Lecture 5a

Thursday 05-02-2008 **09.00-10.30**

Linear Quadratic Theory-I

Lecturer: Paolo Rapisarda

Outline

- **Motivation;**
- **Bilinear- and quadratic differential forms;**
- **Two-variable polynomial representation;**
- **Calculus of B/QDFs;**
- **Lyapunov theory for higher-order systems.**

Motivation

Dynamics and functionals in systems and control

Instances: Lyapunov theory, performance criteria, etc.

Linear case $\Longrightarrow$ *quadratic* and *bilinear* functionals.

Dynamics and functionals in systems and control

Instances: Lyapunov theory, performance criteria, etc.

Linear case $\Longrightarrow$ *quadratic* and *bilinear* functionals.

Usually: state-space equations, constant functionals.

But tearing and zooming do not yield state space equations!

Dynamics and functionals in systems and control

Instances: Lyapunov theory, performance criteria, etc.

Linear case $\Longrightarrow$ *quadratic* and *bilinear* functionals.

Usually: state-space equations, constant functionals.

But tearing and zooming do not yield state space equations!

¡High-order differential equations!

...involving also *latent variables*...

Dynamics and functionals in systems and control

Instances: Lyapunov theory, performance criteria, etc.

Linear case $\Longrightarrow$ *quadratic* and *bilinear* functionals.

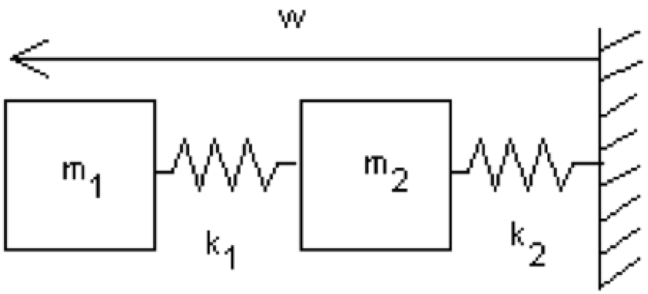
Usually: state-space equations, constant functionals.

But tearing and zooming do not yield state space equations!

¡High-order differential equations!

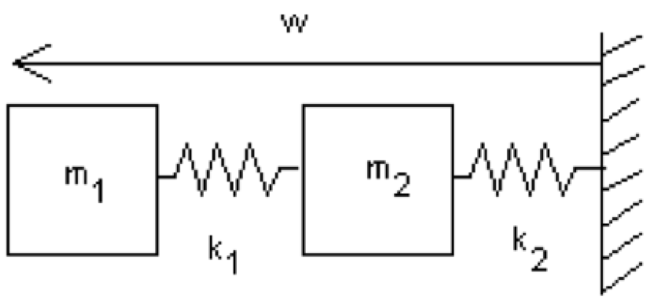
...involving also *latent variables*...

Example : a mechanical system



$$
\begin{aligned}
m_1 \frac{d^2 w_1}{dt^2} + k_1 w_1 - k_2 w_2 &= 0 \\
-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 &= 0
\end{aligned}
$$

Example : a mechanical system

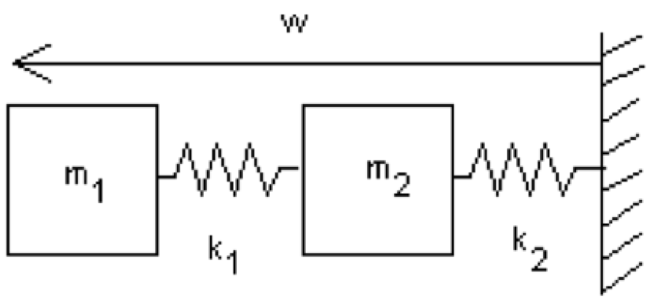


$$
\begin{aligned}
m_1 \frac{d^2 w_1}{dt^2} + k_1 w_1 - k_2 w_2 &= 0 \\
-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 &= 0
\end{aligned}
$$

Eliminate w_2**:**

$$
m_1 m_2 \frac{d^4}{dt^4} w_1 + (k_1 m_1 + k_2 m_1 + k_1 m_2) \frac{d^2}{dt^2} w_1 + k_1 k_2 w_1 = 0
$$

Example : a mechanical system



$$
\begin{aligned}
m_1 \frac{d^2 w_1}{dt^2} + k_1 w_1 - k_2 w_2 &= 0 \\
-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 &= 0
\end{aligned}
$$

Eliminate w_2**:**

$$
m_1 m_2 \frac{d^4}{dt^4} w_1 + (k_1 m_1 + k_2 m_1 + k_1 m_2) \frac{d^2}{dt^2} w_1 + k_1 k_2 w_1 = 0
$$

¿Stability, stored energy, conservation laws?

Aim

An effective algebraic representation
of bilinear and quadratic functionals
of the system variables and their derivatives:

Operations/properties of functionals
$\Updownarrow$
algebraic operations/properties of representation

...a calculus of these functionals!

Bilinear and quadratic differential forms

Bilinear differential forms (BDFs)

$$\Phi := \left\{ \Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2} \right\}_{k,\ell=0,\ldots,L}$$

$$L_\Phi : \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_1}) \times \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_2}) \longrightarrow \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$$

$$L_\Phi(w_1,w_2) := \begin{bmatrix} w_1^\top & \frac{dw_1}{dt}^\top & \ldots \end{bmatrix} \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots \\ \vdots & \vdots & \cdots \\ \Phi_{k,0} & \Phi_{k,1} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} w_2 \\ \frac{dw_2}{dt} \\ \vdots \end{bmatrix}$$

$$= \Sigma_{k,\ell} \left(\frac{d^k}{dt^k} w_1 \right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dt^\ell} w_2 \right)$$

Quadratic differential forms (QDFs)

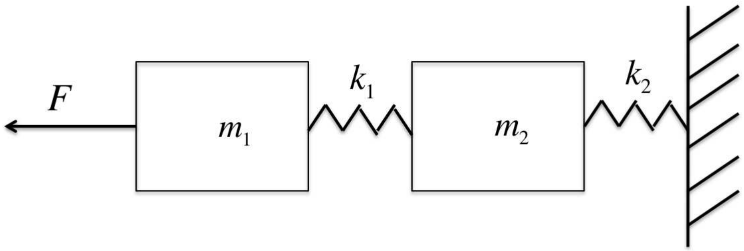
$$\Phi := \left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}\right\}_{k,\ell=0,\ldots,L} \textbf{ symmetric, i.e. } \Phi_{k,\ell} = \Phi_{\ell,k}^{\top}$$

$$Q_\Phi : \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \longrightarrow \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$$

$$Q_\Phi(w) := \begin{bmatrix} w^\top & \frac{dw}{dt}^\top & \cdots \end{bmatrix} \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots \\ \vdots & \vdots & \cdots \\ \Phi_{k,0} & \Phi_{k,1} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} w \\ \frac{dw}{dt} \\ \vdots \end{bmatrix}$$

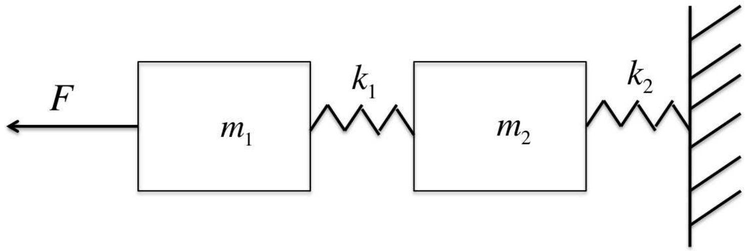
$$= \Sigma_{k,\ell=0}^{L} \left(\frac{d^k w}{dt^k}\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell w}{dt^\ell}\right)$$

Example: total energy in mechanical system



$$
\begin{aligned}
m_1 \frac{d^2 w_1}{dt^2} + k_1(w_1 - w_2) - F &= 0 \\
-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 &= 0
\end{aligned}
$$

Example: total energy in mechanical system



$$
\begin{aligned}
m_1 \frac{d^2 w_1}{dt^2} + k_1(w_1 - w_2) - F &= 0 \\
-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 &= 0
\end{aligned}
$$

