# Lecture 4a

**Wednesday 04-02-2009** **09.00-10.30**

# Rational symbols

**Lecturer: Jan C. Willems**

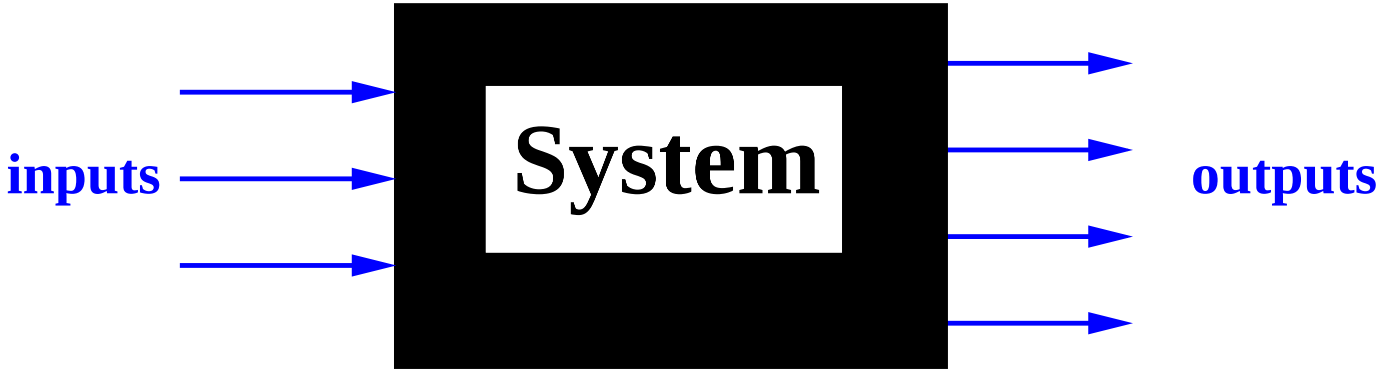
# Outline

- **Behaviors defined by rational symbols**
- **Norm preserving representations**
- **The gap between LITDSs**
- **Model reduction without stability or i/o partition**

# Introduction

# Theme

**In system theory, it is customary to think of dynamical models in terms of inputs and outputs, viz.**




**often with transfer functions** $y = F(s)u$

$F$ **a matrix of rational transfer functions.**

# Theme

$$y = F(s)u$$

**In the present lecture, we will**

- **for good physical and system theoretic reasons, not use an input/output partition**

  $\rightsquigarrow$ **system variables** $w = \begin{bmatrix} u \\ y \end{bmatrix}$

- **interpret** $F$**, not in terms of Laplace transforms, but in terms of differential equations.**

  **Important for pedagogical reasons, among other things.**

# Reminder

**LTIDSs:** $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ **where**

- $\mathbb{T} = \mathbb{R}$ **'time'**
- $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$ **'signal space'**
- **and 'behavior'** $\mathscr{B}$ **= the set of solutions of a system of**

**linear constant coefficient ODEs**

$\mathscr{B}$ **= the** $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$**-solutions of**

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{L}} \frac{d^{\mathtt{L}}}{dt^{\mathtt{L}}} w = 0, \quad R_0, R_1, \ldots \textbf{ matrices}$$

**Polynomial matrix notation** $\rightsquigarrow$ $R\left(\frac{d}{dt}\right) w = 0$

$$R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}, \quad R(\xi) = R_0 + R_1 \xi + \cdots + R_{\mathtt{L}} \xi^{\mathtt{L}}$$

# Representations of LTIDSs

**Behaviors of LTIDSs allow many useful representations**

- **As the set of solutions of** $R\left(\frac{d}{dt}\right) w = 0$ $\quad R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$
- **With input/output partition**
- **Input/state/output representation**

  $\exists$ **matrices** $A, B, C, D$ **such that**
  $\mathscr{B}$ **consists of all** $w's$ **generated by**

$$\frac{d}{dt}x = Ax + Bu, \; y = Cx + Du \quad w \cong \begin{bmatrix} u \\ y \end{bmatrix}$$

- **...**
- **with rational symbols** $\rightsquigarrow$ **this lecture**

# Rational symbols

# ODEs with rational symbols

**Defining what a solution is for ODEs such as**

$$R\left(\frac{d}{dt}\right)w = 0 \quad \textbf{or} \quad \frac{d}{dt}x = Ax + Bu,\ y = Cx + Du, w = \begin{bmatrix} u \\ y \end{bmatrix}$$

**poses no difficulties worth mentioning, but rational functions $\leadsto$ Laplace transforms with domains of convergence, etc.**

# ODEs with rational symbols

**Let $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$, and consider the 'differential equation'**

$$G\left(\frac{d}{dt}\right)w = 0$$

$G$ **is called the associated symbol**

***What do we mean by its solutions*, i.e. by the behavior?**

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***What do we mean by its solutions*, i.e. by the behavior?**

**Recall:**

$[\![M$ **left prime** **(over** $\mathbb{R}[\xi]$**)** $]\!]$

$:\Leftrightarrow [\![\ [\![M = FM']\!] \Rightarrow [\![F$ **unimodular** $]\!]\ ]\!]$

$\Leftrightarrow \quad \exists\, H$ **such that** $MH = I$**.**

**In the scalar case,** $M = [m_1 \;\; m_2 \;\; \cdots \;\; m_{\mathtt{n}}]$**, this means:**

$m_1, m_2, \cdots, m_{\mathtt{n}}$ **have no common root.**

# ODEs with rational symbols

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$$G\left(\frac{d}{dt}\right) w = 0$$

$G$ **is called the associated symbol**

***What do we mean by its solutions*, i.e. by the behavior?**

**Let** $(P, Q)$ **be a left coprime polynomial factorization of** $G$

**i.e.,** $P, Q \in \mathbb{R}[\xi]^{\bullet \times \bullet}, \det(P) \neq 0, G = P^{-1}Q, [P \vdots Q]$ **left prime.**

**In scalar case, this means** $P$ **and** $Q$ **have no common roots.**

# ODEs with rational symbols

**Let $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$, and consider the 'differential equation'**

$$G\left(\frac{d}{dt}\right)w = 0$$

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***What do we mean by its solutions*, i.e. by the behavior?**

**Let $(P, Q)$ be a left coprime polynomial factorization of $G$**

$$[\![G(\frac{d}{dt})w = 0]\!] \Leftrightarrow [\![P^{-1}Q(\frac{d}{dt})w = 0]\!] :\Leftrightarrow [\![\, Q\left(\frac{d}{dt}\right)w = 0 \,]\!]$$

