Lecture 3

Tuesday 03-02-2008 **14.00-17.30**

Linear Time-Invariant Systems, Part 2

Lecturer: Paolo Rapisarda

Outline

1. **Part I:**
   - **Inputs and outputs;**
   - **Autonomous behaviors;**
   - **Input-output representations.**

2. **Part II:**
   - **Controllability;**
   - **Image representations;**
   - **Complementability and decomposition of behaviors;**
   - **Observability.**

Inputs and outputs

Reprise: surjectivity and injectivity of differential operators

Recall that $P\left(\frac{d}{dt}\right) : \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{g}})$ **is surjective** $\Leftrightarrow$ $P(\xi)$ **has full row rank as a polynomial matrix**

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Equivalently, P **admits a left inverse on** $\mathscr{C}^\infty(\mathbb{R})$**:**

$$P = U \begin{bmatrix} I_{\mathtt{m}} \\ 0 \end{bmatrix} V \Longrightarrow V^{-1} \begin{bmatrix} I_{\mathtt{m}} & 0 \end{bmatrix} U^{-1} \text{ is left inverse}$$

Free variables

Given $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **and** $I := \{i_1, \ldots, i_{\mathtt{k}}\} \subseteq \{1, \ldots, \mathtt{w}\}$**, let**

$$\Pi_I \mathscr{B} := \{(\hat{w}_{i_1}, \ldots, \hat{w}_{i_{\mathtt{k}}}) \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{k}}) \mid \exists\, w \in \mathscr{B} \text{ s.t. } w = (w_1, \ldots, \hat{w}_{i_1}, \ldots, \hat{w}_{i_{\mathtt{k}}}, \ldots, w_{\mathtt{w}})\}$$

projection of $\mathscr{B}$ onto variables w_{i_j}, $j = 1, \ldots, \mathtt{k}$

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projection of $\mathscr{B}$ **onto variables** w_{i_j}, $j = 1, \ldots, \mathtt{k}$

Variables w_{i_j}, $j = 1, \ldots, \mathtt{k}$ **are free if**

$$\Pi_I \mathscr{B} = \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{k}})$$

Free variables

***Example*:**

$$p_1\left(\frac{d}{dt}\right)w_1 + p_2\left(\frac{d}{dt}\right)w_2 + p_3\left(\frac{d}{dt}\right)w_3 = 0$$

Assume $p_i \neq 0$**,** $i = 1,2,3$**.**

Let $I = \{1\}$**; since** $\begin{bmatrix} p_2(\xi) & p_3(\xi) \end{bmatrix}$ **is full row rank, for every** $w_1 \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ **there exist** w_2, w_3 **satisfying equation.**

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(w_1, w_2) **(and** (w_2, w_3)**, and** (w_1, w_3)**) are also free.**

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***Example*: In $\frac{d}{dt}x = Ax + Bu$, the variable u is free.**

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$$\mathscr{B} = \ker \begin{bmatrix} \frac{d}{dt} I - A & -B \end{bmatrix} \rightsquigarrow R(\xi) = \begin{bmatrix} \xi I - A & -B \end{bmatrix}$$

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Maximally free sets

Let $I = \{i_1, \ldots, i_{\mathtt{k}}\} \subseteq \{1, \ldots, \mathtt{w}\}$. The variables $w_{i_1}, \ldots, w_{i_{\mathtt{k}}}$ form a maximally free set if

- **they are free; and**
- **for every $I' = \{i'_1, \ldots, i'_{\mathtt{k}}\} \subset \{1, \ldots, \mathtt{w}\}$ such that $I \subset I'$ it holds**

$$\Pi_{I'} \mathscr{B} \subset \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{|I'|})$$

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$$\Pi_{I'} \mathscr{B} \subsetneq \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{|I'|})$$

Maximally free set: every variable in it is free, but any additional variable is not

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Maximally free sets are nonunique!

Inputs and outputs

Theorem: Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$. Assume (if necessary, after permutation of the variables) w partitioned as

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

with w_1 maximally free. Then w_1 are inputs and w_2 outputs.

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***Example*: for $p_1\left(\frac{d}{dt}\right)w_1 + p_2\left(\frac{d}{dt}\right)w_2 + p_3\left(\frac{d}{dt}\right)w_3 = 0$ and assuming $p_i \neq 0$ for $i = 1, 2, 3$, we can choose**

- $\{w_1, w_2\}$ **or**
- $\{w_2, w_3\}$ **or**
- $\{w_1, w_3\}$

as inputs.

Remarks

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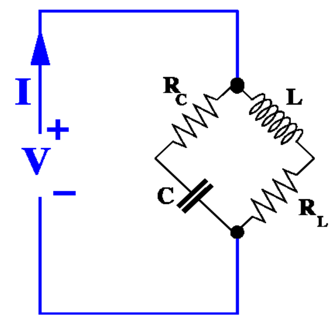
Remarks

- **Nonunicity of i/o partition is *not* an issue. Consider (linear) resistors:**

$$\mathscr{B} = \{(V, I) \mid V = R \cdot I\}$$

Is it voltage- or current-controlled? Consider



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- **Nonunicity of i/o partition is *not* an issue.**
- **'Causality' an issue? What about**

$$w_1 = \frac{d}{dt} w_2?$$

Don't w_1 and w_2 'happen' at the same time?