Total energy is

$$
\frac{1}{2} m_1 \left(\frac{d}{dt} w_1 \right)^2 + \frac{1}{2} m_2 \left(\frac{d}{dt} w_2 \right)^2 + \frac{1}{2} k_1 (w_1 - w_2)^2 + \frac{1}{2} k_2 w_2^2
$$

$$
= \begin{bmatrix} w_1 & w_2 & F & \frac{d}{dt} w_1 & \frac{d}{dt} w_2 & \frac{d}{dt} F \end{bmatrix}
\begin{bmatrix}
\frac{1}{2}k_1 & -\frac{1}{2}k_1 & 0 & 0 & 0 & 0 \\
-\frac{1}{2}k_1 & \frac{1}{2}(k_1 + k_2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}m_1 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2}m_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} w_1 \\ w_2 \\ F \\ \frac{d}{dt} w_1 \\ \frac{d}{dt} w_2 \\ \frac{d}{dt} F \end{bmatrix}
$$

Two-variable polynomial representation

Two-variable polynomial matrices for BDFs

$$\left\{ \Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2} \right\}_{k,\ell=0,\ldots,L}$$

Two-variable polynomial matrices for BDFs

$$\left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}\right\}_{k,\ell=0,\ldots,L}$$

$$L_\Phi(w_1, w_2) = \sum_{k,\ell=0}^{L} \left(\frac{d^k w_1}{dt^k}\right)^\top \Phi_{k,\ell} \frac{d^\ell w_2}{dt^\ell}$$

Two-variable polynomial matrices for BDFs

$$\left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}\right\}_{k,\ell=0,\ldots,L}$$

$$L_\Phi(w_1, w_2) = \sum_{k,\ell=0}^{L} \left(\frac{d^k w_1}{dt^k}\right)^\top \Phi_{k,\ell} \frac{d^\ell w_2}{dt^\ell}$$

$$\Phi(\zeta, \eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell}\ \zeta^k \eta^\ell$$

Two-variable polynomial matrices for BDFs

$$\left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}\right\}_{k,\ell=0,\ldots,L}$$

$$L_\Phi(w_1, w_2) = \sum_{k,\ell=0}^{L} \left(\frac{d^k w_1}{dt^k}\right)^\top \Phi_{k,\ell} \frac{d^\ell w_2}{dt^\ell}$$

$$\Phi(\zeta, \eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell}\, \zeta^k \eta^\ell$$

Two-variable polynomial matrices for BDFs

$$\left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}\right\}_{k,\ell=0,\ldots,L}$$

$$L_\Phi(w_1, w_2) = \sum_{k,\ell=0}^{L} \left(\frac{d^k w_1}{dt^k}\right)^\top \Phi_{k,\ell} \frac{d^\ell w_2}{dt^\ell}$$

$$\Phi(\zeta, \eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell} \, \zeta^k \, \eta^\ell$$

2-variable polynomial matrix associated with L_Φ

Two-variable polynomial matrices for QDFs

$\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}\}_{k,\ell=0,\ldots,L}$ **symmetric** $(\Phi_{k,\ell} = \Phi_{\ell,k}^\top)$

$$Q_\Phi(w) = \sum_{k,\ell=0}^{L} \left(\frac{d^k w}{dt^k}\right)^\top \Phi_{k,\ell} \frac{d^\ell w}{dt^\ell}$$

$$\Phi(\zeta,\eta) = \textstyle\sum_{k,\ell=0}^{L} \Phi_{k,\ell}\, \zeta^k\, \eta^\ell$$

***symmetric*:** $\Phi(\zeta,\eta) = \Phi(\eta,\zeta)^\top$

Example: total energy in mechanical system

$$Q_E(w_1, w_2, F) =$$

$$\begin{bmatrix} w_1 & w_2 & F & \frac{d}{dt}w_1 & \frac{d}{dt}w_2 & \frac{d}{dt}F \end{bmatrix} \begin{bmatrix} \frac{1}{2}k_1 & -\frac{1}{2}k_1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}k_1 & \frac{1}{2}(k_1+k_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ F \\ \frac{d}{dt}w_1 \\ \frac{d}{dt}w_2 \\ \frac{d}{dt}F \end{bmatrix}$$

$$E(\zeta, \eta) = \begin{bmatrix} \frac{1}{2}k_1 & -\frac{1}{2}k_1 & 0 \\ -\frac{1}{2}k_1 & \frac{1}{2}(k_1+k_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\zeta\eta & 0 & 0 \\ 0 & \frac{1}{2}\zeta\eta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example: total energy in mechanical system

$$Q_E(w_1, w_2, F) =$$

$$\begin{bmatrix} w_1 & w_2 & F & \frac{d}{dt}w_1 & \frac{d}{dt}w_2 & \frac{d}{dt}F \end{bmatrix} \begin{bmatrix} \frac{1}{2}k_1 & -\frac{1}{2}k_1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}k_1 & \frac{1}{2}(k_1+k_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ F \\ \frac{d}{dt}w_1 \\ \frac{d}{dt}w_2 \\ \frac{d}{dt}F \end{bmatrix}$$

$$E(\zeta, \eta) = \begin{bmatrix} \frac{1}{2}k_1 & -\frac{1}{2}k_1 & 0 \\ -\frac{1}{2}k_1 & \frac{1}{2}(k_1+k_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\zeta\eta & 0 & 0 \\ 0 & \frac{1}{2}\zeta\eta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example: total energy in mechanical system

$Q_E(w_1, w_2, F) =$

$$\begin{bmatrix} w_1 & w_2 & F & \frac{d}{dt}w_1 & \frac{d}{dt}w_2 & \frac{d}{dt}F \end{bmatrix} \begin{bmatrix} \frac{1}{2}k_1 & -\frac{1}{2}k_1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}k_1 & \frac{1}{2}(k_1+k_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ F \\ \frac{d}{dt}w_1 \\ \frac{d}{dt}w_2 \\ \frac{d}{dt}F \end{bmatrix}$$

$$E(\zeta,\eta) = \begin{bmatrix} \frac{1}{2}k_1 & -\frac{1}{2}k_1 & 0 \\ -\frac{1}{2}k_1 & \frac{1}{2}(k_1+k_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\zeta\eta & 0 & 0 \\ 0 & \frac{1}{2}\zeta\eta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The calculus of B/QDFs

The calculus of B/QDFs

Using powers of ζ and η as placeholders,

B/QDF $\leftrightsquigarrow$ two-variable polynomial matrix

The calculus of B/QDFs

Using powers of ζ and η as placeholders,

B/QDF $\leftrightsquigarrow$ two-variable polynomial matrix

Operations and properties of B/QDF	$\leftrightsquigarrow$	**algebraic operations/properties on two-variable matrix**

The calculus of B/QDFs

Using powers of ζ and η as placeholders,

B/QDF $\leftrightsquigarrow$ two-variable polynomial matrix

Operations and properties of B/QDF	$\leftrightsquigarrow$	**algebraic operations/properties on two-variable matrix**

Differentiation

$\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. $\overset{\bullet}{\Phi}$ **derivative of** Q_Φ**:**

$$Q_{\overset{\bullet}{\Phi}} : \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \to \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$$

$$Q_{\overset{\bullet}{\Phi}}(w) := \frac{d}{dt}(Q_\Phi(w))$$

...a QDF, evidently... what is its two-variable representation?

Differentiation

$\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. $\overset{\bullet}{\Phi}$ **derivative** of Q_Φ:

$$Q_{\overset{\bullet}{\Phi}} : \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \to \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$$

$$Q_{\overset{\bullet}{\Phi}}(w) := \frac{d}{dt}(Q_\Phi(w))$$

...a QDF, evidently... what is its two-variable representation?

