Lecture 7a

Friday 06-02-2008 **9.00-10.30**

LQ-control

Lecturer: Paolo Rapisarda

Outline

- **Optimization of QDFs;**
- **Compact support variations;**
- **Euler-Poisson equations;**
- **One-sided variations;**
- **LQ-control problem.**

Optimization of QDFs

Problem statement

Given $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$**, consider**

$$\begin{array}{rcl} \int Q_\Phi : \mathscr{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) & \longrightarrow & \mathbb{R} \\ w & \longrightarrow & \int_{-\infty}^{+\infty} Q_\Phi(w)dt \end{array}$$

with $\mathscr{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \subset \mathscr{C}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ **compact support functions.**

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Questions:

- **Stationarity and local minima w.r.t. two-sided compact-support variations;**
- **Stationarity and local minima w.r.t. one-sided compact-support variations;**
- **Minimization w.r.t. fixed initial and/or terminal conditions.**

Remark

Assumption w **free is not restrictive. Indeed, if** $w \in \mathscr{B}$ **with** $\mathscr{B}$ **controllable, let observable image representation**

$$w = M\left(\frac{d}{dt}\right)\ell$$

of $\mathscr{B}$**. Given** $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$**, define**

$$\Phi'(\zeta,\eta) := M(\zeta)^\top \Phi(\zeta,\eta) M(\eta)$$

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Observability $\Longrightarrow$ **one-one correspondence between** w **and** ℓ**.**

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Any statement about stationarity, optimality, etc. of $w \in \mathscr{B}$ **w.r.t.** Q_Φ **can be made equivalently about stationarity, optimality, etc. of the free trajectory** ℓ **w.r.t.** $Q_{\Phi'}$**.**

Stationarity

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Given $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **and** $\Phi \in \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$**, cost degradation of adding** $\delta \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **to** w **is**

$$J_w(\delta) := \int_{-\infty}^{+\infty} (Q_\Phi(w+\delta) - Q_\Phi(w))dt.$$

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Now $Q_\Phi(w_1 + w_2) = Q_\Phi(w_1) + Q_\Phi(w_2) + 2L_\Phi(w_1, w_2)$**, where**

$$L_\Phi : \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \times \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$$

$$L_\Phi(w_1, w_2) := \sum_{k,\ell} \left(\frac{d^k w_1}{dt^k} \right)^\top \Psi_{-k,\ell} \left(\frac{d^\ell w_2}{dt^\ell} \right)$$

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Conclude

$$J_w(\delta) = \int_{-\infty}^{+\infty} Q_\Phi(\delta)dt + 2\int_{-\infty}^{+\infty} L_\Phi(w, \delta)dt$$

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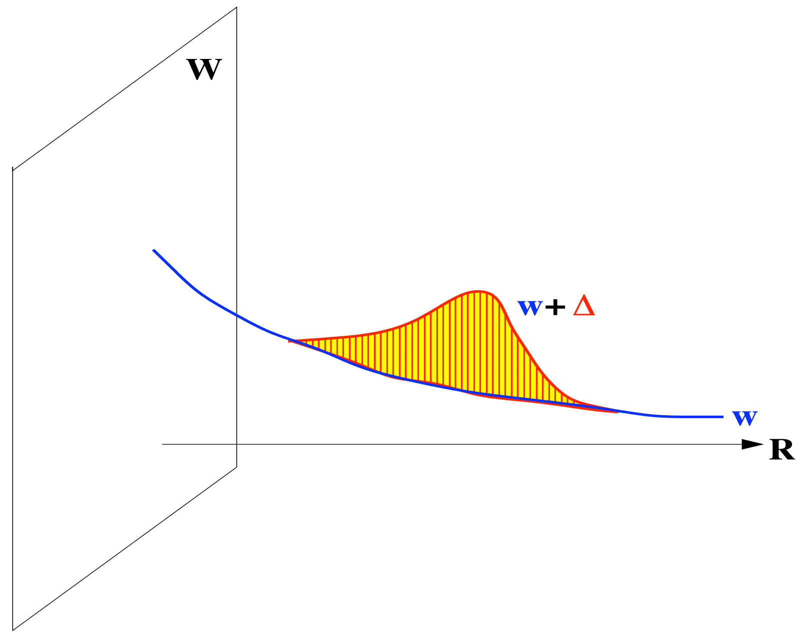
$$\begin{aligned} J_w(\delta) &:= \int_{-\infty}^{+\infty} (Q_\Phi(w+\delta) - Q_\Phi(w))dt \\ &= \int_{-\infty}^{+\infty} Q_\Phi(\delta)dt + 2 \underbrace{\int_{-\infty}^{+\infty} L_\Phi(w,\delta)dt}_{\text{variation associated with } w \text{ and } \delta} \end{aligned}$$