**By definition therefore, the behavior of $G(\frac{d}{dt})w = 0$ is equal to the behavior of $Q(\frac{d}{dt})w = 0$.**

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**By definition therefore, the behavior of $G(\frac{d}{dt})w=0$ is equal to the behavior of $Q(\frac{d}{dt})w=0$.**

**Justification:**

**1. $G$ proper. $G(\xi)=C(I\xi-A)^{-1}B+D$ controllable realization. Consider the output nulling inputs:**

$$\frac{d}{dt}x=Ax+Bw, \quad 0=Cx+Dw$$

**This set of $w$'s are exactly those that satisfy $G\left(\frac{d}{dt}\right)w=0$.**

**Analogous for $\frac{d}{dt}x=Ax+Bw, 0=Cx+D\left(\frac{d}{dt}\right)w, \quad D\in\mathbb{R}[\xi]^{\bullet\times\bullet}$.**

# ODEs with rational symbols

$$[\![G(\frac{d}{dt})w = 0]\!] \Leftrightarrow [\![P^{-1}Q(\frac{d}{dt})w = 0]\!] :\Leftrightarrow [\![\, Q\left(\tfrac{d}{dt}\right) w = 0 \,]\!]$$

**By definition therefore, the behavior of $G(\frac{d}{dt})w = 0$ is equal to the behavior of $Q(\frac{d}{dt})w = 0$.**

**Justification:**

**2. Consider $y = G(s)w$. View $G(s)$ as a transfer f'n.**
**Take your favorite definition of input/output pairs.**

**Output nulling inputs exactly those that satisfy $G\left(\frac{d}{dt}\right)w = 0$.**

**3. ...**

# ODEs with rational symbols

$$[\![G(\frac{d}{dt})w=0]\!] \Leftrightarrow [\![P^{-1}Q(\frac{d}{dt})w=0]\!] :\Leftrightarrow [\![\, Q\left(\frac{d}{dt}\right)w=0 \,]\!]$$

**By definition therefore, the behavior of $G(\frac{d}{dt})w=0$ is equal to the behavior of $Q(\frac{d}{dt})w=0$.**

**Note! With this def., we can deal with transfer functions,**

$$y=F(\frac{d}{dt})u, \quad \textbf{i.e.} \quad \left[F(\tfrac{d}{dt}) \ \vdots \ -I\right]\begin{bmatrix}u\\y\end{bmatrix}=0$$

**with $F$ a matrix of rational functions, and completely avoid Laplace transforms, domains of convergence, and such cumbersome, but largely irrelevant, mathematical traps.**

# Caveats

## $F\left(\frac{d}{dt}\right)$ is not a map!

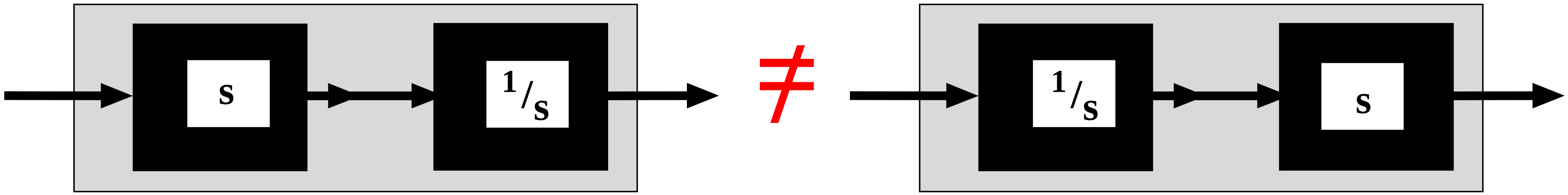
**Consider**

$$y = F\left(\tfrac{d}{dt}\right) u$$

**We now know what it means that** $(u,y) \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^\bullet)$ **satisfies this 'ODE'.**

**Given** $u$, $\exists$ **solution** $y$**, but not unique, unless** $F$ **is polynomial**

## $G_1\left(\frac{d}{dt}\right)$ and $G_2\left(\frac{d}{dt}\right)$ do not commute




$$G_1(s) = \frac{1}{s} \ \textbf{and} \ \ G_2(s) = s$$

**do not commute.**

$$y = \frac{1}{\frac{d}{dt}} v, \ \ v = \frac{d}{dt} u \quad \Rightarrow \quad y(t) = u(t) + \textbf{ constant}$$

$$y = \frac{d}{dt} v, \ \ v = \frac{1}{\frac{d}{dt}} u \quad \Rightarrow \quad y(t) = u(t)$$

# Representations

# Stable representations

**Linear time-invariant differential systems** $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$**.**
$\mathscr{B} = \text{kernel}\left(R\left(\frac{d}{dt}\right)\right)$ **for some** $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$ **by definition.**

**But we may as well take the representation** $G\left(\frac{d}{dt}\right) w = 0$ **for some** $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$ **as the def. of a LTIDS behavior.**

# Stable representations

**Linear time-invariant differential systems $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$.**
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**But we may as well take the representation $G\left(\frac{d}{dt}\right) w = 0$ for some $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$ as the def. of a LTIDS behavior.**
**$R$: all poles at $\infty$, we can take $G$ with no poles at $\infty$, or more generally with all poles in some non-empty set - symmetric w.r.t. $\mathbb{R}$. In particular (many variations on this theme):**