$$\overset{\bullet}{\Phi}(\zeta,\eta) = (\zeta+\eta)\Phi(\zeta,\eta)$$

Two-variable version of Leibniz's rule

Integration

$\mathscr{D}(\mathbb{R},\mathbb{R}^{\bullet})$: **set of $\mathscr{C}^{\infty}$-compact-support trajectories**

$$L_{\Phi} : \mathscr{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}_1}) \times \mathscr{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}_2}) \longrightarrow \mathscr{D}(\mathbb{R},\mathbb{R})$$

$$\int L_{\Phi} : \mathscr{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}_1}) \times \mathscr{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}_2}) \longrightarrow \mathbb{R}$$

$$\int L_{\Phi}(w_1,w_2) := \int_{-\infty}^{+\infty} L_{\Phi}(w_1,w_2)dt$$

Analogous for QDFs

Combining dynamics and functionals: B/QDFs zero along behaviors

Q_Φ **zero on** $\mathfrak{B}$ **(denoted** $Q_\Phi \overset{\mathfrak{B}}{=} 0$**) if**

$$Q_\Phi(w) = 0 \textbf{ for all } w \in \mathfrak{B}$$

Combining dynamics and functionals: B/QDFs zero along behaviors

Q_Φ **zero on** $\mathfrak{B}$ **(denoted** $Q_\Phi \overset{\mathfrak{B}}{=} 0$**) if**

$$Q_\Phi(w) = 0 \textbf{ for all } w \in \mathfrak{B}$$

Theorem: Let $\mathfrak{B} = \ker(R(\frac{d}{dt}))$**. Then**

$$Q_\Phi \overset{\mathfrak{B}}{=} 0 \iff \exists\, F \in \mathbb{R}^{\bullet\times\bullet}[\zeta,\eta] \textbf{ such that}$$

$$\Phi(\zeta,\eta) = R(\zeta)^\top F(\zeta,\eta) + F(\eta,\zeta)^\top R(\eta)$$

Combining dynamics and functionals: B/QDFs zero along behaviors

Q_Φ **zero on** $\mathfrak{B}$ **(denoted** $Q_\Phi \stackrel{\mathfrak{B}}{=} 0$**) if**

$$Q_\Phi(w) = 0 \text{ for all } w \in \mathfrak{B}$$

<u>Theorem</u>: Let $\mathfrak{B} = \ker(R(\frac{d}{dt}))$**. Then**

$$Q_\Phi \stackrel{\mathfrak{B}}{=} 0 \Longleftrightarrow \exists\, F \in \mathbb{R}^{\bullet\times\bullet}[\zeta,\eta] \text{ such that}$$

$$\Phi(\zeta,\eta) = R(\zeta)^\top F(\zeta,\eta) + F(\eta,\zeta)^\top R(\eta)$$

QDF induced by the 2-variable polynomial matrix on RHS is instantaneously zero for $w \in \mathscr{B} = \ker\ R\left(\frac{d}{dt}\right)$**.**

Example: conservation laws

Oscillator: $m\frac{d^2}{dt^2}w + kw = 0 \rightsquigarrow r(\xi) = m\xi^2 + k$

Total energy is $Q_E(w) = \frac{1}{2}m\left(\frac{dw}{dt}\right)^2 + \frac{1}{2}kw^2$**. A QDF.**

Example: conservation laws

Oscillator: $m\frac{d^2}{dt^2}w + kw = 0 \rightsquigarrow r(\xi) = m\xi^2 + k$

Total energy is $Q_E(w) = \frac{1}{2}m\left(\frac{dw}{dt}\right)^2 + \frac{1}{2}kw^2$**. A QDF.**

Two-variable representation is $E(\zeta,\eta) = \frac{1}{2}m\zeta\eta + \frac{1}{2}k$**.**

Example: conservation laws

Oscillator: $m\frac{d^2}{dt^2}w + kw = 0 \rightsquigarrow r(\xi) = m\xi^2 + k$

Total energy is $Q_E(w) = \frac{1}{2}m\left(\frac{dw}{dt}\right)^2 + \frac{1}{2}kw^2$**. A QDF.**

Two-variable representation is $E(\zeta,\eta) = \frac{1}{2}m\zeta\eta + \frac{1}{2}k$**.**

Physics: total energy is conserved

$$\frac{d}{dt}E(w) = 0 \text{ for all } w \in \mathscr{B}$$

Example: conservation laws

Oscillator: $m\frac{d^2}{dt^2}w + kw = 0 \rightsquigarrow r(\xi) = m\xi^2 + k$

Total energy is $Q_E(w) = \frac{1}{2}m\left(\frac{dw}{dt}\right)^2 + \frac{1}{2}kw^2$**. A QDF.**

Two-variable representation is $E(\zeta,\eta) = \frac{1}{2}m\zeta\eta + \frac{1}{2}k$**.**

Physics: total energy is conserved

$$\frac{d}{dt}E(w) = 0 \text{ for all } w \in \mathscr{B}$$

Indeed,

$$(\zeta+\eta)E(\zeta,\eta) = \frac{1}{2}\left(m\zeta^2+k\right)\eta + \left(m\eta^2+k\right)\zeta \;\;=\;\; \zeta r(\eta) + r(\zeta)\eta$$

zero along $\mathscr{B}$**.**

Equivalence of QDFs

$$Q_{\Phi_1} \stackrel{\mathfrak{B}}{=} Q_{\Phi_2} \textbf{ if } Q_{\Phi_1}(w) = Q_{\Phi_2}(w) \textbf{ for all } w \in \mathfrak{B}$$

If $\mathfrak{B} = \ker(R(\frac{d}{dt}))$**,** $Q_{\Phi_1} \stackrel{\mathfrak{B}}{=} Q_{\Phi_2}$ **equivalent with**

$$\Phi_1(\zeta,\eta) - \Phi_2(\zeta,\eta) = R(\zeta)^\top F(\zeta,\eta) + F(\eta,\zeta)^\top R(\eta)$$

¿Canonical representative ?

Preliminary: R-canonical polynomial differential operators

$$\mathscr{B} = \ker(R(\tfrac{d}{dt}))$$ **autonomous:** $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ $\det(R) \neq 0$.

$D(\frac{d}{dt}) \overset{\mathscr{B}}{=} P(\frac{d}{dt})$ **if** $D(\frac{d}{dt})w = P(\frac{d}{dt})w$ **for all** $w \in \mathscr{B}$.

$D(\frac{d}{dt})$ **is** R**-canonical if** DR^{-1} **is strictly proper.**

Every $D(\frac{d}{dt})$ is equivalent along $\mathscr{B}$ to an R-canonical polynomial differential operator:

$$DR^{-1} = \underbrace{P}_{\textbf{polynomial}} + \underbrace{S}_{\textbf{strictly proper}} \Longrightarrow D \overset{\mathscr{B}}{=} D - PR$$

R-canonical quadratic differential forms

Q_Φ R-canonical if $R(\zeta)^{-\top}\Phi(\zeta,\eta)R(\eta)^{-1}$ strictly proper.

R-canonical quadratic differential forms

Q_Φ R-canonical if $R(\zeta)^{-\top}\Phi(\zeta,\eta)R(\eta)^{-1}$ strictly proper.

Theorem: Every Q_Φ is $\mathscr{B}$-equivalent to a R-canonical QDF.

R-canonical quadratic differential forms

Q_Φ R-canonical if $R(\zeta)^{-\top}\Phi(\zeta,\eta)R(\eta)^{-1}$ strictly proper.

Theorem: Every Q_Φ is $\mathscr{B}$-equivalent to a R-canonical QDF.

1. **Compute factorization $\Phi(\zeta,\eta) = N(\zeta)^\top M(\eta)$;**
2. **Compute R-canonical repr. M' for M, and N' for N;**
3. **R-canonical repr. of Φ is $\Phi'(\zeta,\eta) := N'(\zeta)^\top M'(\eta)$**

R-canonical quadratic differential forms

Q_Φ R-canonical if $R(\zeta)^{-\top}\Phi(\zeta,\eta)R(\eta)^{-1}$ strictly proper.

Theorem: Every Q_Φ is $\mathscr{B}$-equivalent to a R-canonical QDF.

1. **Compute factorization $\Phi(\zeta,\eta) = N(\zeta)^\top M(\eta)$;**
2. **Compute R-canonical repr. M' for M, and N' for N;**
3. **R-canonical repr. of Φ is $\Phi'(\zeta,\eta) := N'(\zeta)^\top M'(\eta)$**

Factorization of step 1: factorize coefficient matrix

$$\begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} = \begin{bmatrix} N_0' \\ N_1' \\ \vdots \end{bmatrix} \begin{bmatrix} M_0' & M_1' & \cdots \end{bmatrix}$$

Example: the scalar case

$$r_0 w + r_1 \frac{dw}{dt} + \cdots + \frac{d^n w}{dt^n} w = 0$$

Example: the scalar case

$$r_0 w + r_1 \frac{dw}{dt} + \cdots + \frac{d^n w}{dt^n} w = 0$$

Any quadratic expression of w **and its derivatives with terms containing** $\frac{d^k w}{dt^k}$ **for** $k \geq n$ **rewritable in terms of** w **and its derivatives up to** $\frac{d^{n-1} w}{dt^{n-1}}$**.**

Example: the scalar case

$$r_0 w + r_1 \frac{dw}{dt} + \cdots + \frac{d^n w}{dt^n} w = 0$$

Any quadratic expression of w **and its derivatives with terms containing** $\frac{d^k w}{dt^k}$ **for** $k \geq n$ **rewritable in terms of** w **and its derivatives up to** $\frac{d^{n-1} w}{dt^{n-1}}$**.**