w **is stationary if** $\int_{-\infty}^{+\infty} L_\Phi(w,\delta)dt = 0$ **for all** $\delta \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$**, equivalently** $J_w(\delta) = \int_{-\infty}^{+\infty} Q_\Phi(\delta)dt$ **for all** $\delta \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$

Graphically



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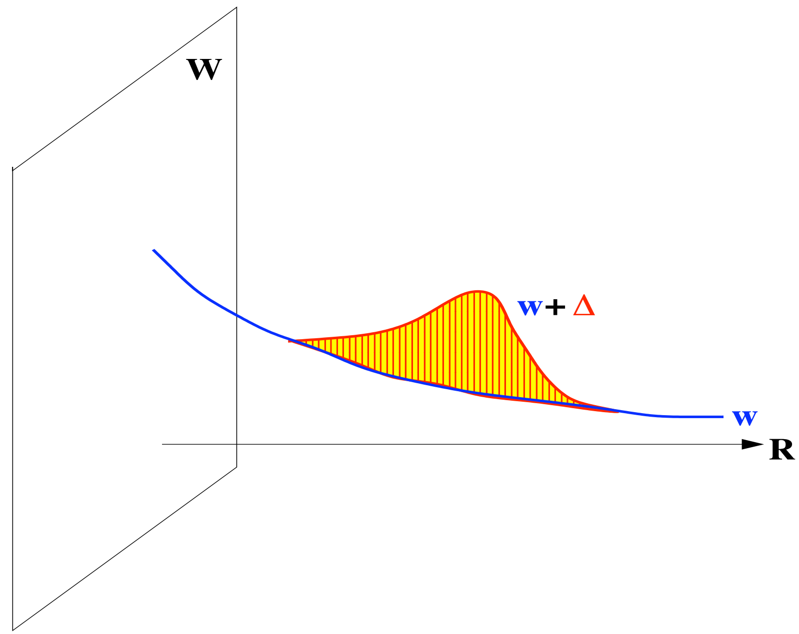

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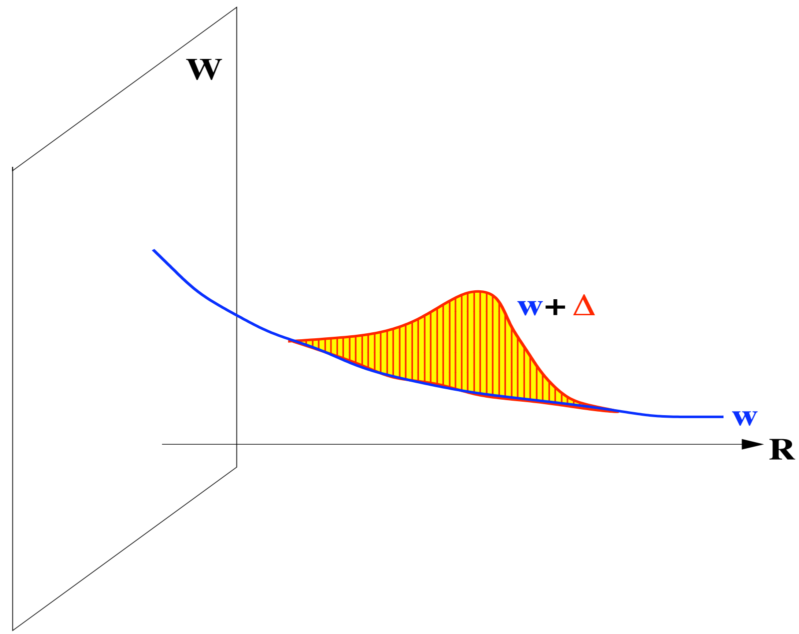

