Exercise 1: Linear models

Call the mathematical model $(\mathbb{R}^{\mathtt{w}}, \mathscr{B})$ linear if $\mathscr{B}$ is a linear subspace of $\mathbb{R}^{\mathtt{w}}$.

1. **Prove that a linear behavior admits a representation**

$$Rw = 0 \qquad R \in \mathbb{R}^{\bullet \times \mathtt{w}} \qquad (\star)$$

Call $(\star)$ a kernel representation of $\mathscr{B}$, and a minimal kernel representation if, among all kernel representations of $\mathscr{B}$, $\mathtt{rowdimension}(R)$ is as small as possible.

2. **Prove that $(\star)$ is minimal iff R has full row rank.**
3. **How are the R's of minimal kernel representations related?**
4. **Define what you mean by an image representation, and prove its existence.**

Exercise 2: Static input/output models

Consider the mathematical model $(\mathbb{U} \times \mathbb{Y}, \mathscr{B})$, with $\mathscr{B}$ the graph of the map

$$f : \mathbb{U} \to \mathbb{Y}, \quad y = f(u)$$

1. **Discuss that it is not illogical to call u the input (cause) and y the output (effect).**
2. **Prove that for the gas law, you can take any of the 4 variables as output, and the other 3 as inputs.**

   Is there anything logical about cause/effect thinking in this example?

Exercise 3: Symmetry

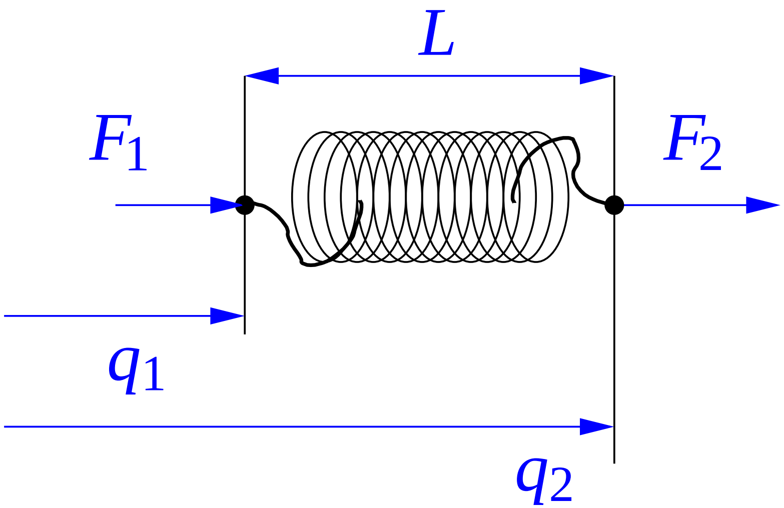
Let $T_g, g \in G$, be a transformation group on $\mathscr{U}$ (see appendix). Call the model $(\mathscr{U}, \mathscr{B})$ symmetric with respect to this transformation group if

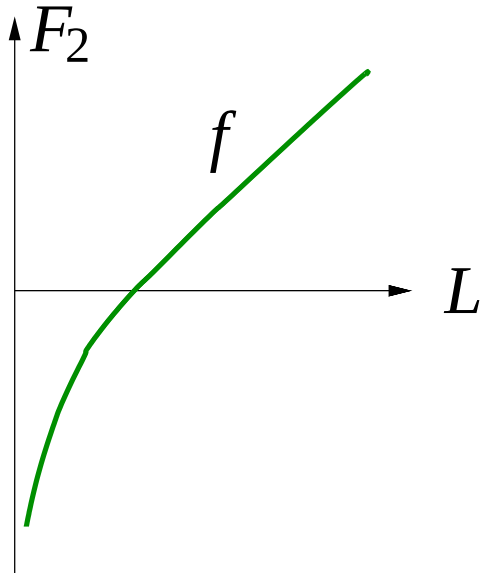
$$T_g(\mathscr{B}) = \mathscr{B} \qquad \text{for all } g \in G$$

1. **Identify an obvious symmetry for the 32-bit strings with a parity check.**
2. **Consider the law of attraction of 2 bodies.**
   Identify a symmetry related to exchanging positions.
3. **Explain in what sense Maxwell's equations are symmetric with respect to translation and rotation.**

Exercise 4: Energy in a spring

Consider the massless spring discussed in lecture 1.





