeling discrete-time data;**
- **The Hankel matrix;**
- **Annihilators;**
- **Recursive computation of the MPUM;**
- **State models from data.**

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This lecture deals with exact data, i.e. not corrupted by noise.

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E.g. in discrete-time, pass from

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to a kernel representation

$$R_0 w(t) + \cdots + R_L w(t+L) = 0$$

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¡No noise, no stochastics!

Modeling from data: a language

Reminder: events, variables, universum

Physical phenomenon $\rightsquigarrow$ 'outcomes', events

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Example: modeling a resistor

Attributes $\rightsquigarrow$ (voltage, current) $\rightsquigarrow$ $\mathbb{R}^2$

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Physical phenomenon $\rightsquigarrow$ 'outcomes', events

Events are described by variables

Example: modeling a gas

Attributes $\rightsquigarrow$ (pressure, temperature,volume) $\rightsquigarrow$ $\mathbb{R}^3_+$

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Physical phenomenon $\rightsquigarrow$ 'outcomes', events

Events are described by variables

Dynamical phenomena: events are maps from time space to variables space

The set of all such maps is the universum $\mathscr{U}$

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Example: modeling a resistor

$$\mathscr{U} = \left\{ (V, I) \in \left(\mathbb{R}^2\right)^{\mathbb{R}} \right\}$$

where $\left(\mathbb{R}^2\right)^{\mathbb{R}} := \{ f : \mathbb{R} \rightarrow \mathbb{R}^2 \}$

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Example: modeling a share value

$$\mathscr{U} = \left\{ V \in (\mathbb{R}_+)^{\mathbb{N}} \right\}$$

a discrete-time phenomenon

Models

Every “good” scientific theory is prohibition: it forbids certain things to happen...

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Example: Ohm's resistor

$$
\begin{aligned}
\mathscr{U} &= \left\{ (V,I) \in \left(\mathbb{R}^2\right)^{\mathbb{R}} \right\} \\
\mathscr{M} &= \{ \mathscr{B} \subset \mathscr{U} \mid \exists\, R \in \mathbb{R}_+ \textbf{ s.t. } (V,I) \in \mathscr{B} \Longrightarrow V = R\, I \}
\end{aligned}
$$

Models

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Example: Linear models

$$\begin{aligned} \mathscr{U} &= \mathbb{R}^{\mathtt{w}} \\ \mathscr{M} &= \{\text{Linear subspaces of } \mathscr{U}\} \end{aligned}$$

The Most Powerful Unfalsified Model

Modeling from data: the Most Powerful Unfalsified Model

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Fewer possible outcomes, more discriminating model, better!

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Given D **and** $\mathscr{M}$, $\mathscr{B}$ **is Most Powerful Unfalsified Model if**

- $\mathscr{B} \in \mathscr{M}$ **(i.e. admissible);**

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- $\mathscr{B} \in \mathscr{M}$**;**
- $D \subseteq \mathscr{B}$**;**
- $\mathscr{B}' \in \mathscr{M}, D \subset \mathscr{B}' \Longrightarrow \mathscr{B} \subset \mathscr{B}'$ **(i.e. most powerful).**