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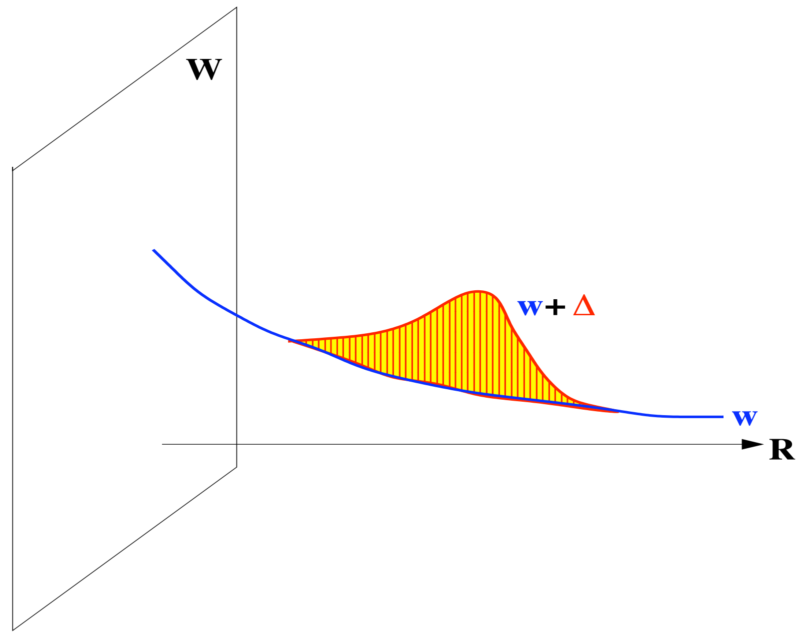

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w **stationary if** $\int_{-\infty}^{+\infty} L_\Phi(w,\delta)dt = 0$ **for all** $\delta \in \mathscr{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$**.**

Stationarity and Euler-Poisson equation

Theorem: Given $\Phi(\zeta,\eta) \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, $w \in \mathscr{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ is stationary w.r.t. $Q_\Phi \Leftrightarrow w$ satisfies

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Proof: Factor $\Phi(\zeta,\eta) = M^T(\zeta)\Sigma_\Phi M(\eta)$, with Σ_Φ nonsingular signature matrix, and $M(\xi) = M_0 + M_1\xi + M_2\xi^2 + \cdots + M_L\xi^L$.

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Integrate variation $\int_{-\infty}^{+\infty}(M(\frac{d}{dt})w)^T\Sigma\left(M(\frac{d}{dt})\delta\right)dt$ by parts on $\delta \in \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. Results in

$$\sum_{k=1}^{L}\sum_{j=k}^{L}(-1)^{k-1}\delta^{(j-k)}M_j^T\Sigma\left(M\left(\frac{d}{dt}\right)w\right)^{(k-1)}\Bigg|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty}\delta^T\left(M\left(-\frac{d}{dt}\right)^T\Sigma M\left(\frac{d}{dt}\right)w\right)dt.$$

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Conclude variation is zero iff for all $\delta\in\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ holds

$$\int_{-\infty}^{+\infty}\delta^T\left(M\left(-\frac{d}{dt}\right)^T\Sigma M\left(\frac{d}{dt}\right)w\right)dt=0$$

Arbitrariness of $\delta \rightsquigarrow M\left(-\frac{d}{dt}\right)^T\Sigma M\left(\frac{d}{dt}\right)w=0$.