**Theorem: Every linear time-invariant differential systems has a representation**

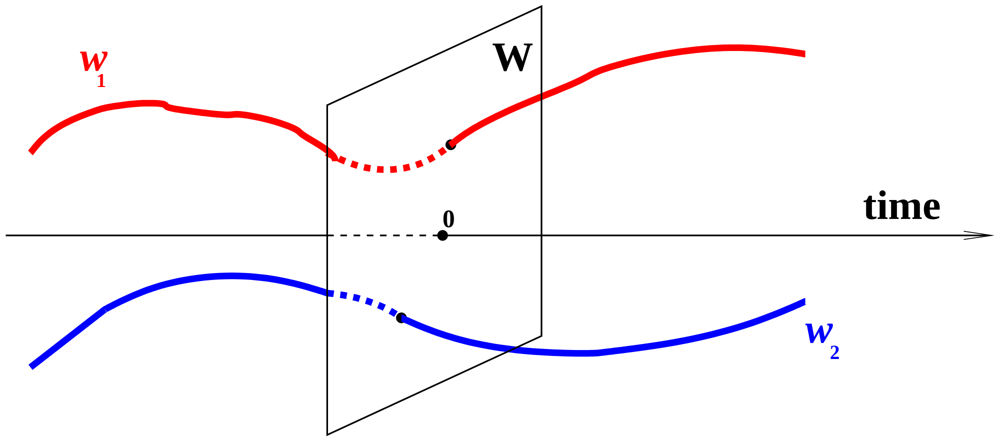
$$G\left(\frac{d}{dt}\right) w = 0$$

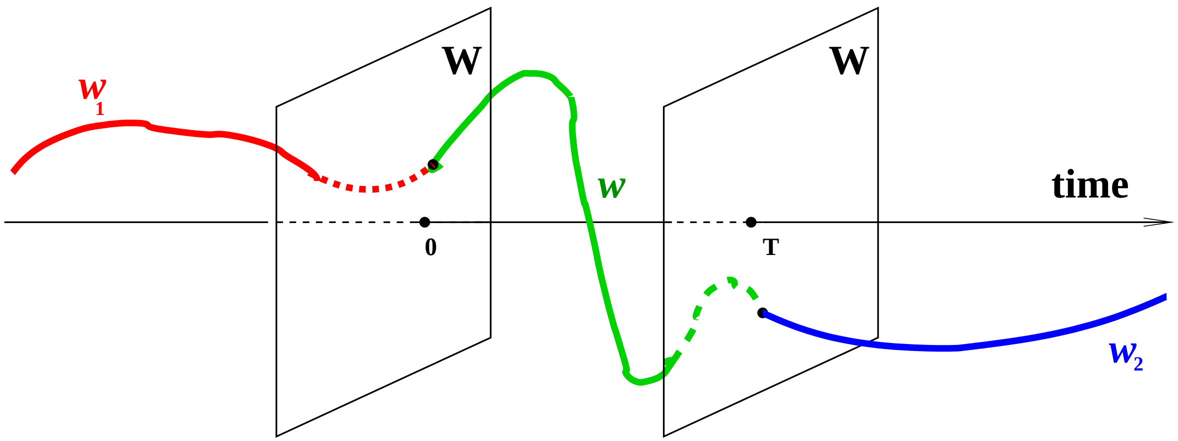
**with $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$ strictly proper stable rational.**
**<u>Proof</u>: Take $G(s) = \frac{R(s)}{(s+\lambda)^{\mathtt{n}}}$, suitable $\lambda \in \mathbb{R}, \mathtt{n} \in \mathbb{N}$.**

# Controllability and stabilizability

$\mathscr{B}$ **is said to be controllable** $:\Leftrightarrow$

$$\forall\, w_1, w_2 \in \mathscr{B},\ \exists\, T \geq 0 \text{ and } w \in \mathscr{B} \text{ such that ...}$$



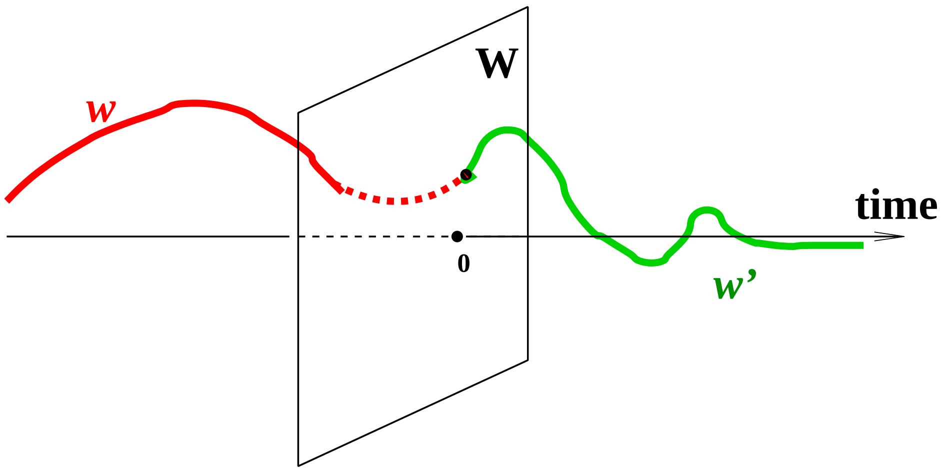



# Controllability and stabilizability

$\mathscr{B}$ **is said to be** **controllable** $:\Leftrightarrow$

$\mathscr{B}$ **is said to be** **stabilizable** $:\Leftrightarrow$

$\forall\, w \in \mathscr{B},\ \exists\, w' \in \mathscr{B}$ **such that ...**




**(asymptotic) stability in the sense of Lyapunov**

# Rational representations

**What properties on $G$ imply that the system with rational representation**

$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$$

**has any of these properties?**

**Under what conditions on $G$ does $G\left(\frac{d}{dt}\right)w = 0$ define a controllable or a stabilizable system?**

# Rational representations

**What properties on $G$ imply that the system with rational representation**

$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$$

**has any of these properties?**

**Under what conditions on $G$ does $G\left(\frac{d}{dt}\right)w = 0$ define a controllable or a stabilizable system?**

**Can a rational representation be used to put one of these properties in evidence?**

# Tests

**Theorem:** **The LTIDS**

$$G\left(\tfrac{d}{dt}\right)w = 0 \qquad\qquad G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$$

**is controllable if and only if**

$$G(\lambda) \text{ has the same rank } \forall \lambda \in \mathbb{C}$$

**Interpret carefully in cases like**

$$G(s) = \begin{bmatrix} s & 0 \\ 0 & \frac{1}{s} \end{bmatrix}, G(s) = \begin{bmatrix} s \\ \frac{1}{s} \end{bmatrix}, G(s) = \begin{bmatrix} s & \frac{1}{s} \end{bmatrix}$$

# Tests

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**Theorem:** **The LTIDS**

$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$$

**is stabilizable if and only if**

$$G(\lambda) \text{ has the same rank } \forall \lambda \in \mathbb{C} \text{ with } \mathbb{R}\text{ealpart}(\lambda) \geq 0$$

# Rational image representations

**Theorem:** **A LTIDS is controllable if and only if its behavior allows an image representation**

$$w = M(\tfrac{d}{dt})\ell \qquad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$$

**For example,**

$$y = F(\frac{d}{dt})u \qquad \rightsquigarrow w = \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} \ell \\ F(\frac{d}{dt})\ell \end{bmatrix}$$