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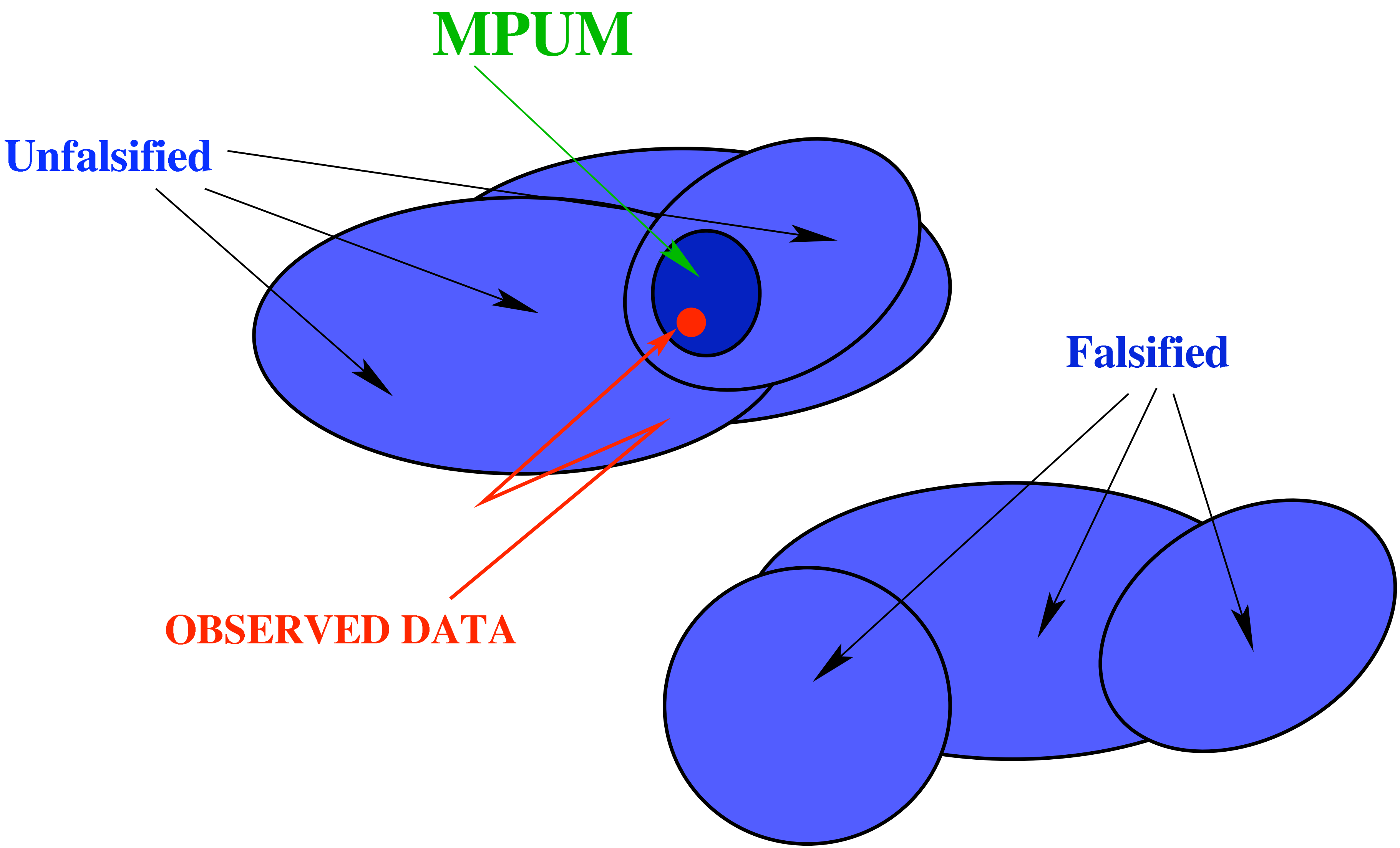
Existence? Uniqueness? Representations? Algorithms?

Graphically



The intersection property

***Example*: Consider**

$$\begin{aligned} \mathscr{U} &= \mathbb{R}^n \\ \mathscr{M} &= \textbf{Linear subspaces of } \mathbb{R}^n \end{aligned}$$

Given measurements

$$D = \{w_1, \cdots, w_k\}$$

MPUM is

$$\textbf{span } \{w_i \mid i = 1, \cdots, k\}$$

the intersection of all subspaces containing ($\equiv$ unfalsified by) data.

The intersection property

Theorem: Assume that $\mathscr{M}$ satisfies

- **The intersection property i.e.**

$$\mathscr{M}' \subset \mathscr{M} \Longrightarrow \left(\bigcap_{\mathscr{B} \in \mathscr{M}'} \mathscr{B} \right) \in \mathscr{M}$$

- **For each $D \in 2^{\mathscr{U}}$ there exists $\mathscr{B} \in \mathscr{M}$ such that $D \subseteq \mathscr{B}$.**

Then for each D there exists a unique MPUM $\mathscr{B}^*$, namely

$$\mathscr{B}^* := \bigcap_{\mathscr{B} \in \mathscr{M},\, D \subseteq \mathscr{B}} \mathscr{B}$$