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$$\partial\Phi(\xi) = \partial\Phi(-\xi)^\top$$

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- **If** $\det\ \Phi(-\xi,\xi)\neq 0$**, then** $\lambda\in\mathbb{C}$ **is characteristic frequency of** $\ker\ \partial\Phi\left(\frac{d}{dt}\right)\implies -\overline{\lambda}\in\mathbb{C}$ **is also characteristic frequency: 'time-symmetry'.**

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- **Time-symmetry** $\rightsquigarrow$ **stability is not part of the picture!**

Stationarity and duality

Duality

Given $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **and** $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ **signature matrix,**
the dual behavior of $\mathscr{B}$ **is**

$$\mathscr{B}^{\perp_\Sigma} := \{v \in \mathscr{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \quad | \quad \int_{-\infty}^{+\infty} v^\top \Sigma w dt = 0$$

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If $\mathscr{B} = \ker R\left(\frac{d}{dt}\right) = \operatorname{im} M\left(\frac{d}{dt}\right)$**, then**

$$\mathscr{B}^{\perp_\Sigma} = \operatorname{im} \Sigma R\left(-\frac{d}{dt}\right)^\top = \ker M\left(-\frac{d}{dt}\right)^\top \Sigma$$

Stationarity and duality

<u>Theorem</u>: Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ controllable, and $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ be a signature matrix. The set of stationary trajectories of $\mathscr{B}$ with respect to Σ is

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Proof: Let $\mathscr{B} = \operatorname{im} M\left(\frac{d}{dt}\right)$, define $\Phi(\zeta,\eta) := M(\zeta)^\top \Sigma M(\eta)$. Observe that

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$$\iff \begin{cases} R\left(\frac{d}{dt}\right) w = 0 \\ M\left(-\frac{d}{dt}\right)^\top \Sigma w = 0 \end{cases}$$

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- **Set of stationary trajectories has kernel representation**

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- **Interconnection of behaviors: $\ker M\left(-\frac{d}{dt}\right)^{\top}\Sigma$ is optimal controller for $\mathscr{B}$;**

- **Analogous to Hamiltonian matrix**

$$\begin{bmatrix} A & -BR^{-1}B^{\top} \\ -Q & -A^{\top} \end{bmatrix}$$

of LQ-control problem $\min \int_0^T x(t)^{\top}Qx(t) + u(t)^{\top}Ru(t)dt$ **subject to** $\frac{d}{dt}x = Ax + Bu$**.**

Local minima

Local minima

Given $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **and** $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$**,** w **is local minimum of** Q_Φ **if**

$$J_w(\delta) = \int_{-\infty}^{+\infty} (Q_\Phi(w+\delta) - Q_\Phi(w))dt \geq 0$$

for all $\delta \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$**.**

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Theorem: w **is local minimum of** Q_Φ **if and only if**

1. $\Phi\left(-\frac{d}{dt}, \frac{d}{dt}\right) w = 0$**; (stationarity)**

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Theorem: w **is local minimum of** Q_Φ **if and only if**

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2. $\int_{-\infty}^{+\infty} Q_\Phi(\delta)dt \geq 0$ **for all** $\delta \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **(average non-negativity)**

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<u>Theorem</u>: w **is local minimum of** Q_Φ **if and only if**

1. $\Phi\left(-\frac{d}{dt},\frac{d}{dt}\right)w = 0$**;**
2. $\Phi(-i\omega,i\omega) \geq 0$ **for all** $\omega \in \mathbb{R}$**.**

Example

Take `w` $= 1$**, and** $\Phi(\zeta,\eta) = 1 - \zeta\eta$**:**

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No local minimum: $\Phi(-i\omega, i\omega) = 1 - \omega^2$**, not always nonnegative.**

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Theorem: Let $\Phi \in \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$**; assume that** $\Phi(-i\omega, i\omega) > 0$ **for all** $\omega \in \mathbb{R}$**. Let** H **be a Hurwitz spectral factor of** $\Phi(-\xi, \xi)$**, i.e.**