**Systems defined by transfer functions are controllable**

**Transfer functions can only deal with controllable systems**

# Rational image representations

**Theorem: A LTIDS is controllable if and only if its behavior allows an image representation**

$$w = M(\tfrac{d}{dt})\ell \qquad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$$

**Theorem: A LTIDS is stabilizable if and only if its behavior allows a kernel representation**

$$R(\tfrac{d}{dt})w = 0$$

**with $R \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$ left prime over the ring of (proper) stable rationals**

# Raison d'être of rational representations

**LTIDSs are defined in terms of polynomial symbols**

$$R\left(\frac{d}{dt}\right)w = 0 \qquad R \in \mathbb{R}[\xi]^{\bullet\times \mathtt{w}}$$

**(behavior $\mathscr{B}$:= the $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ solutions) but can also be represented by rational symbols**

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$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet\times \mathtt{w}}$$

**Behavior := the set of solutions of**

$$Q\left(\frac{d}{dt}\right)w = 0 \qquad Q \in \mathbb{R}[\xi]^{\bullet\times \mathtt{w}}$$

**where** $G = P^{-1}Q, \quad P,Q \in \mathbb{R}[\xi]^{\bullet\times\bullet}, \quad P$ **and** $Q$ **left coprime**

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$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet\times \mathtt{w}}$$

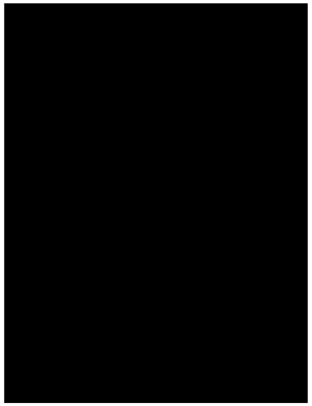
**This added flexibility $\leadsto$ better adapted to certain applications**

- **e.g. (series, parallel, ...) interconnections**
- **e.g. distance between systems**
- **e.g. behavioral model reduction**
- **e.g. parametrization of the set of stabilizing controllers**

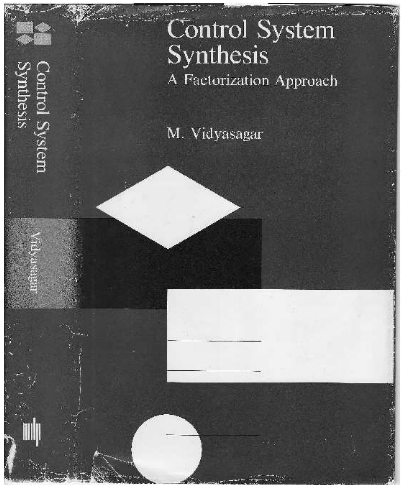
# Parametrization of stabilizing controllers

**One of the main applications where rational representations are used is for the Kučera-Youla parametrization of stabilizing controllers cfr. the book by Vidyasagar**

**Vladimir Kučera**



**Dante Youla**




**M. Vidyasagar**

# Norm-preserving representations

# Norm-preserving representations

**Let $\mathscr{B}$ be the behavior of a controllable LTIDS.**

**Then it allows a rational symbol based image representation**

$$w = M(\frac{d}{dt})\ell \quad \textbf{with} \quad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet} \;\; \textbf{\&} \;\; M(-\xi)^\top M(\xi) = I$$

**i.e.,** $||\ell||^2_{\mathscr{L}_2(\mathbb{R},\mathbb{R}^\bullet)} = ||w||^2_{\mathscr{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}})}$ **'norm preserving image repr.'**

$$\int_{-\infty}^{+\infty} ||w(t)||^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} ||\hat{w}(i\omega)||^2 \, d\omega =$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} ||M(i\omega)\hat{\ell}(i\omega)||^2 \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} ||\hat{\ell}(i\omega)||^2 \, d\omega = \int_{-\infty}^{+\infty} ||\ell(t)||^2 \, dt$$

**Note:** $M$ **cannot be polynomial, it must be rational**
**Obviously $M$ must be proper. Can also make it stable.**

# Norm-preserving representations

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**Idea of proof: Start with obs. polynomial im. representation**

$$w = M\left(\frac{d}{dt}\right)\ell.$$

**Factor** $M^\top(-\xi)M(\xi) = F^\top(-\xi)F(\xi)$

**Now take rational symbol based image representation**

$$w = MF^{-1}\left(\frac{d}{dt}\right)\ell$$

# Distance between systems

# Motivation

**What is a good, computable, definition for the distance between two (LTID) systems?**

**Basic issue underlying model simplification, robustness, etc.**

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**What is a good, computable, definition for the distance between two (LTID) systems?**

**Basic issue underlying model simplification, robustness, etc.**

- **Approximate a system by a simpler one.**
- **If a system has a particular property (e.g., stabilized by a controller), will this also hold for close-by systems?**
- **Does a sequence of systems converge?**

**What is meant**

**by 'approximate', by 'close-by', by 'converge'?**

# The gap

# Distance between linear subspaces

**In the behavioral theory, we identify a dynamical system with its behavior, that is, a set of trajectories. For LTIDSs, with a subspace $\mathscr{B} \subseteq \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.**