E.g. since for $w \in \ker\ r\left(\frac{d}{dt}\right)$

$$\frac{d^n w}{dt^n} = -\left(r_0 w + r_1 \frac{dw}{dt} + \cdots + \frac{d^{n-1} w}{dt^{n-1}} w\right)$$

it holds

$$\left(\frac{d^n w}{dt^n}\right)^2 = \left(r_0 w + r_1 \frac{dw}{dt} + \cdots + \frac{d^{n-1} w}{dt^{n-1}} w\right)^2$$

Example: the scalar case

$$r_0 w + r_1 \frac{dw}{dt} + \cdots + \frac{d^n w}{dt^n} w = 0$$

Any quadratic expression of w **and its derivatives with terms containing** $\frac{d^k w}{dt^k}$ **for** $k \geq n$ **rewritable in terms of** w **and its derivatives up to** $\frac{d^{n-1} w}{dt^{n-1}}$**.**

Same for terms containing $\frac{d^{n+1} w}{dt^{n+1}}$**, etc.**

Example: the scalar case

$$r_0 w + r_1 \frac{dw}{dt} + \cdots + \frac{d^n w}{dt^n} w = 0$$

Any quadratic expression of w and its derivatives with terms containing $\frac{d^k w}{dt^k}$ for $k \geq n$ rewritable in terms of w and its derivatives up to $\frac{d^{n-1} w}{dt^{n-1}}$.

"Rewriting in terms of lower order derivatives" equivalent to "taking the r-canonical representative".

Nonnegativity and positivity

$Q_\Phi \geq 0$ **if** $Q_\Phi(w) \geq 0$ **for all** $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$

Prop.: $Q_\Phi \geq 0$ **if and only if exists** $D \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ **s.t.**

$$\Phi(\zeta, \eta) = D(\zeta)^\top D(\eta)$$

Nonnegativity and positivity

$Q_\Phi \geq 0$ **if** $Q_\Phi(w) \geq 0$ **for all** $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$

Prop.: $Q_\Phi \geq 0$ **if and only if exists** $D \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ **s.t.**

$$\Phi(\zeta, \eta) = D(\zeta)^\top D(\eta)$$

$Q_\Phi > 0$ **if** $Q_\Phi \geq 0$**, and** $[Q_\Phi(w) = 0] \implies [w = 0]$

Prop.: $Q_\Phi > 0$ **if and only if exists** $D \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ **s.t.**

$$\Phi(\zeta, \eta) = D(\zeta)^\top D(\eta)$$

$$\mathtt{rank}(D(\lambda)) = \mathtt{w} \text{ for all } \lambda \in \mathbb{C}$$

Nonnegativity and positivity along a behavior

$Q_\Phi \overset{\mathscr{B}}{\geq} 0$ **if** $Q_\Phi(w) \geq 0 \; \forall \; w \in \mathscr{B}$

Prop.: $Q_\Phi \overset{\mathscr{B}}{\geq} 0$ **iff** $\exists \; \Phi' \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ **such that**

$\Phi(\zeta,\eta) \overset{\mathscr{B}}{=} \Phi'(\zeta,\eta)$ **and** $Q_{\Phi'} \geq 0$

Nonnegativity and positivity along a behavior

$Q_\Phi \overset{\mathscr{B}}{\geq} 0$ **if** $Q_\Phi(w) \geq 0 \ \forall\, w \in \mathscr{B}$

Prop.: $Q_\Phi \overset{\mathscr{B}}{\geq} 0$ **iff** $\exists\ \Phi' \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ **such that** $\Phi(\zeta,\eta) \overset{\mathscr{B}}{=} \Phi'(\zeta,\eta)$ **and** $Q_{\Phi'} \geq 0$

Prop.: **Let** $\mathscr{B} = \ker R(\frac{d}{dt})$**. Then** $Q_\Phi \overset{\mathscr{B}}{\geq} 0$ **iff there exist** $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$**,** $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\zeta,\eta]$ **such that**

$$\Phi(\zeta,\eta) = \underbrace{D(\zeta)^\top D(\eta)}_{\geq 0 \text{ for all } w} + \underbrace{R(\zeta)^\top X(\zeta,\eta) + X(\eta,\zeta)^\top R(\eta)}_{= 0 \text{ if evaluated on } \mathscr{B}}$$

Nonnegativity and positivity along a behavior

$Q_\Phi \overset{\mathscr{B}}{>} 0$ **if** $Q_\Phi \overset{\mathscr{B}}{\geq} 0$**, and** $[Q_\Phi(w) = 0] \Longrightarrow [w = 0]$

Prop.: $Q_\Phi \overset{\mathscr{B}}{>} 0$ **iff** $\exists\ \Phi' \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ **such that**

$\Phi(\zeta, \eta) \overset{\mathscr{B}}{=} \Phi'(\zeta, \eta)$ **and** $\Phi'(\zeta, \eta) = D(\zeta)^\top D(\eta)$ **with**
$\mathtt{rank}\ \mathrm{col}(R(\lambda), D(\lambda)) = \mathtt{w}$ **for all** $\lambda \in \mathbb{C}$

Nonnegativity and positivity along a behavior

$Q_\Phi \overset{\mathscr{B}}{>} 0$ **if** $Q_\Phi \overset{\mathscr{B}}{\geq} 0$**, and** $[Q_\Phi(w) = 0] \Longrightarrow [w = 0]$

Prop.: $Q_\Phi \overset{\mathscr{B}}{>} 0$ **iff** $\exists\ \Phi' \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ **such that**

$\Phi(\zeta,\eta) \overset{\mathscr{B}}{=} \Phi'(\zeta,\eta)$ **and** $\Phi'(\zeta,\eta) = D(\zeta)^\top D(\eta)$ **with**
rank $\operatorname{col}(R(\lambda), D(\lambda)) = \mathtt{w}$ **for all** $\lambda \in \mathbb{C}$

Prop.: **Let** $\mathscr{B} = \ker R(\frac{d}{dt})$**. Then** $Q_\Phi \overset{\mathscr{B}}{>} 0$ **iff there exist** $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$**,** $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\zeta,\eta]$ **such that**

$$\Phi(\zeta,\eta) = \underbrace{D(\zeta)^\top D(\eta)}_{\geq 0 \text{ for all } w} + \underbrace{R(\zeta)^\top X(\zeta,\eta) + X(\eta,\zeta)^\top R(\eta)}_{=0 \text{ if evaluated on } \mathscr{B}}$$

and moreover rank $\operatorname{col}(R(\lambda), D(\lambda)) = \mathtt{w}$ **for all** $\lambda \in \mathbb{C}$

Remarks

- **Property of functional (positivity, nonnegativity along a behavior, etc.) translated into algebraic properties of its two-variable representation;**

Remarks

- **Property of functional (positivity, nonnegativity along a behavior, etc.) translated into algebraic properties of its two-variable representation;**
- **Standard polynomial computations to obtain R-canonical representative $\Longrightarrow$ positivity, negativity along behaviors easy to check.**

Lyapunov Theory

Lyapunov theory

$\mathscr{B}$ **is *asymptotically stable*** $\Leftrightarrow \lim_{t \to \infty} w(t) = 0 \ \forall \ w \in \mathscr{B}$

This implies $\mathscr{B}$ **is autonomous.**

Lyapunov theory

$\mathscr{B}$ **is *asymptotically stable*** $\Leftrightarrow \lim_{t\to\infty} w(t) = 0 \ \forall\, w \in \mathscr{B}$

This implies $\mathscr{B}$ is autonomous.

Algebraic characterization:

<u>Theorem</u>: $\mathscr{B} = \ker R(\frac{d}{dt})$ **stable,** $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$
$\Longleftrightarrow \det(R)$ **Hurwitz**

Lyapunov theory

$\mathscr{B}$ **is *asymptotically stable*** $\Leftrightarrow \lim_{t\to\infty} w(t) = 0 \; \forall \; w \in \mathscr{B}$

This implies $\mathscr{B}$ is autonomous.