$$\Phi(-\xi, \xi) = H(-\xi)^\top H(\xi)$$

and $\det(H)$ **is a Hurwitz polynomial.**

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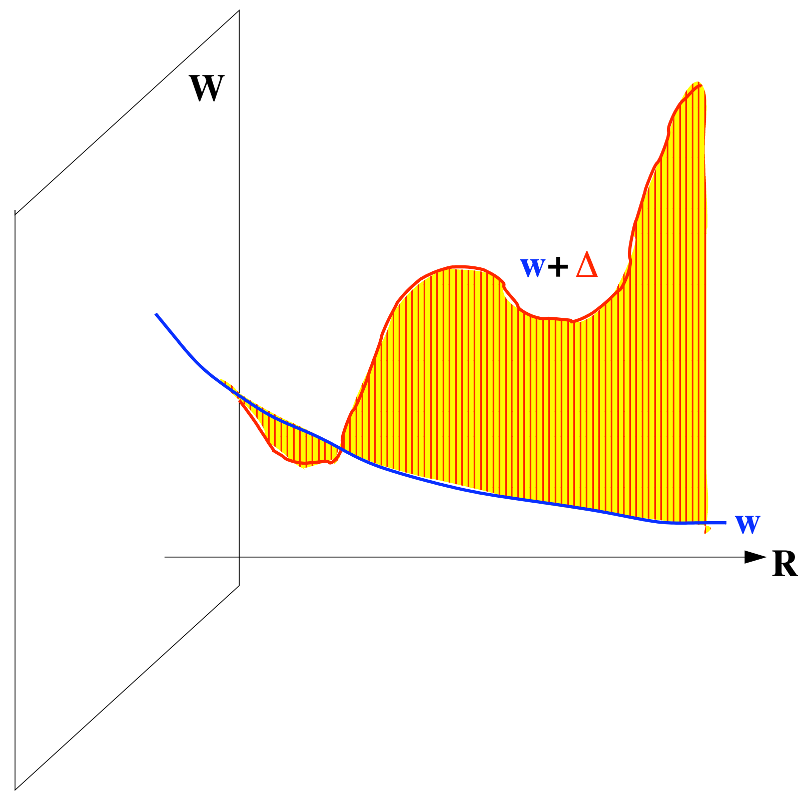
Then the set of stable local minima for Q_Φ **is** $\ker H\left(\frac{d}{dt}\right)$**.**

Remarks

- **Stability also direct consequence of stating problem with one-sided variations, i.e. variations δ with support in $[t, +\infty)$ for some $t \in \mathbb{R}$.**



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- $\Phi(-\xi,\xi) = H(-\xi)^\top H(\xi) \Longrightarrow \exists\, \Psi_- \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ **such that**

$$\Phi(\zeta,\eta) = H(\zeta)^\top H(\eta) + (\zeta+\eta)\Psi_-(\zeta,\eta)$$

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Consequently

$$\int_0^{+\infty} Q_\Phi(w)dt \quad = \quad \int_0^{+\infty} \|H\left(\frac{d}{dt}\right) w\|^2 dt + \int_0^{+\infty} \frac{d}{dt} Q_{\Psi_-}(w)dt$$

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Consequently

$$\int_0^{+\infty} Q_\Phi(w)dt \quad = \quad -Q_{\Psi_-}(w)(0)dt$$

for the stable trajectories of $\ker\, H\left(\frac{d}{dt}\right)$**.**

Minimal storage function yields optimal cost computation!

LQ-control
in a behavioral setting

Problem statement

Given $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$**, find**

$$\textbf{inf} \int_0^{+\infty} Q_\Phi(w)dt$$

subject to the initial conditions

$$\left(I\left(\frac{d}{dt}\right)w\right)(0) = a$$

where $I \in \mathbb{R}^{\mathtt{k}\times\mathtt{w}}[\xi]$**,** $a \in \mathbb{R}^{\mathtt{k}}$**.**

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Additional conditions on w **and derivatives as** $t \to \infty$ **possible.**

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Additional conditions on w **and derivatives as** $t \to \infty$ **possible.**

¿Existence? Uniqueness? When is infimum=minimum?

On the initial conditions

$$\textbf{inf} \int_0^{+\infty} Q_\Phi(w)dt$$

$$\textbf{subject to} \left(I \left(\frac{d}{dt} \right) w \right) (0) = a$$

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Example: Take $I(\xi) = \begin{bmatrix} I_{\mathtt{w}} & \xi I_{\mathtt{w}} \dots & \xi^L I_{\mathtt{w}} \end{bmatrix}$ for expressing conditions on w and its derivatives at $t = 0$.