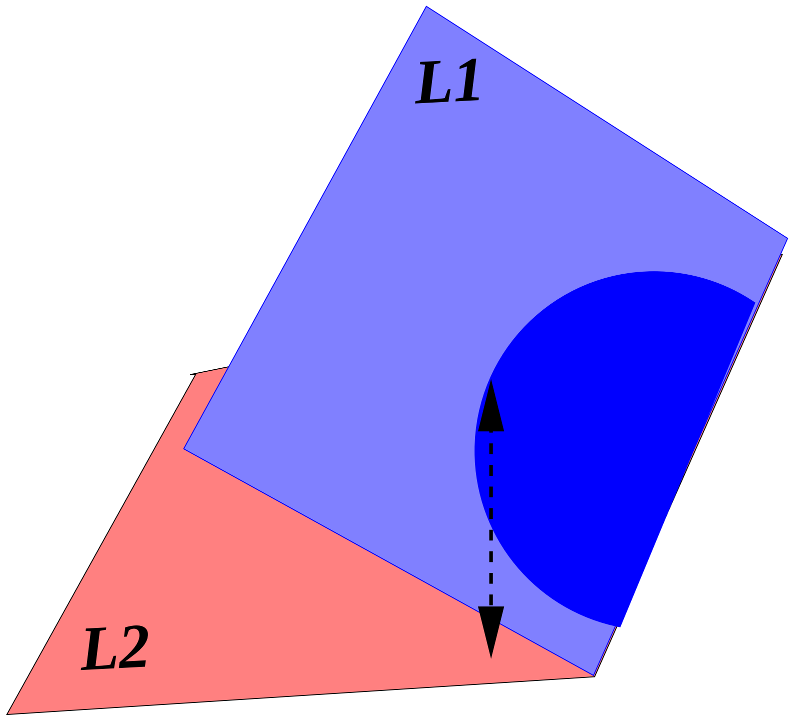
**Distance between systems**

**$\cong$ distance between linear subspaces.**

# Distance between linear subspaces of $\mathbb{R}^n$

$\mathscr{L}_1, \mathscr{L}_2 \subseteq \mathbb{R}^n$, **linear subspaces**

$$\overrightarrow{d}(\mathscr{L}_1, \mathscr{L}_2) := \max_{x_1 \in \mathscr{L}_1, ||x_1||=1} \; \min_{x_2 \in \mathscr{L}_2} ||x_1 - x_2||$$



# Distance between linear subspaces of $\mathbb{R}^{\mathtt{n}}$

$\mathscr{L}_1, \mathscr{L}_2 \subseteq \mathbb{R}^{\mathtt{n}}$, **linear subspaces**

$$d(\mathscr{L}_1, \mathscr{L}_2) := \max\left(\left\{ \overrightarrow{d}(\mathscr{L}_1, \mathscr{L}_2), \overrightarrow{d}(\mathscr{L}_2, \mathscr{L}_1) \right\}\right)$$

$$0 \leq d(\mathscr{L}_1, \mathscr{L}_2) \leq 1$$

# Distance between linear subspaces of $\mathbb{R}^{\mathtt{n}}$

$\mathscr{L}_1, \mathscr{L}_2 \subseteq \mathbb{R}^{\mathtt{n}}$, **linear subspaces**
$P_{\mathscr{L}}$ $\perp$ **projection onto** $\mathscr{L}$

$S_1, S_2$ **matrices, columns orthonormal basis for** $\mathscr{L}_1, \mathscr{L}_2$
**Note:** $S_1 S_1^\top, S_2 S_2^\top$ **orthogonal projectors**

$$
\begin{aligned}
d(\mathscr{L}_1, \mathscr{L}_2) &= ||P_{\mathscr{L}_1} - P_{\mathscr{L}_2}|| \qquad \text{`gap', `aperture'} \\
&= ||S_1 S_1^\top - S_2 S_2^\top|| \\
&= \min_{\text{matrices } U} ||S_1 - S_2 U|| \\
&= \min_{U \text{ such that } U\mathscr{L}_1 = \mathscr{L}_2} ||I - U||
\end{aligned}
$$

# Distance between linear subspaces of $\mathbb{R}^{\mathtt{n}}$

$\mathscr{L}_1, \mathscr{L}_2 \subseteq \mathbb{R}^{\mathtt{n}}$, **linear subspaces**
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\end{aligned}
$$

**Therefore,** $d(\mathscr{L}_1, \mathscr{L}_2) = ||S_1 S_1^\top - S_2 S_2^\top|| \leq ||S_1 - S_2||$

# Distance between LTIDSs

# Distance between controllable behaviors

min $\rightarrow$ **inf**, max $\rightarrow$ **sup**, etc., readily generalized to linear subspaces of Hilbert space, ...... and to LTIDSs.

**Which subspace of which Hilbert space should we associate with a LTIDS with behavior $\mathscr{B} \subseteq \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$?**

# Distance between controllable behaviors

$\min \to \texttt{inf}, \max \to \texttt{sup}$, **etc., readily generalized to linear subspaces of Hilbert space, ...... and to LTIDSs.**

**Which subspace of which Hilbert space should we associate with a LTIDS with behavior $\mathscr{B} \subseteq \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$?**

**For LTIDS, behaviors** $\mathscr{B} \mapsto (\mathscr{B} \cap \mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}}))^{\text{closure}}$

**Defines a** $1 \leftrightarrow 1$ **relation between controllable systems and 'certain' closed subspaces of** $\mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$**.**

# Distance between controllable behaviors

**Define the distance between two controllable behaviors as**

$$d(\mathscr{B}_1, \mathscr{B}_2) := \mathtt{gap}\left(\left(\mathscr{B}_1 \cap \mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})\right)^{\text{closure}}, \left(\mathscr{B}_2 \cap \mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})\right)^{\text{closure}}\right)$$

**We consider only the $\mathscr{L}_2$-trajectories for measuring distance.**

**Henceforth, keep notation $\mathscr{B}$ for** $\left(\mathscr{B} \cap \mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})\right)^{\text{closure}}$

# Distance between controllable behaviors

**Define the distance between two controllable behaviors as**

$$d(\mathscr{B}_1,\mathscr{B}_2) := \mathtt{gap}((\mathscr{B}_1 \cap \mathscr{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}}))^{\text{closure}}, (\mathscr{B}_2 \cap \mathscr{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}})^{\text{closure}}))$$

**We consider only the $\mathscr{L}_2$-trajectories for measuring distance.**

$$\forall w_1 \in \mathscr{B}_1, \exists w_2 \in \mathscr{B}_2 \text{ **such that** } ||w_1 - w_2|| \leq \mathtt{gap}(\mathscr{B}_1,\mathscr{B}_2)||w_1||$$

**and vice-versa.** **Small gap $\Rightarrow$ the models are 'close'.**

- **How to compute the gap?**
- **Model reduce according to the gap!**

# The gap and norm-preserving representations

**Let $\mathscr{B}$ be the behavior of a controllable LTIDS.**

**Then it allows a rational symbol based image representation**

$$w = M(\frac{d}{dt})\ell \quad \textbf{with} \quad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet} \;\; \textbf{\&} \;\; M(-\xi)^\top M(\xi) = I$$

**i.e.,** $||\ell||^2_{\mathscr{L}_2(\mathbb{R},\mathbb{R}^\bullet)} = ||w||^2_{\mathscr{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}})}$ **'norm preserving image repr.'**