Algebraic characterization:

Theorem: $\mathscr{B} = \ker R(\frac{d}{dt})$ **stable,** $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$
$\Longleftrightarrow \det(R)$ **Hurwitz**

Lyapunov functional characterization:

Theorem: $\mathscr{B}$ **asymptotically stable iff**

exists Q_Φ **such that** $Q_\Phi \overset{\mathscr{B}}{\geq} 0$ **and** $Q_{\overset{\bullet}{\Phi}} \overset{\mathscr{B}}{<} 0$

Lyapunov theory

$\mathscr{B}$ **is *asymptotically stable*** $\Leftrightarrow \lim_{t\to\infty} w(t) = 0 \; \forall \; w \in \mathscr{B}$

This implies $\mathscr{B}$ **is autonomous.**

Algebraic characterization:

Theorem: $\mathscr{B} = \ker R(\frac{d}{dt})$ **stable,** $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$
$\iff \det(R)$ **Hurwitz**

Lyapunov functional characterization:

Theorem: $\mathscr{B}$ **asymptotically stable iff**

exists Q_Φ **such that** $Q_\Phi \overset{\mathscr{B}}{\geq} 0$ **and**$Q_{\overset{\bullet}{\Phi}} \overset{\mathscr{B}}{<} 0$

Example

$$\mathscr{B} = \ker\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) \rightsquigarrow r(\xi) = \xi^2 + 3\xi + 2$$

Example

$$\mathscr{B} = \ker\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) \rightsquigarrow r(\xi) = \xi^2 + 3\xi + 2$$

Choose $\Psi(\zeta,\eta)$ **s.t.** $Q_\Psi \overset{\mathscr{B}}{<} 0$**, e.g.** $\Psi(\zeta,\eta) = -\zeta\eta$**;**

Example

$$\mathscr{B} = \ker\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) \rightsquigarrow r(\xi) = \xi^2 + 3\xi + 2$$

Choose $\Psi(\zeta,\eta)$ **s.t.** $Q_\Psi \overset{\mathscr{B}}{<} 0$**, e.g.** $\Psi(\zeta,\eta) = -\zeta\eta$**;**

Find $\Phi(\zeta,\eta)$ **s.t.** $\frac{d}{dt}Q_\Phi(w) = Q_\Psi(w)$ **for all** $w \in \mathscr{B}$**:**

$$(\zeta+\eta)\Phi(\zeta,\eta) = \Psi(\zeta,\eta) + \underbrace{r(\zeta)x(\eta) + x(\zeta)r(\eta)}_{=0 \textbf{ on } \mathscr{B}}$$

Example

$$\mathscr{B} = \ker\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) \rightsquigarrow r(\xi) = \xi^2 + 3\xi + 2$$

Choose $\Psi(\zeta,\eta)$ **s.t.** $Q_\Psi \overset{\mathscr{B}}{<} 0$**, e.g.** $\Psi(\zeta,\eta) = -\zeta\eta$**;**

Find $\Phi(\zeta,\eta)$ **s.t.** $\frac{d}{dt}Q_\Phi(w) = Q_\Psi(w)$ **for all** $w \in \mathscr{B}$**:**

$$(\zeta+\eta)\Phi(\zeta,\eta) = \Psi(\zeta,\eta) + \underbrace{r(\zeta)x(\eta) + x(\zeta)r(\eta)}_{=0 \textbf{ on } \mathscr{B}}$$

$$\frac{d}{dt}Q_\Phi(w) = Q_\Psi(w) \textbf{ for all } w \in \mathscr{B}$$

Example

$$\mathscr{B} = \ker\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) \rightsquigarrow r(\xi) = \xi^2 + 3\xi + 2$$

Choose $\Psi(\zeta,\eta)$ **s.t.** $Q_\Psi \overset{\mathscr{B}}{<} 0$**, e.g.** $\Psi(\zeta,\eta) = -\zeta\eta$**;**

Find $\Phi(\zeta,\eta)$ **s.t.** $\frac{d}{dt}Q_\Phi(w) = Q_\Psi(w)$ **for all** $w \in \mathscr{B}$**:**

$$(\zeta+\eta)\Phi(\zeta,\eta) = \Psi(\zeta,\eta) + \underbrace{r(\zeta)x(\eta) + x(\zeta)r(\eta)}_{=0 \text{ on } \mathscr{B}}$$

Equivalent to solving polynomial Lyapunov equation

$$0 = \underbrace{\Psi(-\xi,\xi)}_{\xi^2} + \underbrace{r(-\xi)}_{\xi^2-3\xi+2}\, x(\xi) + x(-\xi)\, \underbrace{r(\xi)}_{\xi^2+3\xi+2}$$

$\rightsquigarrow x(\xi) = \frac{1}{6}\xi$

Example

$$\mathscr{B} = \ker\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right) \rightsquigarrow r(\xi) = \xi^2 + 3\xi + 2$$

Choose $\Psi(\zeta,\eta)$ **s.t.** $Q_\Psi \overset{\mathscr{B}}{<} 0$**, e.g.** $\Psi(\zeta,\eta) = -\zeta\eta$**;**

Find $\Phi(\zeta,\eta)$ **s.t.** $\frac{d}{dt}Q_\Phi(w) = Q_\Psi(w)$ **for all** $w \in \mathscr{B}$**:**

$$(\zeta+\eta)\Phi(\zeta,\eta) = \Psi(\zeta,\eta) + \underbrace{r(\zeta)x(\eta) + x(\zeta)r(\eta)}_{=0 \textbf{ on } \mathscr{B}}$$

$$\begin{aligned}\Phi(\zeta,\eta) &= \frac{-\zeta\eta + (\zeta^2+3\zeta+2)\frac{1}{6}\eta + \frac{1}{6}\zeta(\eta^2+3\eta+2)}{\zeta+\eta} \\ &= \frac{1}{6}\zeta\eta + \frac{1}{3} > 0\end{aligned}$$

Main points

- **Functionals of system variables and derivatives thereof;**

Main points

- Functionals of system variables and derivatives thereof;
- Two-variable polynomial representation;

Main points

- **Functionals of system variables and derivatives thereof;**
- **Two-variable polynomial representation;**
- **Operations/properties in time domain $\rightsquigarrow$ algebraic operations on two-variable representation.**

Main points

- **Functionals of system variables and derivatives thereof;**
- **Two-variable polynomial representation;**
- **Operations/properties in time domain $\rightsquigarrow$ algebraic operations on two-variable representation;**
- **Lyapunov theory for higher-order systems.**

End of Lecture 5a

Lecture 5b

Thursday 05-02-2008 **11.00-12.30**

Linear Quadratic Theory-II

Lecturer: Paolo Rapisarda

Outline

- **Dissipative systems;**
- **Spectral factorization;**
- **Storage functions;**
- **Distributed dissipative systems.**

Dissipative Systems

Dissipation inequality

Physical examples:

- **Resistive electrical circuits;**
- **Mechanical systems with friction;**
- **...**

Dissipation inequality

Physical examples:

- **Resistive electrical circuits;**
- **Mechanical systems with friction;**
- **...**

Energy supplied to system $\rightsquigarrow$ **supply rate variable** F_Σ

Dissipation inequality

Physical examples:

- **Resistive electrical circuits;**
- **Mechanical systems with friction;**
- **...**

Energy supplied to system $\rightsquigarrow$ **supply rate variable** F_Σ

- **Electrical circuits:** $V^\top I$ **with** V **(resp.** I**) vector of voltages (resp. currents)**
- **Mechanical systems:** $F^\top \frac{d}{dt}x$ **with** F **(resp.** x**) vector of forces (resp. displacements)**

Dissipation inequality

Energy supplied to system $\rightsquigarrow$ **supply rate variable** F_Σ

Energy stored in system $\rightsquigarrow$ **storage variable** F_S

Dissipation inequality

Energy supplied to system $\rightsquigarrow$ **supply rate variable** F_Σ

Energy stored in system $\rightsquigarrow$ **storage variable** F_S

- **Electrical circuits:** $\frac{1}{2}C \cdot V^2$ **for capacitor,** $\frac{1}{2}L \cdot I^2$ **for inductor**
- **Mechanical systems:** $\frac{1}{2}K \cdot x^2$ **for spring**