On the initial conditions

$$\textbf{inf} \int_0^{+\infty} Q_\Phi(w)dt$$

$$\textbf{subject to} \left(I\left(\frac{d}{dt}\right) w \right)(0) = a$$

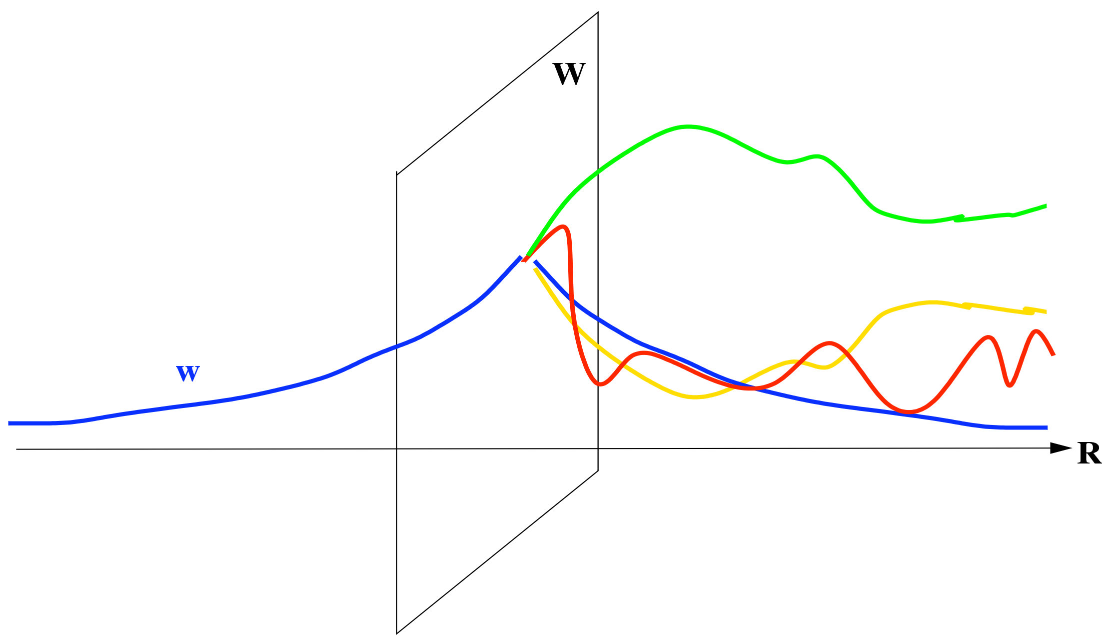
Example: Take $I(\xi) = X(\xi)$, a state map, for expressing conditions on the initial state of the system.

On the initial conditions

$$\textbf{inf} \int_0^{+\infty} Q_\Phi(w)dt$$

$$\textbf{subject to} \left(I\left(\frac{d}{dt}\right)w\right)(0) = a$$

Example: find inf $\int_0^{+\infty} \|w\|^2 dt$ **subj. to** $w \in \mathscr{B}$**,** $w_{|(-\infty,0]}$ **given:**



On the initial conditions

$$\textbf{inf} \int_0^{+\infty} Q_\Phi(w)\,dt$$

$$\textbf{subject to} \left(I\left(\frac{d}{dt}\right) w \right)(0) = a$$

Affine constraints on $\frac{d^k w}{dt^k}(0)$**,** $k = 0, \ldots$

Outline of solution

$$\textbf{inf} \int_0^{+\infty} Q_\Phi(w)dt$$

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Infimum is $> -\infty \Longrightarrow \Phi(-i\omega, i\omega) \geq 0$ **for all** $\omega \in \mathbb{R}$**.**

Outline of solution

$$\textbf{inf} \int_0^{+\infty} Q_\Phi(w)dt$$

$$\textbf{subject to} \left(I\left(\frac{d}{dt}\right)w\right)(0) = a$$

Assume $\Phi(-i\omega, i\omega) > 0$ for all $\omega \in \mathbb{R}$. Then Hurwitz spectral factorization $\Phi(-\xi, \xi) = H(-\xi)^\top H(\xi)$ yields

$$Q_\Phi(w) = \left(\frac{d}{dt}Q_{\Psi_-}\right)(w) + \|H\left(\frac{d}{dt}\right)w\|^2$$