$\mathscr{B}_1 \mapsto M_1, \mathscr{B}_2 \mapsto M_2$, **both norm preserving & stable, then**

$$\text{gap}(\mathscr{B}_1, \mathscr{B}_2) \quad = ||M_1(i\omega)M_1(-i\omega)^\top - M_2(i\omega)M_2(-i\omega)^\top||_{\mathscr{L}_\infty}$$

$$\leq ||M_1(i\omega) - M_2(i\omega)||_{\mathscr{H}_\infty}$$

# Model reduction

# Reducing the state dimension

**There is an elegant theory for reducing the state space dimension of stable LTI input/output systems.**

**Let** $\mathscr{B}$ **be described by** $\frac{d}{dt}x = Ax + Bu, \; y = Cx + Du$
**with** $A$ **Hurwitz (**$:\Leftrightarrow$ **eigenvalues in left half plane).**

**There are effective methods (balancing, AAK) with good error bounds (in terms of the** $\mathscr{H}_\infty$ **norm) for approximating** $\mathscr{B}$ **by a (stable) system with a lower dimensional state space.**

# Reducing the state dimension

**There is an elegant theory for reducing the state space dimension of stable LTI input/output systems.**

**Let** $\mathscr{B}$ **be described by** $\frac{d}{dt}x = Ax + Bu,\ y = Cx + Du \quad w \cong \begin{bmatrix} u \\ y \end{bmatrix}$

**with** $A$ **Hurwitz.**

**Balanced model reduction** $\Rightarrow$

$$||F(i\omega) - F_{\text{reduced}}(i\omega)||_{\mathscr{L}_\infty} \leq 2\ \left(\Sigma_{\textbf{neglected Hankel SVs}}\ \sigma_k\right)$$

**Keith Glover**

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**Let** $\mathscr{B}$ **be described by** $\frac{d}{dt}x = Ax + Bu,\ y = Cx + Du$ **with** $A$ **Hurwitz.**

$F(s)$ **proper stable rational** $\Rightarrow$ **reducible.**

**¡¡ Extend this to situations where we do not make a distinction between inputs and outputs, and to unstable systems.**

# Model reduction by balancing

**Start with $\mathscr{B}$. Take representatation**

$$w = M(\frac{d}{dt})\ell \quad \text{with} \quad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$$ **norm preserving, stable**

**Now model reduce $w = M(\frac{d}{dt})\ell$ (viewed as a stable input/output system) using, for example, balancing**

$$\rightsquigarrow \quad w = M_{\mathtt{reduced}}(\frac{d}{dt})\ell$$

**and an error bound**

$$||M - M_{\mathtt{reduced}}||_{\mathscr{H}_\infty} \leq 2 \left(\Sigma_{\textbf{neglected SVs of } M}\, \sigma_{\mathtt{k}}\right)$$

# Behavioral error bound

**Start with stable norm preserving representation of $\mathscr{B}$**

$$w = M(\frac{d}{dt})\ell \quad \textbf{with} \quad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$$

**Model reduce using balancing** $\rightsquigarrow \quad w = M_{\mathtt{reduced}}(\frac{d}{dt})\ell$.
**Call behavior $\mathscr{B}_{\text{reduced}}$. Error bound**

$$\begin{aligned} \mathrm{gap}(\mathscr{B}, \mathscr{B}_{\text{reduced}}) &= \|MM^\top - M_{\mathtt{reduced}}M_{\mathtt{reduced}}^\top\|_{\mathscr{L}_\infty} \\ &\leq \|M - M_{\mathtt{reduced}}\|_{\mathscr{H}_\infty} \\ &\leq 2\ \left(\Sigma_{\textbf{neglected SVs of } M}\ \sigma_{\mathrm{k}}\right) \end{aligned}$$

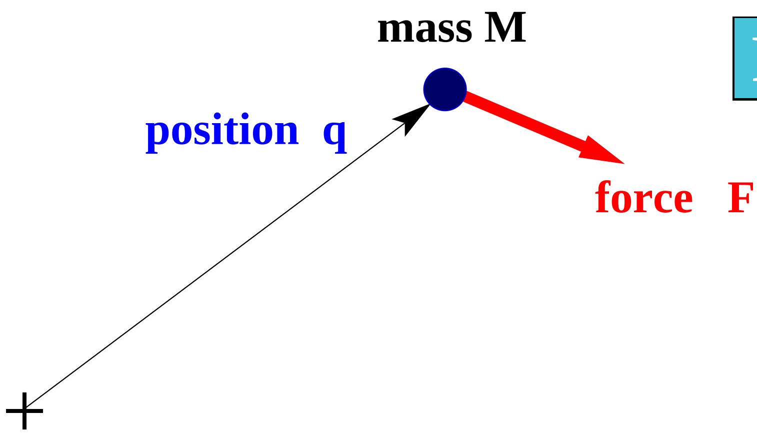
$\forall w \in \mathscr{B} \exists w' \in \mathscr{B}_{\text{red}}$ **such that** $\|w - w'\| \leq 2(\Sigma_{\textbf{neglected SVs}}\ \sigma_{\mathrm{k}})\|w\|$

**and vice-versa.**

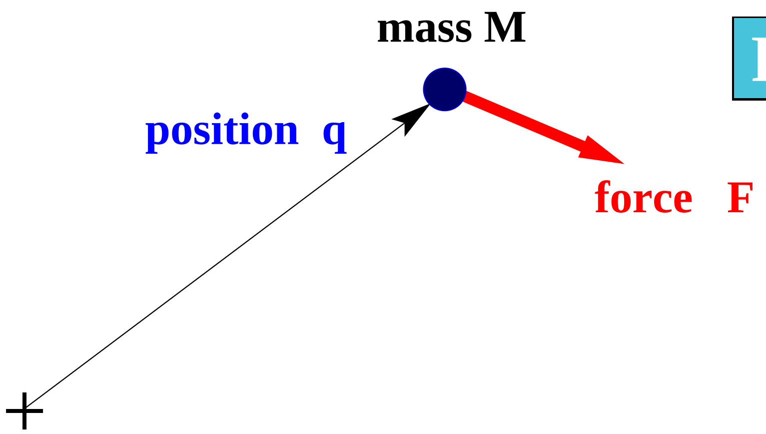
$\Sigma_{\textbf{neglected SVs of } M}\ \sigma_{\mathrm{k}}$ **small** $\Rightarrow$ **good approximation in the gap.**

# Examples

# Pointmass




$$F = M\frac{d^2}{dt^2}q, \qquad w = \begin{bmatrix} F \\ q \end{bmatrix} \cong \begin{bmatrix} M\frac{d^2}{dt^2} \\ 1 \end{bmatrix} \ell$$