Dissipation inequality

Energy supplied to system $\rightsquigarrow$ supply rate variable F_Σ

Energy stored in system $\rightsquigarrow$ storage variable F_S

In a dissipative system,
energy cannot be stored faster than it is supplied

Dissipation inequality

Energy supplied to system $\rightsquigarrow$ supply rate variable F_Σ

Energy stored in system $\rightsquigarrow$ storage variable F_S

In a dissipative system,
energy cannot be stored faster than it is supplied

$$\frac{d}{dt}F_S \leq F_\Sigma$$

Dissipation inequality

Energy supplied to system $\rightsquigarrow$ **supply rate variable** F_Σ

Energy stored in system $\rightsquigarrow$ **storage variable** F_S

In a dissipative system,
energy cannot be stored faster than it is supplied

$$\frac{d}{dt}F_S \leq F_\Sigma$$

Dissipation inequality

Dissipation equality

$$\frac{d}{dt} F_S \leq F_\Sigma$$

Dissipation equality

$$\frac{d}{dt} F_S \leq F_\Sigma$$

$$F_\Delta := F_\Sigma - \frac{d}{dt} F_S$$ **dissipation rate (nonnegative)**

Dissipation equality

$$\frac{d}{dt}F_S \leq F_\Sigma$$

$F_\Delta := F_\Sigma - \frac{d}{dt}F_S$ **dissipation rate (nonnegative)**

$$F_\Sigma = F_\Delta + \frac{d}{dt}F_S$$

Dissipation equality

Dissipation equality

$$\frac{d}{dt}F_S \leq F_\Sigma$$

$$F_\Delta := F_\Sigma - \frac{d}{dt}F_S$$ **dissipation rate (nonnegative)**

$$F_\Sigma = F_\Delta + \frac{d}{dt}F_S$$

Dissipation equality

Lossless systems: $F_\Sigma = \frac{d}{dt}F_S$

Dissipation equality

$$\frac{d}{dt}F_S \leq F_\Sigma$$

$$F_\Delta := F_\Sigma - \frac{d}{dt}F_S$$ **dissipation rate (nonnegative)**

$$F_\Sigma = F_\Delta + \frac{d}{dt}F_S$$

Dissipation equality

Lossless systems: $F_\Sigma = \frac{d}{dt}F_S$

Now, linear time-invariant finite-dimensional systems, with quadratic supply rates

Setting the stage

LTI systems $\rightsquigarrow$ **supply, dissipation, storage are quadratic functionals of the system variables and their derivatives**

Dissipation equality:

$$Q_\Phi(w) = Q_\Delta(w) + \frac{d}{dt} Q_\Psi(w)$$

where $w \in \mathscr{B}$

Setting the stage

LTI systems $\rightsquigarrow$ **supply, dissipation, storage are quadratic functionals of the system variables and their derivatives**

Dissipation equality:

$$Q_\Phi(w) = Q_\Delta(w) + \frac{d}{dt} Q_\Psi(w)$$

where $w \in \mathscr{B}$

...equalities along $\mathscr{B}$ are cumbersome to work with...

Setting the stage

Controllable system

$w = M(\frac{d}{dt})\ell \rightsquigarrow M(\xi)$

Power ('supply rate')

$Q_\Phi \rightsquigarrow \Phi(\zeta,\eta)$

Setting the stage

Controllable system

$$w = M(\tfrac{d}{dt})\ell \rightsquigarrow M(\xi)$$

Power ('supply rate')

$$Q_\Phi \rightsquigarrow \Phi(\zeta,\eta)$$

$$Q_\Phi(w) = Q_\Phi(M(\tfrac{d}{dt})\ell)$$

$$\Phi'(\zeta,\eta) := M(\zeta)^\top \Phi(\zeta,\eta) M(\eta)$$

$Q_{\Phi'}$ **acts on free variable** ℓ**, i.e.** $\mathscr{C}^\infty$

When is a system dissipative?

Dissipation equality:

$$Q_\Phi = \frac{d}{dt} Q_\Psi + Q_\Delta$$

When is a system dissipative?

Dissipation equality:

$$Q_\Phi = \frac{d}{dt} Q_\Psi + Q_\Delta$$

Integrate along compact-support trajectory:

$$\int_{-\infty}^{+\infty} Q_\Phi(w)dt = Q_\Psi(w) \mid_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} Q_\Delta(w)dt$$

When is a system dissipative?

Dissipation equality:

$$Q_\Phi = \frac{d}{dt} Q_\Psi + Q_\Delta$$

Integrate along compact-support trajectory:

$$\int_{-\infty}^{+\infty} Q_\Phi(w)dt = \underbrace{Q_\Psi(w) \mid_{-\infty}^{+\infty}}_{=0} + \int_{-\infty}^{+\infty} \underbrace{Q_\Delta(w)}_{\geq 0} dt$$

When is a system dissipative?

Dissipation equality:

$$Q_\Phi = \frac{d}{dt} Q_\Psi + Q_\Delta$$

$$\int_{-\infty}^{+\infty} Q_\Phi(w)dt \geq 0$$

for all compact-support trajectories $w \in \mathscr{B}$

When is a system dissipative?

$$\int_{-\infty}^{+\infty} Q_{\Phi}(w)dt \geq 0$$

for all compact-support trajectories $w \in \mathscr{B}$

If $w = M(\frac{d}{dt})\ell$**, equivalent to**

$$Q_{\Phi'}(\ell) \geq 0 \text{ **for all** } \ell \in \mathscr{C}^{\infty}$$

with $\Phi'(\zeta,\eta) = M(\zeta)^{\top}\Phi(\zeta,\eta)M(\eta)$

When is a system dissipative?

$$\int_{-\infty}^{+\infty} Q_\Phi(w)dt \geq 0$$

for all compact-support trajectories $w \in \mathscr{B}$

If $w = M(\frac{d}{dt})\ell$**, equivalent to**

$$Q_{\Phi'}(\ell) \geq 0 \text{ **for all** } \ell \in \mathscr{C}^\infty$$

with $\Phi'(\zeta,\eta) = M(\zeta)^\top \Phi(\zeta,\eta) M(\eta)$

Fourier transformation leads to

$$\Phi'(-i\omega, i\omega) = M(-i\omega)^\top \Phi(-i\omega, i\omega) M(i\omega) \geq 0$$

for all $\omega \in \mathbb{R}$

¡A frequency-domain inequality!

When is a system dissipative?

We just proved:

<u>Theorem</u>: $\mathrm{im}\, M(\frac{d}{dt})$ **is Φ-dissipative if and only if** $M(-i\omega)^\top \Phi(-i\omega, i\omega) M(i\omega) \geq 0$ **for all** $\omega \in \mathbb{R}$

Characterizations of dissipativity

<u>Theorem:</u> The following conditions are equivalent:

- $\int_{-\infty}^{+\infty} Q_\Phi(\ell)dt \geq 0$ **for all** $\mathscr{C}^\infty$ **compact-support** ℓ**;**
- Q_Φ **admits a storage function;**
- Q_Φ **admits a dissipation rate**

Given Q_Φ**, storage and dissipation are one-one:**

$$\begin{aligned} \frac{d}{dt}Q_\Psi &= Q_\Phi - Q_\Delta \\ (\zeta+\eta)\Psi(\zeta,\eta) &= \Phi(\zeta,\eta) - \Delta(\zeta,\eta) \end{aligned}$$

Characterizations of dissipativity

<u>Theorem:</u> The following conditions are equivalent:

- $\int_{-\infty}^{+\infty} Q_\Phi(\ell)dt \geq 0$ **for all** $\mathscr{C}^\infty$ **compact-support** ℓ**;**
- Q_Φ **admits a storage function;**
- Q_Φ **admits a dissipation rate**

Given Q_Φ**, storage and dissipation are one-one:**

$$\begin{aligned} \frac{d}{dt} Q_\Psi &= Q_\Phi - Q_\Delta \\ (\zeta+\eta)\Psi(\zeta,\eta) &= \Phi(\zeta,\eta) - \Delta(\zeta,\eta) \end{aligned}$$

¿Given Φ, how to find dissipation/storage functions?

Spectral factorization

Dissipation in an algebraic setting: spectral factorization

$$(\zeta+\eta)\Psi(\zeta,\eta)+\Delta(\zeta,\eta)=\Phi(\zeta,\eta)$$

¿How to compute Δ and Ψ?

Dissipation in an algebraic setting: spectral factorization

$$(\zeta+\eta)\Psi(\zeta,\eta)+\Delta(\zeta,\eta)=\Phi(\zeta,\eta)$$

¿How to compute Δ and Ψ?

Let $\zeta=-\xi$, $\eta=\xi$; then $\Delta(-\xi,\xi)=\Phi(-\xi,\xi)$

Dissipation in an algebraic setting: spectral factorization

$$(\zeta+\eta)\Psi(\zeta,\eta)+\Delta(\zeta,\eta)=\Phi(\zeta,\eta)$$

¿How to compute Δ and Ψ?