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The following are instances in which the intersection property holds:

- $\mathscr{M} = 2^{\mathscr{U}}$**, whatever** $\mathscr{U}$ **is;**
- $\mathscr{U} = \mathbb{R}^{\mathtt{n}}$**,** $\mathscr{M} = \{V \mid V$ **is linear subspace of** $\mathscr{U}\}$**;**
- $\mathscr{U}$ **topological vector space, and model class is** $\mathscr{M} = \{V \mid V$ **is closed linear subspace of** $\mathscr{U}\}$**.**

Dynamical modeling from data

Time-series modeling

Problem: given $\mathtt{w}$-dimensional time series

$$w := \{w(0), w(1), \cdots\}$$

find LTI complete behavior $\mathscr{B}$ containing w.

Universum $\mathscr{U} = (\mathbb{R}^{\mathtt{w}})^{\mathbb{R}}$. Model class $\mathscr{M} = \mathscr{L}^{\mathtt{w}}$.

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$\mathscr{L}^{\mathtt{w}}$ satisfies the intersection property: MPUM exists.

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Any unfalsified model is shift-invariant: must contain

$$\begin{aligned} w &= \{w(0), w(1), \cdots\} \\ \sigma w &= \{w(1), w(2), \cdots\} \\ \sigma^2 w &= \{w(2), w(3), \cdots\} \\ &\vdots \end{aligned}$$

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Intersection of all linear unfalsified models yields

$$\mathscr{B}^* = \left(\textbf{span}\ \{w, \sigma w, \sigma^2 w, \cdots\}\right)^{\textbf{closure}}$$

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¿What about representations?

The Hankel matrix

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MPUM is subspace spanned by rows of

$$\mathscr{H}(w) := \begin{bmatrix} w(0) & w(1) & \cdots & w(t'') & \cdots \\ w(1) & w(2) & \cdots & w(t''+1) & \cdots \\ w(2) & w(3) & \cdots & w(t''+2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w(t') & w(t'+1) & \cdots & w(t'+t''-1) & \cdots \\ w(t'+1) & w(t'+2) & \cdots & w(t'+t'') & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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Constant along the block-antidiagonal: Hankel structure

The left kernel of $\mathscr{H}(w)$

Let $R(\xi) = R_0 + R_1\xi + \cdots + R_L\xi^L \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$**.**

Then $R(\sigma)w = 0 \rightsquigarrow$

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Infinite dimensional problem? Not quite!

Annihilators

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Associate polynomials with left kernel vectors:

$$\begin{bmatrix} r_0 & r_1 & \cdots & r_L & 0 & \cdots \end{bmatrix} \rightsquigarrow r(\xi) := r_0 + r_1\xi + \cdots + r_L\xi^L$$

Then $r(\xi), \xi r(\xi), \cdots$ **also represent left annihilators of** $\mathscr{H}(w)$

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Module structure of annihilators

Denote the set of annihilators of $\mathscr{H}(w)$ with

$$\mathscr{N}(\mathscr{H}(w)) := \{r_0 + r_1\xi + \cdots + r_n\xi^n \in \mathbb{R}^{1\times \mathtt{w}}[\xi] \mid \begin{bmatrix} r_0 & r_1 & \cdots & r_n & 0 & \cdots \end{bmatrix} \in \textbf{ left kernel } \mathscr{H}(w)\}$$

Then $\mathscr{N}(\mathscr{H}(w))$ is a submodule of $\mathbb{R}^{1\times \mathtt{w}}[\xi]$, and consequently it is finitely generated: there exist basis elements $a_1(\xi), \cdots, a_p(\xi) \in \mathbb{R}^{1\times \mathtt{w}}[\xi]$ such that for every $b \in \mathscr{N}(\mathscr{H}(w))$

$$\exists\, g_1(\xi), \cdots, g_p(\xi) \in \mathbb{R}[\xi] \textbf{ s.t. } b(\xi) = \sum_{i=1}^{p} g_i(\xi) a_i(\xi)$$

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Not quite "finite-dimensional", but "almost".