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- **'Smoothness', meaning**

$$(u,y) \in \mathscr{B} \quad \textbf{and} \quad u \ k\textbf{-times differentiable}$$
$$\implies \quad y \ k\textbf{-times differentiable}$$

if and only if $P^{-1}Q$ is proper. Strict properness $\Leftrightarrow$

$$(u,y) \in \mathscr{B} \quad \textbf{and} \quad u \ k\textbf{-times differentiable}$$
$$\implies \quad y \ (k+1)\textbf{-times differentiable}$$

Causality in discrete-time systems

Consider a linear $\mathscr{B} \subset (\mathbb{R}^{\mathtt{w}_1+\mathtt{w}_2})^{\mathbb{Z}}$**. Let** $\mathscr{B}_1 = \Pi_{\mathtt{w}_1}\mathscr{B}$**.**

w_2 **does not anticipate** $w_1 \Leftrightarrow$

$$w_1 \in \mathscr{B}_1 \quad \textbf{and} \quad w_{1|\mathbb{Z}_-} = 0$$

$$\Longrightarrow \quad \textbf{exists } w_2' \textbf{ s.t. } w_{2|\mathbb{Z}_-} = 0 \textbf{ and } (w_1, w_2) \in \mathscr{B}$$

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Theorem: Assume w_2 **is output and** w_1 **is input, and let**

$$P(\sigma)w_2 = Q(\sigma)w_1$$

be an i/o representation of $\mathscr{B}$**. Then** w_2 **does not anticipate** w_1 $\Leftrightarrow P^{-1}Q$ **is proper.**

Input-output representations

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Theorem: Consider

$$\mathscr{B} = \left\{ (u,y) \mid P\left(\frac{d}{dt}\right) y = Q\left(\frac{d}{dt}\right) u \right\}$$

with P square and nonsingular. Then y is output and u is input.

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Surjectivity of $P\left(\frac{d}{dt}\right) \Longrightarrow u$ is free.

u maximally free: add one component of y to those of u, resulting set satisfies differential equation $\Longrightarrow$ it is not free.

Input-output representations

<u>Theorem</u>: Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$. There exists (possibly after permuting components) a partition of $w = (u, y)$ and $P \in \mathbb{R}^{\mathtt{y}\times\mathtt{y}}[\xi]$ nonsingular, $Q \in \mathbb{R}^{\mathtt{y}\times\mathtt{u}}[\xi]$ such that

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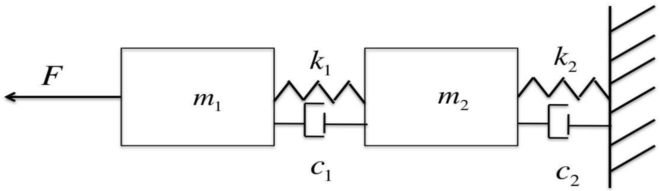
***Proof*: Use minimal kernel representation $\mathscr{B} = \ker\, R\left(\frac{d}{dt}\right)$.**

R of full row rank $\Longrightarrow$ exists nonsingular submatrix P.

For $P^{-1}Q$ proper, select P to be a maximal determinantal degree (nonsingular) submatrix of R.

Example



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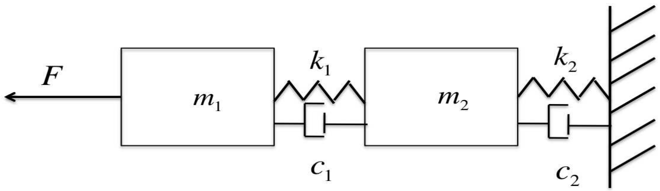

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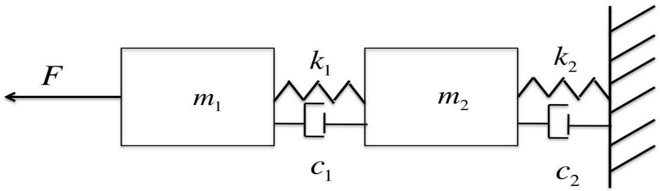

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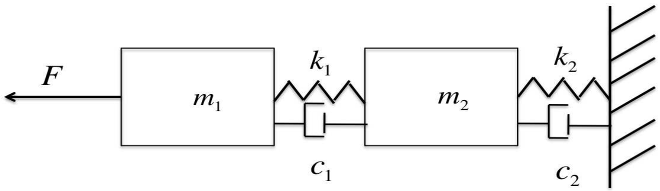