Let $\zeta=-\xi$**,** $\eta=\xi$**; then** $\Delta(-\xi,\xi)=\Phi(-\xi,\xi)$

$Q_\Delta(\ell)\geq 0\ \forall\ \ell\in\mathscr{C}^\infty(\mathbb{R},\mathbb{R}^\bullet)\implies\exists$ **square** $D\in\mathbb{R}^{\bullet\times\bullet}[\xi]$ **such that**

$$\Delta(\zeta,\eta)=D(\zeta)^\top D(\eta)$$

Dissipation in an algebraic setting: spectral factorization

$$(\zeta+\eta)\Psi(\zeta,\eta)+\Delta(\zeta,\eta)=\Phi(\zeta,\eta)$$

¿How to compute Δ and Ψ?

Let $\zeta=-\xi$**,** $\eta=\xi$**; then** $\Delta(-\xi,\xi)=\Phi(-\xi,\xi)$

$Q_\Delta(\ell)\geq 0\ \forall\ \ell\in\mathscr{C}^\infty(\mathbb{R},\mathbb{R}^\bullet)\implies\exists$ **square** $D\in\mathbb{R}^{\bullet\times\bullet}[\xi]$ **such that**

$$\Delta(\zeta,\eta)=D(\zeta)^\top D(\eta)$$

Spectral factorization: given $\Phi(-\xi,\xi)$**, find square** D **s.t.**

$$\Phi(-\xi,\xi)=D(-\xi)^\top D(\xi)$$

Example

$$\Phi(\zeta,\eta) = 4 + 6\eta + 2\eta^2 + 6\zeta + 9\zeta\eta + 4\zeta\eta^2 + 2\zeta^2 + 4\zeta^2\eta + \eta^2\zeta^2$$

Example

$$\Phi(\zeta,\eta) = 4 + 6\eta + 2\eta^2 + 6\zeta + 9\zeta\eta + 4\zeta\eta^2 + 2\zeta^2 + 4\zeta^2\eta + \eta^2\zeta^2$$

Check if $\Phi(-i\omega, i\omega) \geq 0$ **for all** $\omega \in \mathbb{R}$**:**

$$\Phi(-i\omega, i\omega) = 4 + 5\omega^2 + \omega^4$$

a sum of squares, always nonnegative.

Example

$$\Phi(\zeta,\eta) = 4 + 6\eta + 2\eta^2 + 6\zeta + 9\zeta\eta + 4\zeta\eta^2 + 2\zeta^2 + 4\zeta^2\eta + \eta^2\zeta^2$$

Check if $\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$:

$$\Phi(-i\omega, i\omega) = 4 + 5\omega^2 + \omega^4$$

a sum of squares, always nonnegative.

Note that

$$\Phi(-\xi,\xi) = 4 - 5\xi^2 + \xi^4 = (\xi - 2)(\xi - 1)(\xi + 1)(\xi + 2)$$

Example

$$\Phi(\zeta,\eta) = 4 + 6\eta + 2\eta^2 + 6\zeta + 9\zeta\eta + 4\zeta\eta^2 + 2\zeta^2 + 4\zeta^2\eta + \eta^2\zeta^2$$

Check if $\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$:

$$\Phi(-i\omega, i\omega) = 4 + 5\omega^2 + \omega^4$$

a sum of squares, always nonnegative.

Note that

$$\Phi(-\xi,\xi) = 4 - 5\xi^2 + \xi^4 = (\xi - 2)(\xi - 1)(\xi + 1)(\xi + 2)$$

We can choose

$$\Delta(\zeta,\eta) = (\zeta + 1)(\zeta - 2)(\eta + 1)(\eta - 2)$$

Example

$$\Phi(\zeta,\eta)=4+6\eta+2\eta^2+6\zeta+9\zeta\eta+4\zeta\eta^2+2\zeta^2+4\zeta^2\eta+\eta^2\zeta^2$$

Check if $\Phi(-i\omega,i\omega)\geq 0$ for all $\omega\in\mathbb{R}$:

$$\Phi(-i\omega,i\omega)=4+5\omega^2+\omega^4$$

a sum of squares, always nonnegative.

Note that

$$\Phi(-\xi,\xi)=4-5\xi^2+\xi^4=(\xi-2)(\xi-1)(\xi+1)(\xi+2)$$

We can also choose

$$\Delta'(\zeta,\eta)=(\zeta+1)(\zeta+2)(\eta+1)(\eta+2)$$

and so forth...

Spectral factorization

Spectral factorization: given $\Phi(-\xi,\xi)$, find square matrix D s.t.

$$\Phi(-\xi,\xi) = D(-\xi)^\top D(\xi)$$

Spectral factorization

Spectral factorization: given $\Phi(-\xi,\xi)$, find square matrix D s.t.

$$\Phi(-\xi,\xi) = D(-\xi)^\top D(\xi)$$

Solvable if and only if $\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$.

¡Frequency domain condition for dissipativity!

Spectral factorization

Spectral factorization: given $\Phi(-\xi,\xi)$, find square matrix D s.t.

$$\Phi(-\xi,\xi) = D(-\xi)^\top D(\xi)$$

Spectral factorize $\Phi(-\xi,\xi) = D(-\xi)^\top D(\xi)$, define

$$\Delta(\zeta,\eta) := D(\zeta)^\top D(\eta)$$

Spectral factorization

Spectral factorization: given $\Phi(-\xi,\xi)$, find square matrix D s.t.

$$\Phi(-\xi,\xi) = D(-\xi)^\top D(\xi)$$

Spectral factorize $\Phi(-\xi,\xi) = D(-\xi)^\top D(\xi)$, define

$$\Delta(\zeta,\eta) := D(\zeta)^\top D(\eta)$$

$\Phi(-\xi,\xi) = \Delta(-\xi,\xi) \Longrightarrow$ **there exists $\Psi(\zeta,\eta)$ s.t.**

$$\Phi(\zeta,\eta) - \Delta(\zeta,\eta) = (\zeta+\eta)\Psi(\zeta,\eta)$$

Then storage function is

$$\Psi(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - \Delta(\zeta,\eta)}{\zeta+\eta}$$

Remarks

- **Many ways of spectral factorizing the same matrix**

  $\rightsquigarrow$ **many dissipation functions**

  $\rightsquigarrow$ **many storage functions.**

- **Set of storage functions is convex:**

  Q_{Ψ_1}, Q_{Ψ_2} **storage functions and** $\alpha \in [0,1]$

  $\Longrightarrow \alpha Q_{\Psi_1} + (1-\alpha) Q_{\Psi_2}$ **is storage function**

Example

$$\Phi(\zeta,\eta) = 4 + 6\eta + 2\eta^2 + 6\zeta + 9\zeta\eta + 4\zeta\eta^2 + 2\zeta^2 + 4\zeta^2\eta + \eta^2\zeta^2$$

Example

$$\Phi(\zeta,\eta)=4+6\eta+2\eta^2+6\zeta+9\zeta\eta+4\zeta\eta^2+2\zeta^2+4\zeta^2\eta+\eta^2\zeta^2$$

Since

$$\Phi(-\xi,\xi)=4-5\xi^2+\xi^4=(\xi-2)(\xi-1)(\xi+1)(\xi+2)$$

if we choose the dissipation function

$$\Delta(\zeta,\eta)=(\zeta+1)(\zeta-2)(\eta+1)(\eta-2)$$

we obtain the storage function

$$\Psi(\zeta,\eta)=\frac{\Phi(\zeta,\eta)-\Delta(\zeta,\eta)}{\zeta+\eta}=4+4\eta+4\zeta+5\zeta\eta$$

Example

$$\Phi(\zeta,\eta)=4+6\eta+2\eta^2+6\zeta+9\zeta\eta+4\zeta\eta^2+2\zeta^2+4\zeta^2\eta+\eta^2\zeta^2$$

Since also

$$\Phi(-\xi,\xi)=4-5\xi^2+\xi^4=(\xi-2)(\xi-1)(\xi+1)(\xi+2)$$

if we choose the dissipation function

$$\Delta'(\zeta,\eta)=(\zeta+1)(\zeta+2)(\eta+1)(\eta+2)$$

we obtain the storage function

$$\Psi'(\zeta,\eta)=\frac{\Phi(\zeta,\eta)-\Delta'(\zeta,\eta)}{\zeta+\eta}=\zeta\eta$$