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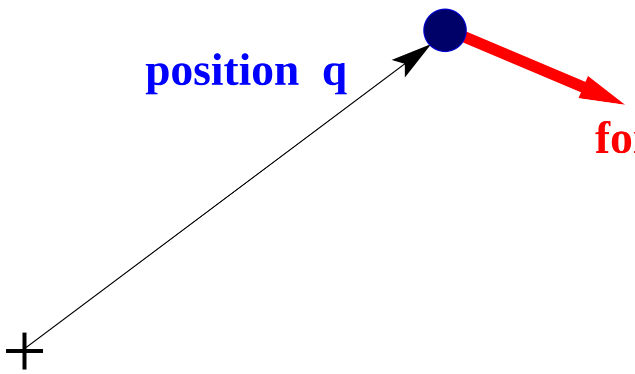
**Norm preserving, stable**

$$\begin{bmatrix} F \\ q \end{bmatrix} = \begin{bmatrix} \dfrac{M\frac{d^2}{dt^2}}{M\frac{d^2}{dt^2}+\sqrt{2M}\frac{d}{dt}+1} \\ \dfrac{1}{M\frac{d^2}{dt^2}+\sqrt{2M}\frac{d}{dt}+1} \end{bmatrix} \ell$$

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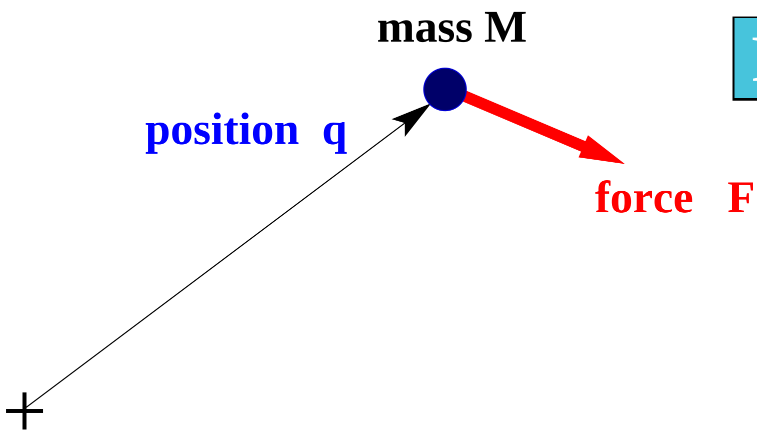
**reduced model**

$$\begin{bmatrix} F \\ q \end{bmatrix} = \begin{bmatrix} \dfrac{\sqrt{M}\frac{d}{dt}-\frac{1}{2}}{\sqrt{M}\frac{d}{dt}+\frac{1}{\sqrt{2}}} \\ \dfrac{\frac{1}{2}}{\sqrt{M}\frac{d}{dt}+\frac{1}{\sqrt{2}}} \end{bmatrix} \ell$$

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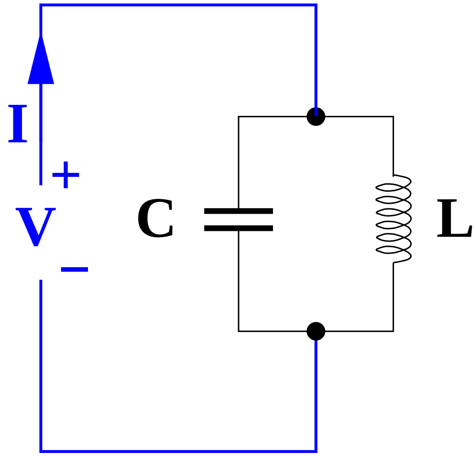
$F = \frac{d^2}{dt^2}q$ **first order approximation** $F = 2\sqrt{M}\frac{d}{dt}q - q$



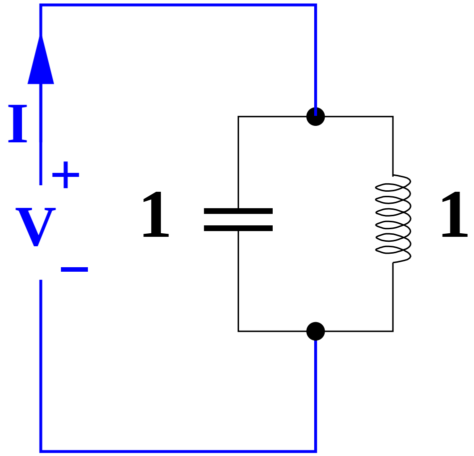

# LC circuit

**kernel**

$$\left(1 + LC\frac{d^2}{dt^2}\right)V = C\frac{d}{dt}I$$

**image**

$$\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} 1 + LC\frac{d^2}{dt^2} \\ C\frac{d}{dt} \end{bmatrix} \ell$$

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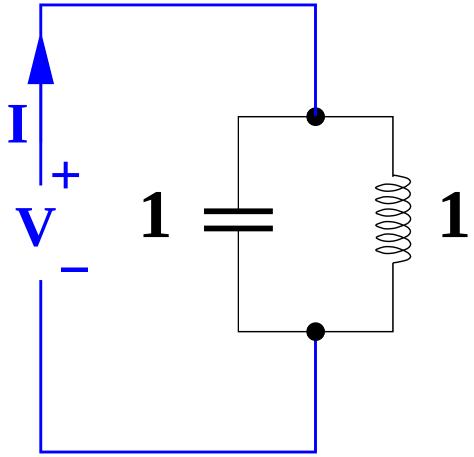
**Take** $L = C = 1$**.**

**stable norm-preserving** $$\begin{bmatrix} I \\ V \end{bmatrix} = \frac{1}{\frac{d^2}{dt^2}+\frac{d}{dt}+1}\begin{bmatrix} \frac{d^2}{dt^2}+1 \\ \frac{d}{dt} \end{bmatrix}\ell$$




## LC circuit

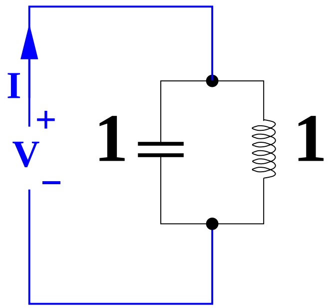
**Take** $L = C = 1$**.**

**stable norm-preserving**

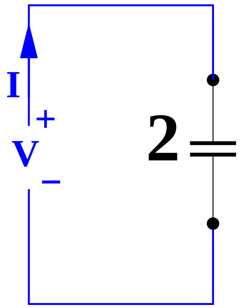
$$\begin{bmatrix} I \\ V \end{bmatrix} = \frac{1}{\frac{d^2}{dt^2} + \frac{d}{dt} + 1} \begin{bmatrix} \frac{d^2}{dt^2} + 1 \\ \frac{d}{dt} \end{bmatrix} \ell$$