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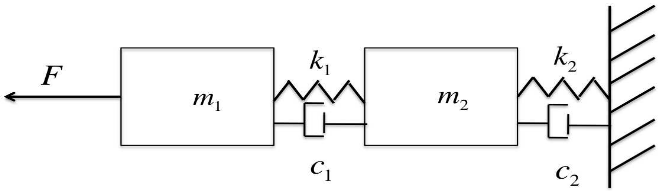

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w_1 and w_2 outputs, F input; $P^{-1}Q$ strictly proper

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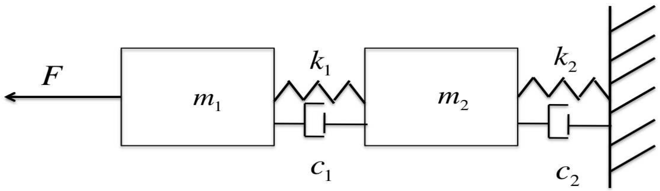

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w_1 **and** F **outputs,** w_2 **input;** $P^{-1}Q$ **not proper**

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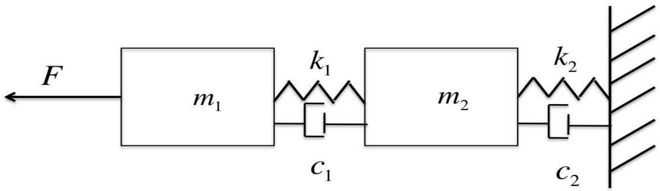

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- **In discrete-time case, there always exists a causal input-output partition!**

Autonomous behaviors

No inputs: autonomous systems

Recall that $\mathscr{B}$ is autonomous if

$$w_1, w_2 \in \mathscr{B} \quad \textbf{and} \quad w_1 \mid_{(-\infty,0]} = w_2 \mid_{(-\infty,0]}$$
$$\Longrightarrow \quad w_1 = w_2$$

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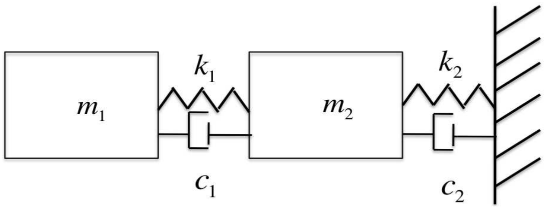
Equivalent with

- $\mathtt{m}(\mathscr{B}) = 0$ **(no inputs);**
- **there exists** $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ **nonsingular such that** $\mathscr{B} = \ker\ R\left(\frac{d}{dt}\right)$

Example: a mechanical system



$$
\begin{aligned}
m_1 \frac{d^2 w_1}{dt^2} + c_1 \left(\frac{d}{dt} w_1 - \frac{d}{dt} w_2 \right) + k_1 (w_1 - w_2) &= 0 \\
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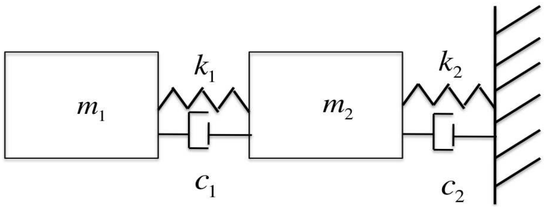

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Classical mechanics:
motion depends only on ‘initial conditions’

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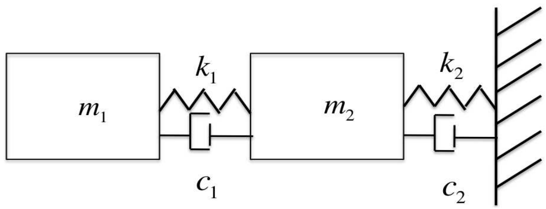

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$$R(\xi)=\begin{bmatrix} m_1\xi^2+c_1\xi+k_1 & -c_1\xi-k_1\\ -c_1\xi-k_1 & m_2\xi^2+(c_1+c_2)\xi+k_1+k_2 \end{bmatrix}$$

R **nonsingular** $\rightsquigarrow$ **autonomous system**

Example: state-space systems

Let (A,C) **observable and consider**

$$\mathscr{B} := \{y \in \mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathrm{y}}) \mid \exists\, x \text{ s.t. } \frac{d}{dt}x = Ax, y = Cx\}$$

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$\mathscr{B}$ is autonomous: there are no free variables in y.

Remarks

- **For autonomous $\mathscr{B}$,** $\det(R)$ **is invariant for all** R **such that** $\mathscr{B} = \ker\, R\left(\frac{d}{dt}\right)$**.**

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***Proof*: Take R s.t. $\mathscr{B} = \ker\, R\left(\frac{d}{dt}\right)$, w.l.o.g. minimal.**
Compute Smith form $R = U\Delta V$:

$$R\left(\frac{d}{dt}\right) w = 0 \Longleftrightarrow \Delta\left(\frac{d}{dt}\right) \underbrace{V\left(\frac{d}{dt}\right) w}_{=:w'} = 0$$

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Set of solutions of linear differential equation is finite-dimensional. Also w is!