Outline of solution

$$\textbf{inf} \int_0^{+\infty} Q_\Phi(w)dt$$

$$\textbf{subject to} \left(I\left(\frac{d}{dt}\right) w \right)(0) = a$$

Consequently

$$\int_0^{+\infty} Q_\Phi(w)dt = \int_0^{+\infty} \left(\frac{d}{dt} Q_{\Psi_-}\right)(w)dt + \int_0^{+\infty} \|H\left(\frac{d}{dt}\right) w\|^2 dt$$

Outline of solution

$$\textbf{inf} \int_0^{+\infty} Q_\Phi(w)dt$$

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$$\begin{aligned}\int_0^{+\infty} Q_\Phi(w)dt &= \int_0^{+\infty}\left(\frac{d}{dt}Q_{\Psi_-}\right)(w)dt + \int_0^{+\infty}\|H\left(\frac{d}{dt}\right)w\|^2dt \\ &= -Q_{\Psi_-}(w)(0) + \int_0^{+\infty}\|H\left(\frac{d}{dt}\right)w\|^2dt\end{aligned}$$

Outline of solution

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Outline of solution

$$\textbf{inf} \quad -Q_{\Psi_-}(w)(0) + \int_0^{+\infty} \|H\left(\frac{d}{dt}\right)w\|^2 dt$$

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Outline of solution

$$\textbf{inf}\ -Q_{\Psi_-}(w)(0)+\int_0^{+\infty}\left\|H\left(\frac{d}{dt}\right)w\right\|^2 dt$$

$$\textbf{subject to}\ \left(I\left(\frac{d}{dt}\right)w\right)(0)=a$$

- $-Q_{\Psi_-}(w)(0)$ **depends only on** $\frac{d^k w}{dt^k}(0)$, $k=0,\ldots$:

$$\begin{bmatrix} w(0)^\top & \frac{dw}{dt}(0)^\top & \ldots \end{bmatrix}\begin{bmatrix} \Psi_{-0,0} & \Psi_{-0,1} & \cdots \\ \Psi_{-1,0} & \Psi_{-1,1} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix}\begin{bmatrix} w(0) \\ \frac{dw}{dt}(0) \\ \vdots \end{bmatrix}$$

A quadratic form induced by the coefficient matrix $\mathrm{mat}(\Psi_-)$ on the independent variables $\frac{d^k w}{dt^k}(0)$, $k=0,\ldots$.

Outline of solution

$$\textbf{inf} \; -Q_{\Psi_-}(w)(0) + \int_0^{+\infty} \|H\left(\frac{d}{dt}\right)w\|^2 dt$$

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- $\int_0^{+\infty} \|H\left(\frac{d}{dt}\right)w\|^2 dt$ **term is** ≥ 0**, zero if** $w \in \ker\, H\left(\frac{d}{dt}\right)$**.**

Outline of solution

$$\textbf{inf} \; -Q_{\Psi_-}(w)(0) + \int_0^{+\infty} \|H\left(\frac{d}{dt}\right)w\|^2 dt$$

$$\textbf{subject to} \; \left(I\left(\frac{d}{dt}\right)w\right)(0) = a$$

Assume that

- **infimum** $\frac{d^k w^*}{dt^k}(0)$**,** $k = 0, \ldots$ **of** $\mathrm{mat}(\Psi_-)$ **is** $> -\infty$**;**
- **infimum** $\frac{d^k w^*}{dt^k}(0)$**,** $k = 0, \ldots$ **satisfies the initial condition;**
- $\exists\, \overline{w} \in \ker\, H\left(\frac{d}{dt}\right)$ **such that** $\frac{d^k \overline{w}}{dt^k}(0) = \frac{d^k w^*}{dt^k}(0)$**,** $k = 0, \ldots$

Then $\overline{w}$ **is infimum.**

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- **Algorithms: representation $\rightsquigarrow$ stationary/optimal behavior?**

Main points

- **Optimality problems for systems described by higher-order equations, with no *a priori* representation;**
- **QDFs and their calculus are essential tools;**
- **Variational approach;**
- **Stationarity;**
- **Local minimality;**
- **LQ-control.**

End of Lecture 7a