Storage functions

Maximal and minimal storage functions

<u>Theorem</u>: Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ be controllable and Φ-dissipative. There exist storage functions Q_{Ψ_-} and Q_{Ψ_+} such that for any storage function Q_Ψ it holds

$$Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}$$

Maximal and minimal storage functions

Theorem: Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ be controllable and Φ-dissipative. There exist storage functions Q_{Ψ_-} and Q_{Ψ_+} such that for any storage function Q_Ψ it holds

$$Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}$$

Q_{Ψ_-} is **minimal-**, Q_{Ψ_+} is **maximal storage function**

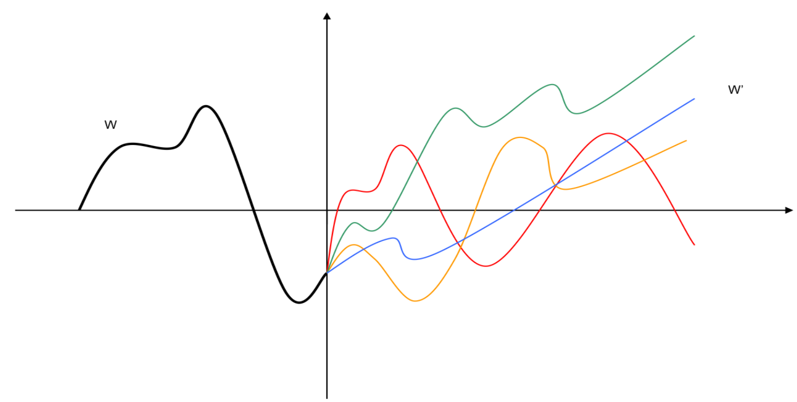
Maximal and minimal storage functions

Theorem: **Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ be controllable and Φ-dissipative. There exist storage functions Q_{Ψ_-} and Q_{Ψ_+} such that for any storage function Q_Ψ it holds**

$$Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}$$

Q_{Ψ_-} **is available storage:**

$$Q_{\Psi_-}(w)(0) = \sup_{\substack{w' \text{ s.t.} \\ w \wedge w' \in \mathscr{B}}} \left(-\int_0^\infty Q_\Phi(w')dt \right)$$



Maximum amount of energy extractable from system.

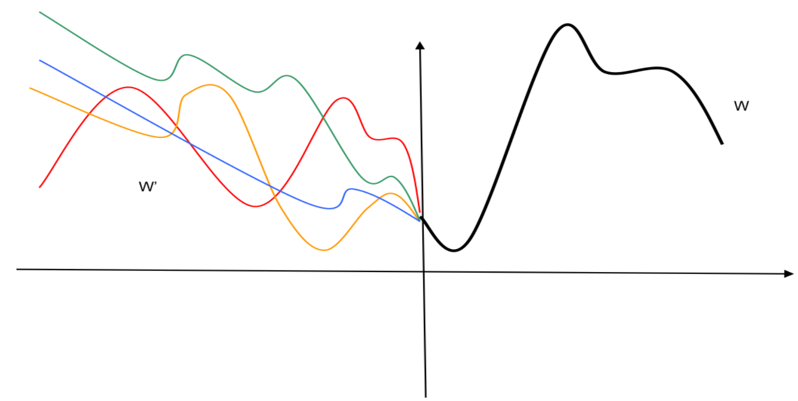
Maximal and minimal storage functions

Theorem: Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ be controllable and Φ-dissipative. There exist storage functions Q_{Ψ_-} and Q_{Ψ_+} such that for any storage function Q_Ψ it holds

$$Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}$$

Q_{Ψ_+} is **required supply**:

$$Q_{\Psi_+}(w)(0) = \inf_{\substack{w' \text{ s.t.} \\ w' \wedge w \in \mathscr{B}}} \left(\int_{-\infty}^{0} Q_\Phi(w')dt \right)$$



Minimum energy needed to produce w **from** $t = 0$

Spectral factorization and extremal storage functions

If $\det\Phi(-\xi,\xi)\neq 0$ **and** $\Phi(-i\omega,i\omega)\geq 0$ **for all** $\omega\in\mathbb{R}$**, there exist** H**,** A **s.t.**

$$\Phi(-\xi,\xi)=H(-\xi)^\top H(\xi)=A(-\xi)^\top A(\xi)$$

where

$$\det(H(\lambda))=0\Longrightarrow\lambda\in\mathbb{C}^0_-\quad\text{(“semi-Hurwitz polynomial”)}$$

$$\det(A(\lambda))=0\Longrightarrow\lambda\in\mathbb{C}^0_+\quad\text{(“semi-anti-Hurwitz polynomial”)}$$

Spectral factorization and extremal storage functions

If $\det \Phi(-\xi,\xi) \neq 0$ **and** $\Phi(-i\omega, i\omega) \geq 0$ **for all** $\omega \in \mathbb{R}$**, there exist** H**,** A **s.t.**

$$\Phi(-\xi,\xi) = H(-\xi)^\top H(\xi) = A(-\xi)^\top A(\xi)$$

where

$$\det(H(\lambda)) = 0 \Longrightarrow \lambda \in \mathbb{C}_-^0$$ **("semi-Hurwitz polynomial")**

$$\det(A(\lambda)) = 0 \Longrightarrow \lambda \in \mathbb{C}_+^0$$ **("semi-anti-Hurwitz polynomial")**

In this case,

$$\Psi_-(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - H(\zeta)^\top H(\eta)}{\zeta+\eta}$$

$$\Psi_+(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - A(\zeta)^\top A(\eta)}{\zeta+\eta}$$

Storage functions and the state

Circuit theory folklore: state variables are associated with energy storing elements (capacitors, inductors)

Physics: potential energy in a field dependent on position (and velocity/acceleration)

Storage functions and the state

Circuit theory folklore: state variables are associated with energy storing elements (capacitors, inductors)

Physics: potential energy in a field dependent on position (and velocity/acceleration)

¿Can we give rational foundation to the intuition that “storage” is related with “memory”?

Storage functions and the state

<u>Theorem</u>: **Let** $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ **be nonsingular. Assume that** $\mathfrak{B} = \operatorname{im}\left(M(\frac{d}{dt})\right)$ **is** Σ**-dissipative.**

Let $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ **be a storage function, and let** $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ **be a state map for** $\mathfrak{B}$**.**

Then $\exists\, K = K^\top \in \mathbb{R}^{\bullet\times\bullet}$**,** $E = E^\top \in \mathbb{R}^{\bullet\times\bullet}$ **such that**

$$\Psi(\zeta,\eta) = X(\zeta)^\top K X(\eta)$$

$$\Delta(\zeta,\eta) = \begin{bmatrix} M(\zeta) \\ X(\zeta) \end{bmatrix}^\top E \begin{bmatrix} M(\eta) \\ X(\eta) \end{bmatrix}$$

Storage functions and the state

<u>Theorem</u>: Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. Assume that $\mathscr{B} = \text{im}\,(M(\frac{d}{dt}))$ is Σ-dissipative.

Let $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be a storage function, and let $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ be a state map for $\mathscr{B}$.

Then $\exists\, K = K^\top \in \mathbb{R}^{\bullet\times\bullet}$, $E = E^\top \in \mathbb{R}^{\bullet\times\bullet}$ such that

$$\Psi(\zeta,\eta) = X(\zeta)^\top K X(\eta)$$

$$\Delta(\zeta,\eta) = \begin{bmatrix} M(\zeta) \\ X(\zeta) \end{bmatrix}^\top E \begin{bmatrix} M(\eta) \\ X(\eta) \end{bmatrix}$$

¡The storage function is a quadratic function of the state!

Storage functions and the state

Theorem: Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. Assume that $\mathscr{B} = \mathrm{im}\,(M(\frac{d}{dt}))$ is Σ-dissipative.

Let $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be a storage function, and let $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ be a state map for $\mathscr{B}$.

Then $\exists\, K = K^\top \in \mathbb{R}^{\bullet\times\bullet}$, $E = E^\top \in \mathbb{R}^{\bullet\times\bullet}$ such that

$$\Psi(\zeta,\eta) = X(\zeta)^\top K X(\eta)$$

$$\Delta(\zeta,\eta) = \begin{bmatrix} M(\zeta) \\ X(\zeta) \end{bmatrix}^\top E \begin{bmatrix} M(\eta) \\ X(\eta) \end{bmatrix}$$

¡The dissipation function is a quadratic function of the state and of the input!

Main points

- Dissipative systems: storage and dissipation;

Main points

- **Dissipative systems: storage and dissipation;**
- **Spectral factorization and storage functions;**

Main points

- **Dissipative systems: storage and dissipation;**
- **Spectral factorization and storage functions;**
- **Extremal storage functions;**

Main points

- **Dissipative systems: storage and dissipation;**
- **Spectral factorization and storage functions;**
- **Extremal storage functions;**
- **Storage function is a function of the state.**

Main points

End of Lecture 5b