On autonomous system trajectories

Scalar case:

$$p\left(\frac{d}{dt}\right)w = 0 \Longleftrightarrow w(t) = \sum_{i=1}^{n}\sum_{j=0}^{n_i} \alpha_{ij} t^j e^{\lambda_i t}$$

where

- n **is number of distinct roots of** $p(\xi)$**;**
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λ_i **are the characteristic frequencies of** p**.**

On autonomous system trajectories

For $\mathtt{w} > 1$**, resort to Smith form** $R = U\Delta V$**:**

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Assume for simplicity all roots of $\det(R)$ are simple:

$$w = V\left(\frac{d}{dt}\right)^{-1} w' \Longleftrightarrow w(t) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t}$$

with $\alpha_i \in \mathbb{C}^{\mathtt{w}}$ such that $R(\lambda_i)\alpha_i = 0$, $i = 1, \dots, n$.

Remarks

- **Linear combinations of polynomial exponential vector trajectories**

$$\sum_{i=1}^{n} \sum_{j=0}^{n_i} \alpha_{ij} t^j e^{\lambda_i t}$$

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- **Characteristic frequencies** λ_i **are roots of** $\det(R)$**.**

Together with corresponding multiplicities, they determine $\mathscr{B}$ **uniquely.**

Stability

$\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **is** **asymptotically stable** $\Leftrightarrow$

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<u>Theorem</u>: $\mathscr{B} = \ker\, R\left(\frac{d}{dt}\right)$ **is asymptotically stable**
$\Leftrightarrow$ $\mathtt{rank}(R(\lambda)) = \mathtt{w}(\mathscr{B})$ **for all** $\lambda \in \mathbb{C}$ **s.t. Re**$(\lambda) \geq 0$.

Stability

$\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **is stable** $\Leftrightarrow$

$$w \in \mathscr{B} \Longrightarrow w_{|\mathbb{R}_+} \text{ is bounded.}$$

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Theorem: $\mathscr{B} = \ker R\left(\frac{d}{dt}\right)$ **is stable** $\Leftrightarrow$

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Stability=roots in closed left half-plane, and semisimplicity.

End of Part I

Controllability

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$\mathscr{B}$ **controllable** $\Leftrightarrow$ **for all** $w_1, w_2 \in \mathscr{B}$ **there exists** $w \in \mathscr{B}$ **and** $T \geq 0$ **such that**

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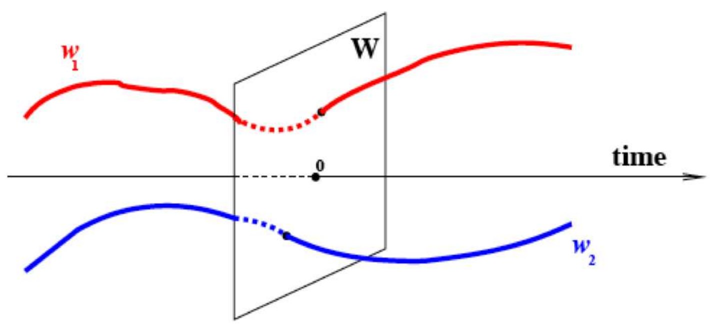

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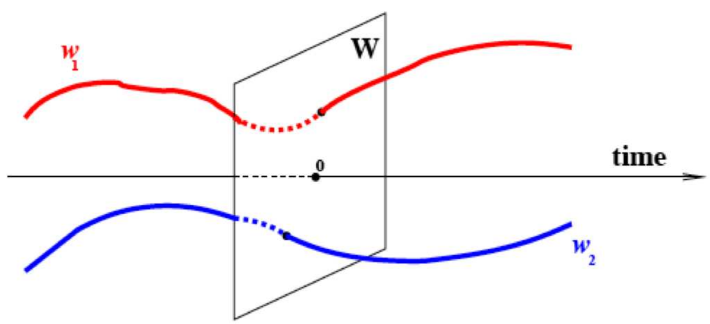

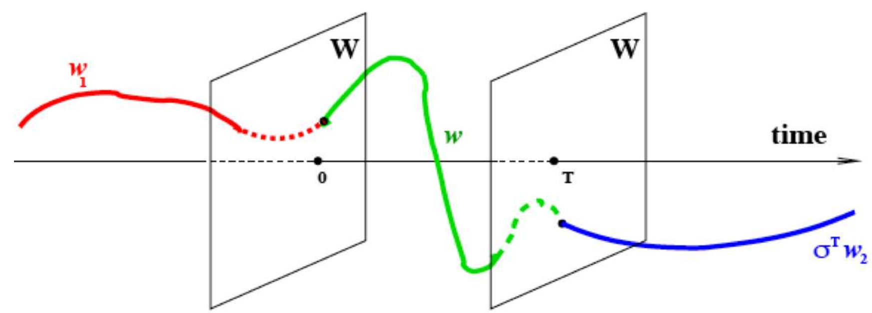

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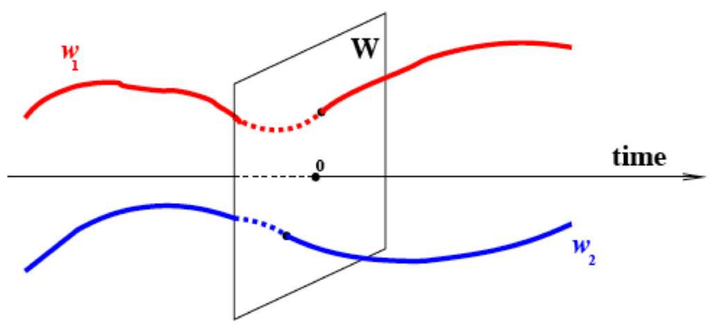