Recursive computation of the MPUM

Recursive computation of kernel representation of MPUM

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Equivalent formulation:

Problem: find basis for the submodule $\mathscr{N}\left(\mathscr{H}\left(w\right)\right)$

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Basic idea: compute annihilators one by one, at each step using the previous annihilators in order to get a new one.

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Basic idea: compute annihilators one by one, at each step using the previous annihilators in order to get a new one.

Basic technique: unimodular completion of a polynomial matrix

Unimodular completion of a polynomial matrix

$R \in \mathbb{R}^{\mathtt{p} \times \mathtt{w}}[\xi]$ **is left-prime if** $R(\lambda)$ **has full row rank** $\forall\ \lambda \in \mathbb{C}$**.**

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- **Unimodular completion**: $\exists\ E \in \mathbb{R}^{(\mathtt{w}-\mathtt{p})\times\mathtt{w}}[\xi]$ **such that**

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Special case $\mathtt{w} = 2$ **leads to Bézout equation**

$$\det\left(\begin{bmatrix} r_1(\xi) & r_2(\xi) \\ e_1(\xi) & e_2(\xi) \end{bmatrix}\right) = r_1(\xi)e_2(\xi) - r_2(\xi)e_1(\xi) = 1$$

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Completion is not unique. Algorithms to compute one available.

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  is unimodular.
- **Behavioral interpretation: If** $\mathscr{B} := \ker\, R(\sigma)$ **is controllable, then there exists** $\mathscr{B}' := \ker\, E(\sigma)$ **such that**

$$\mathscr{B} \oplus \mathscr{B}' = (\mathbb{R}^{\mathtt{w}})^{\mathbb{N}}$$

Recursive computation of an MPUM representation

Let $r(\xi) = r_0 + r_1\xi + \cdots + r_L\xi^n \in \mathcal{N}(\mathcal{H}(w))$**, i.e.**

$$\begin{bmatrix} r_0 & r_1 & \cdots & r_L \end{bmatrix} \begin{bmatrix} w(0) & w(1) & \cdots & w(t'') & \cdots \\ w(1) & w(2) & \cdots & w(t''+1) & \cdots \\ w(2) & w(3) & \cdots & w(t''+2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w(L) & w(L+1) & \cdots & w(L+t'') & \cdots \end{bmatrix} = 0$$

Recursive computation of an MPUM representation

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$$r'(\sigma)E_r(\sigma)w = r'(\sigma)\left(E_r(\sigma)w\right) = r'(\sigma)e = 0$$

i.e. $r'(\xi)E_r(\xi)$ **is annihilator of** w**.**

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Continue until error is zero.

From data to state model

Identification of state models of the MPUM

Problem: Compute from an infinite time-series

$$w = w(0), w(1). \cdots$$

a state space model $E\sigma x + Fx + Gw = 0$ **of the MPUM**

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Refinement: compute i/s/o model of MPUM

$$\begin{aligned} \sigma x &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

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Classical approaches based on identifying transfer function model, and then realizing it in state space form.

Subspace identification approach: construct a state sequence

$$x(0), x(1), \cdots$$

from w **by oblique projection of "past" onto "future" of** w

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Lag of $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$, denoted $\mathtt{L}(\mathscr{B})$, is smallest integer L such that

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Under suitable assumptions ("persistency of excitation"), if $\Delta \geq \mathtt{L}(\mathscr{B})$ then left kernel of

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Subspace identification

$$\begin{bmatrix} \mathscr{H}_- \\ \mathscr{H}_+ \end{bmatrix} = \begin{bmatrix} w(0) & w(1) & \cdots & w(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ w(\Delta) & w(\Delta+1) & \cdots & w(t+\Delta-1) & \cdots \\ w(\Delta+1) & w(\Delta+2) & \cdots & w(t+\Delta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ w(2\Delta) & w(2\Delta+1) & \cdots & w(t+2\Delta-1) & \cdots \end{bmatrix}$$

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Basis for row span($\mathscr{H}_-$) $\cap$ row span($\mathscr{H}_+$) induces state sequence

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Refinements using i/o partition obtaining A, B, C, D.

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Not surprising: state map matrix appears!

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