**reduced model order = 1**

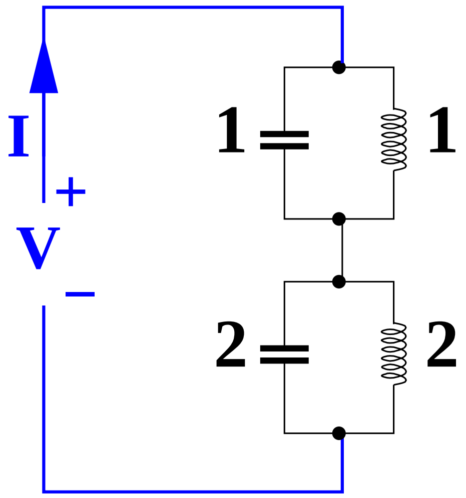
$$\begin{bmatrix} I \\ V \end{bmatrix} = \frac{1}{\frac{d}{dt} + \frac{1}{2}} \begin{bmatrix} \frac{d}{dt} \\ \frac{1}{2} \end{bmatrix} \ell$$




$$\left(\frac{d^2}{dt^2} + 1\right) V = \frac{d}{dt} I \quad \rightsquigarrow \quad \frac{d}{dt} V = \frac{1}{2} I$$



# LCLC circuit




**kernel**

$$\left(1+5\frac{d^2}{dt^2}+4\frac{d^4}{dt^4}\right)V=\left(3\frac{d}{dt}+6\frac{d^3}{dt^3}\right)I$$

**image**

$$\begin{bmatrix} I \\ V \end{bmatrix}=\begin{bmatrix} 1+5\frac{d^2}{dt^2}+4\frac{d^4}{dt^4} \\ 3\frac{d}{dt}+6\frac{d^3}{dt^3} \end{bmatrix}\ell$$

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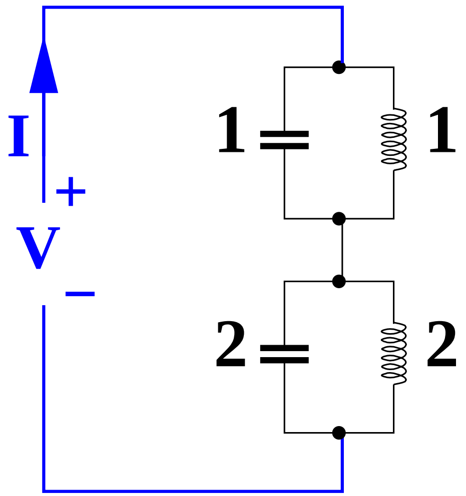


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**image**

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**stable norm-preserving image**

$$\begin{bmatrix} I \\ V \end{bmatrix} = \frac{1}{1 + 3\frac{d}{dt} + 5\frac{d^2}{dt^2} + 6\frac{d^3}{dt^3} + 4\frac{d^4}{dt^4}} \begin{bmatrix} 1 + 5\frac{d^2}{dt^2} + 4\frac{d^4}{dt^4} \\ 3\frac{d}{dt} + 6\frac{d^3}{dt^3} \end{bmatrix} \ell$$

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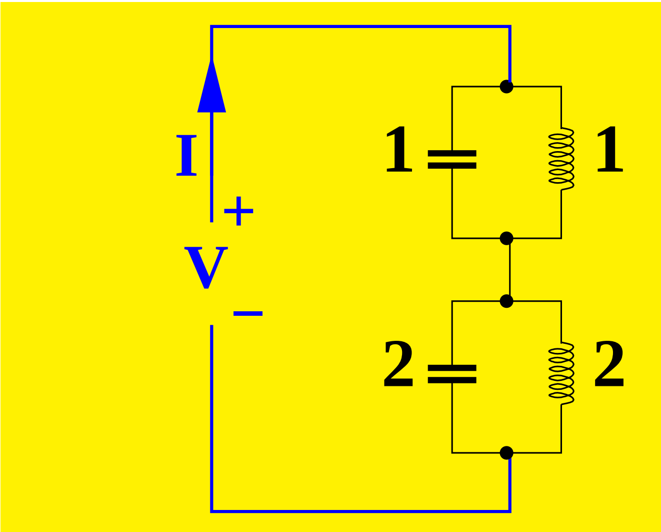
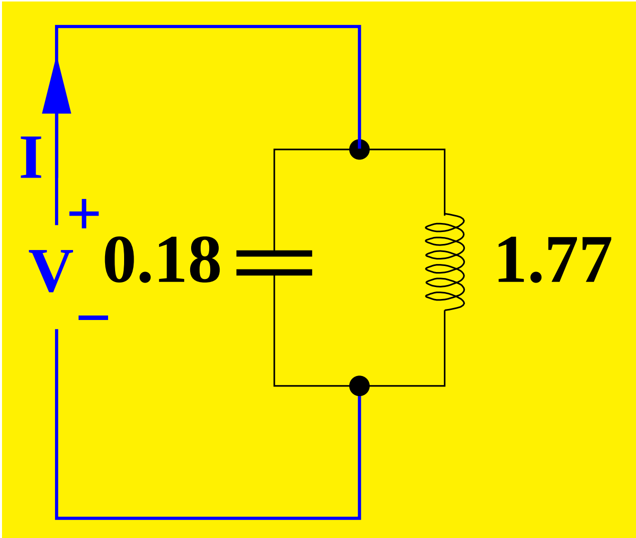
**red. order = 2** $$\begin{bmatrix} I \\ V \end{bmatrix} = \frac{1}{\frac{d^2}{dt^2}+0.1861\frac{d}{dt}+0.3298} \begin{bmatrix} \frac{d^2}{dt^2}+0.3298 \\ 0.1861\frac{d}{dt} \end{bmatrix} \ell$$




$\rightsquigarrow$



# Summary of Lecture 4a

# The main points

- $G(\frac{d}{dt})w = 0$ **defined in terms left-coprime factorization of rational** $G$**.**
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- **Numerous other applications of rational symbols**

# End of Lecture 4a