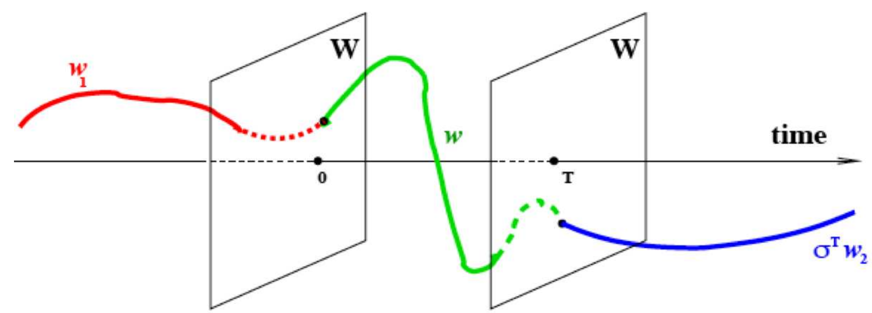

Past of any trajectory can be "patched up" with future of any trajectory

Examples

$$r\left(\frac{d}{dt}\right)w = 0$$

where $0 \neq r \in \mathbb{R}[\xi]$ **has degree** n**.**

System **autonomous: every solution uniquely determined by ‘initial conditions’** $\frac{d^i w}{dt^i}(t)$**,** $i = 0, \ldots, n-1$**, so no patching possible among d** $\Leftrightarrow$ **erent trajectories.**

Past of trajectory uniquely determines its future.

Examples

Classical state-space system

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\begin{aligned}
\frac{d}{dt}x &= Ax + Bu \\
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"State point-controllability": for all $x_1, x_2 \in \mathbb{R}^{\mathtt{n}}$ $\exists$ $x \in \mathscr{B}_x$ **and** $T \geq 0$ **s.t.** $x(0) = x_0$ **and** $x(T) = x_1$**.**

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Algebraic characterization of controllability

Theorem: $\mathscr{B} = \ker\, R\left(\frac{d}{dt}\right)$ **is controllable**

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$$R = U \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} V \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$$

$U\left(\frac{d}{dt}\right)$, $V\left(\frac{d}{dt}\right)$ **bijective** $\Longrightarrow \ker\, R\left(\frac{d}{dt}\right)$ **controllable** $\Leftrightarrow \ker\, \Delta\left(\frac{d}{dt}\right)$ **is.**

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Change variables $w \rightsquigarrow w' := V\left(\frac{d}{dt}\right) w$**, define** $\mathscr{B}' := V\left(\frac{d}{dt}\right) \mathscr{B} = \ker\ \Delta\left(\frac{d}{dt}\right)$**.**

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Proof: **Last** $\mathtt{p} - \mathtt{rank}(R)$ **trajectories of** $\mathscr{B}' = \ker\ \Delta\left(\frac{d}{dt}\right)$ **are free, since equations are** $0 \cdot w_i' = 0$**.**

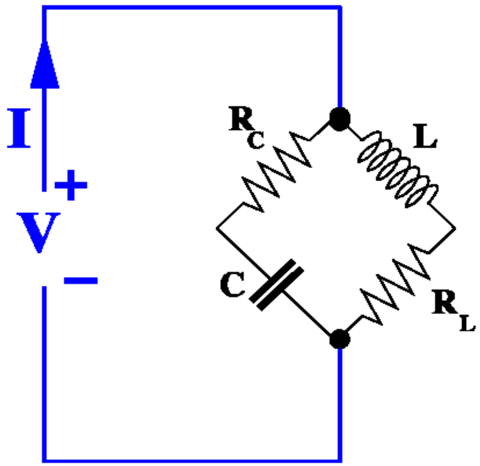
First $\mathtt{rank}(R)$ **equations are**

$$\delta_i\left(\frac{d}{dt}\right) w_i' = 0$$

with δ_i i**-th invariant polynomial of** R**.**

Evidently, w_i' **controllable if and only if** $\delta_i = 1$**.**

Example



Case 1: $CR_C \neq \frac{L}{R_L}$

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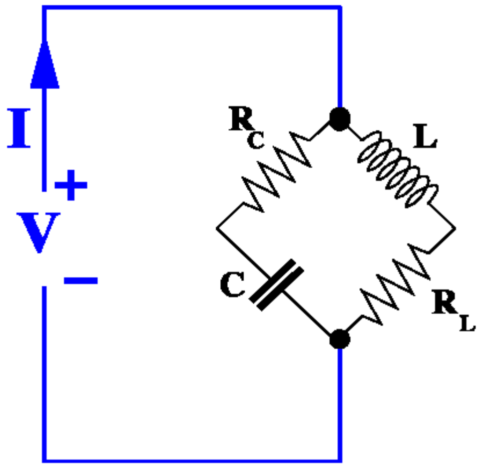

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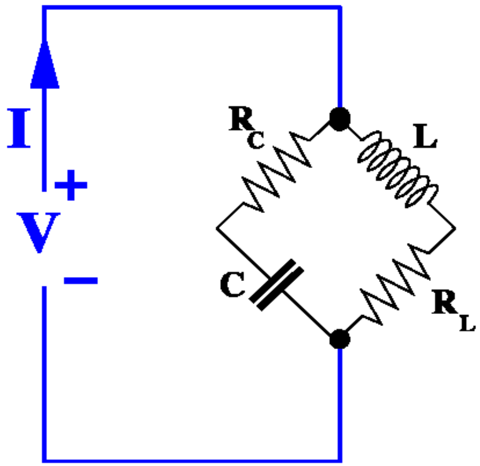
¿Is system controllable?

Example



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Are there common roots among the two polynomials?

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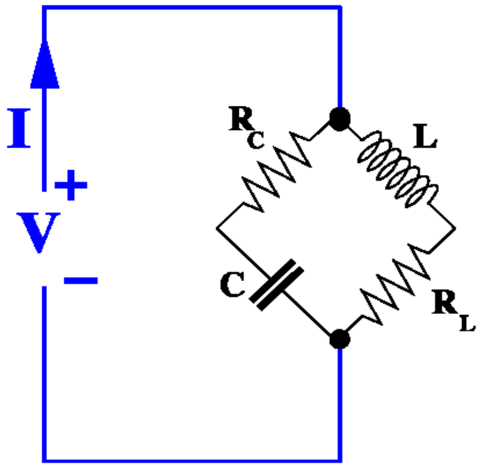

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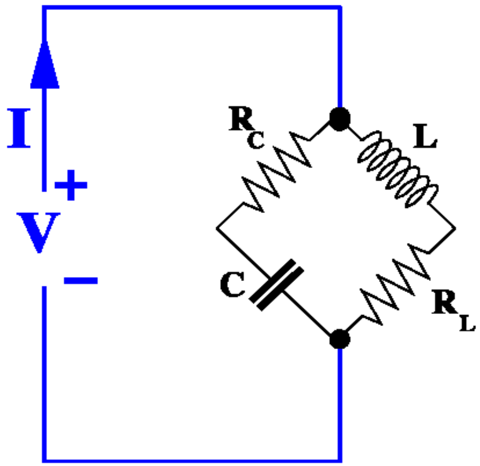
No $\Longrightarrow$ system is controllable

Example



Case 2: $CR_C = \frac{L}{R_L}$

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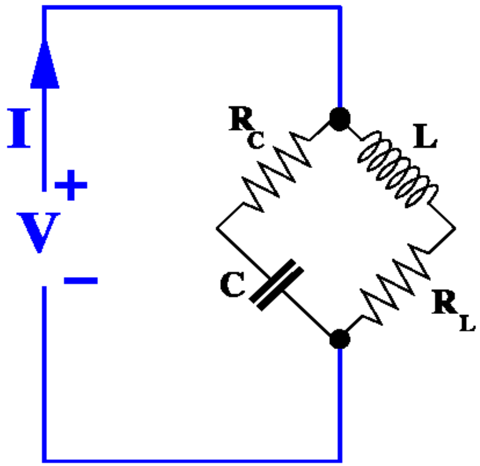

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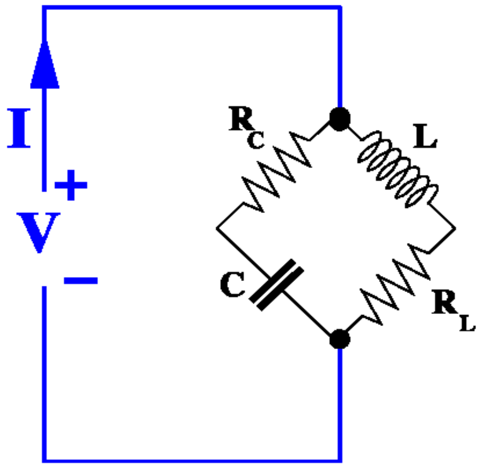

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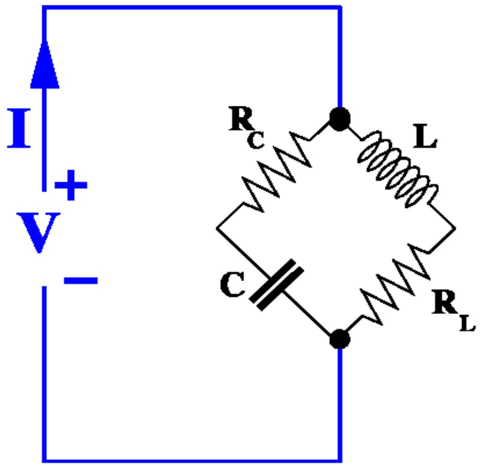

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¿Is system controllable?

If $R_C = R_L \implies$ system is not controllable

Remarks

- $\mathscr{B} = \ker\ R\left(\frac{d}{dt}\right)$**, with** $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ **nonsingular, is controllable** $\iff$ R **is unimodular** $\iff$ $\mathscr{B} = \{0\}$

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Image representations

Image representations and controllability

<u>Theorem</u>: There exists $M \in \mathbb{R}^{\mathtt{w}\times\bullet}[\xi]$ **such that** $\mathscr{B} = \operatorname{im} M\left(\frac{d}{dt}\right) \Leftrightarrow \mathscr{B}$ **is controllable.**

Image representations and controllability

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***Only if*: Full behavior is controllable, since has kernel representation induced by**

$$\begin{bmatrix} I_{\mathtt{w}} & -M(\xi) \end{bmatrix}$$

with constant rank over $\mathbb{C}$.

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***If*: Take R for minimal kernel representation of $\mathscr{B}$. Apply constancy of rank to conclude Smith form of R is**

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Now $U\left(\frac{d}{dt}\right)\begin{bmatrix} I_{\mathtt{p}} & 0_{\mathtt{p}\times\mathtt{m}}\end{bmatrix}\underbrace{V\left(\frac{d}{dt}\right)w}_{=:w'} = 0 \Leftrightarrow \begin{bmatrix} I_{\mathtt{p}} & 0_{\mathtt{p}\times\mathtt{m}}\end{bmatrix} w' = 0 \Leftrightarrow$

$$w' = \begin{bmatrix} 0_{\mathtt{p}} \\ I_{\mathtt{m}} \end{bmatrix} \ell$$

with $\ell \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$ free.

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Consequently,

$$w' = V\left(\frac{d}{dt}\right) w = \begin{bmatrix} 0_{\mathtt{p}} \\ I_{\mathtt{m}} \end{bmatrix} \ell$$

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$$w = V\left(\frac{d}{dt}\right)^{-1} \begin{bmatrix} 0_{\mathtt{p}} \\ I_{\mathtt{m}} \end{bmatrix} \ell =: M\left(\frac{d}{dt}\right) \ell$$

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Note also that M can be chosen with $\mathtt{m}(B)$ columns.

Complementability and decomposition of behaviors

Complementability

Theorem: **Let** $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **be controllable. There exists** $\mathscr{B}' \in \mathscr{L}^{\mathtt{w}}$ **such that**

$$\mathscr{B} \oplus \mathscr{B}' = \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$$

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Define

$$R' := U \begin{bmatrix} 0 & I_{\mathtt{w}-\mathtt{p}} \end{bmatrix} V$$

and $\mathscr{B}' := \ker\ R'\left(\frac{d}{dt}\right)$. $\mathscr{B}'$ is also controllable.

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***Proof*: Observe that $\mathscr{B} \cap \mathscr{B}'$ is represented in kernel form by**

$$U \begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & I_{\mathtt{w}-\mathtt{p}} \end{bmatrix} V$$

a unimodular matrix. Consequently, $\mathscr{B} \cap \mathscr{B}' = \{0\}$.

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<u>Theorem</u>: Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ be controllable. There exists $\mathscr{B}' \in \mathscr{L}^{\mathtt{w}}$ such that

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***Proof*: Easy to see image representations of $\mathscr{B}$, $\mathscr{B}'$ given by**

$$\mathscr{B} = \operatorname{im} V^{-1} \begin{bmatrix} 0 \\ I_{\mathtt{w}-\mathtt{p}} \end{bmatrix} \qquad \mathscr{B}' = \operatorname{im} V^{-1} \begin{bmatrix} I_{\mathtt{p}} \\ 0 \end{bmatrix}$$

Consequently $\mathscr{B} + \mathscr{B}'$ represented by

$$V^{-1} \begin{bmatrix} 0 & I_{\mathtt{p}} \\ I_{\mathtt{w}-\mathtt{p}} & 0 \end{bmatrix}$$

unimodular, consequently bijective.

Decomposition of behaviors

Theorem: Let $\mathscr{B} = \ker\ R\left(\frac{d}{dt}\right)$**, with** $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ **full row rank. There exist** $\mathscr{B}_{aut} \subseteq \mathscr{B}$ **and** $\mathscr{B}_{contr} \subseteq \mathscr{B}$ **such that**

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$$w' \in \mathscr{B}' \Longleftrightarrow w' = \begin{bmatrix} w'_1 \\ w'_2 \end{bmatrix}$$

with $w'_1 \in \ker\ D\left(\frac{d}{dt}\right)$**,** w'_2 **free.**

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If $D = I_{\mathtt{p}} \Longrightarrow$ **take** $\mathscr{B}'_{\mathrm{contr}} = \mathscr{B}'$**,** $\mathscr{B}'_{\mathrm{aut}} = \{0\}$**.**

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If $D \neq I_{\mathtt{p}}$**, define**

$$\mathscr{B}'_{\text{contr}} = \left\{ \begin{bmatrix} w'_1 \\ 0 \end{bmatrix} \mid w'_1 \in \ker\ D\left(\frac{d}{dt}\right) \right\}$$

$$\mathscr{B}'_{\text{aut}} = \left\{ \begin{bmatrix} 0 \\ w'_2 \end{bmatrix} \mid w'_2 \in \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}-\mathtt{p}}) \right\}.$$

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Then transform back to w **variables.**

Observability

Observability

w_1
observed variables

w_2
to-be-deduced variables

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¿Can w_2 be determined knowing w_1 and the system dynamics?

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$\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$, $w = (w_1, w_2)$. w_2 **is observable from** w_1 **if**

$$(w_1, w_2'), (w_1, w_2'') \in \mathscr{B} \Longrightarrow w_2' = w_2''$$

Algebraic characterization of observability

Assume $\mathscr{B}$ represented in kernel form as

$$R_1\left(\frac{d}{dt}\right)w_1 + R_2\left(\frac{d}{dt}\right)w_2 = 0$$

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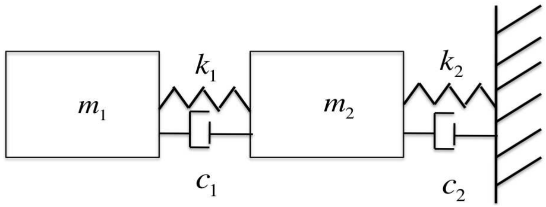
It has $\Leftrightarrow$ $R_2\left(\frac{d}{dt}\right)$ injective $\Leftrightarrow$ $R_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$

Example



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\begin{aligned}
m_1 \frac{d^2 w_1}{dt^2} + c_1 \left(\frac{d}{dt} w_1 - \frac{d}{dt} w_2 \right) + k_1 (w_1 - w_2) &= 0 \\
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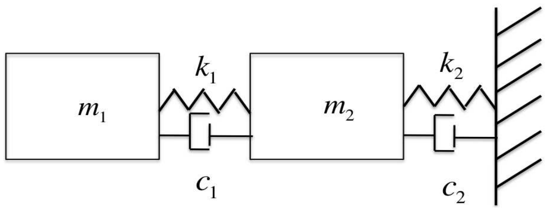

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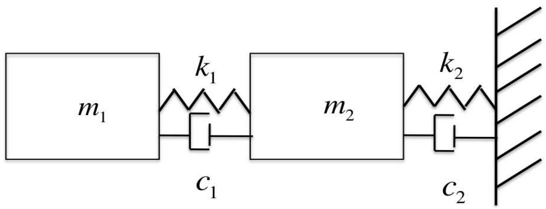

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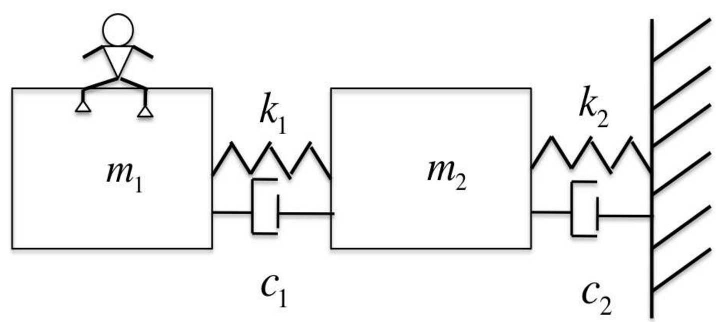

Example

$$\begin{bmatrix} m_1 \frac{d^2}{dt^2} + c_1 \frac{d}{dt} + k_1 \\ -c_1 \frac{d}{dt} - k_1 \end{bmatrix} w_1 = \begin{bmatrix} c_1 \frac{d}{dt} + k_1 \\ -m_2 \frac{d^2}{dt^2} - (c_2 + c_1) \frac{d}{dt} - (k_1 + k_2) \end{bmatrix} w_2$$

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Is polynomial differential operator on RHS injective?

Example

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Is polynomial differential operator on RHS injective?

$$\begin{bmatrix} c_1 \lambda + k_1 \\ -m_2 \lambda^2 - (c_2 + c_1) \lambda - (k_1 + k_2) \end{bmatrix}$$

has full column rank $\forall \lambda \in \mathbb{C}$ **(**$\Longleftrightarrow$ **observability)** $\Leftrightarrow$

$$-m_2 k_1^2 + c_1 c_2 k_1 - k_2 c_2^2 \neq 0$$

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Summary of Lecture 3

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- **Algebraic characterizations;**
- **Image representations.**

End of Lecture 3