The behavioral equations are

$$q_1 - q_2 = L, \;\; F_1 + F_2 = 0, \;\; L = f(F_2) \qquad (\spadesuit)$$

with f typically looking as shown in the above figure.

Exercise 4: Energy in a spring - continued

1. **Define an associated variable, E, the energy in the spring, and give an expression for it as a function of L.**

Now consider the spring as a dynamical system, by assuming that q_1, q_2, F_1, F_2, L are functions of time, and that equations (♠) hold for all $t \in \mathbb{R}$.

2. **The energy is now also a function of time. Prove that**

$$\frac{d}{dt}E = F_1 \frac{d}{dt}q_1 + F_2 \frac{d}{dt}q_2.$$

3. **Consider $p = F_1 \frac{d}{dt}q_1 + F_2 \frac{d}{dt}q_2$, the power absorbed by the spring. Of course, $\frac{d}{dt}E = p$. Consider the behavior of the joint variables $F_1, F_2, q_1, q_2, L, E, p$. Reason that this is a genuine dynamical system: energy and power can bring in dynamics, even in a static system.**

Exercise 5: Conservation law

Consider the dynamical system $(\mathbb{R}, \mathbb{W} \times \mathbb{R}, \mathscr{B})$. Assume that it is autonomous. Let (w, c) be a typical element of $\mathscr{B}$. Call c a conservation law if c is observable from w and
$[\![(w,c) \in \mathscr{B}]\!] \Rightarrow [\![c =$ constant (as a function of time)$]\!]$.

Define for the following examples (discussed in the lecture) an additional variable such that it is a conservation law for extended behavior.

1. **Newton's second law, made into an autonomous system by setting $F = 0$.**
2. **A pointmass moving in a gravitational field.**
3. **The diffusion equation, made autonomous by setting $q = 0$, viewed as a dynamical system with $w(t) = T(t, \cdot)$, and assuming $\int_{-\infty}^{+\infty} T(0, x)\, dx < \infty$.**

Exercise 6: State controllability

Consider

$$\frac{d}{dt}x = Ax + Bu, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}.$$

1. **Prove that this system is controllable in the sense of behaviors iff it is state controllable in the sense of Kalman.**

Consider

$$\frac{d}{dt}x = Ax + Bu, y = Cx + Du, \quad w = \begin{bmatrix} u \\ y \\ x \end{bmatrix}.$$

2. **Recall the standard definition of state observability in terms of deducing the initial state from the input and output. Give a formal definition using the notation from behaviors.**

3. **Prove that state observability is equivalent to behavioral observability with (u, y) observed and x to-be-deduced.**

Exercise 7: Time-reversibility

$\Sigma = (\mathbb{R}, \mathbb{W}, \mathscr{B})$ **is said to be time-reversible if** $w \in \mathscr{B}$ **implies** $\texttt{reverse}(w) \in \mathscr{B}$**, where** $\texttt{reverse}(w)$ **is defined by** $\texttt{reverse}(w)(t) := w(-t)$.

1. **Do Kepler's laws define a time-reversible system?**
2. **Let** $0 \neq p \in \mathbb{R}[\xi]$**. Prove that** $p(\frac{d}{dt})w = 0, p \in \mathbb{R}[\xi]$**, is time-reversible iff** p **is either an even or an odd polynomial. (Hint: in the time-reversible case,** $p(-\frac{d}{dt})w = 0$ **is also a kernel representation.)**
3. **Generalize the 'if' part to** $p(\frac{d}{dt})w_1 = q(\frac{d}{dt})w_2, p, q \in \mathbb{R}[\xi]$**.**
4. **Assume in addition that** p **and** q **are co-prime. (We will see in lecture 3 that this means that** $p(\frac{d}{dt})w_1 = q(\frac{d}{dt})w_2$ **defines a controllable system.) Prove that time-reversibility then implies that** p **and** q **are both even.**

Exercise 8: Covers

A behavior $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **is said to be a cover of** $\mathscr{B}' \in \mathscr{L}^{\mathtt{w}}$ **if** $\mathscr{B}' \subseteq \mathscr{B}$.

Consider the set of siso systems $\mathscr{B} \in \mathscr{L}^2$**, defined by**

$$p_1(\frac{d}{dt})w_1 = p_2(\frac{d}{dt})w_2 \quad p_1, p_2 \in \mathbb{R}[\xi], \text{ not both zero}.$$

1. **Take** $p_1(\xi) = (\xi+1)(\xi^2+1), p_2(\xi) = (\xi-1)(\xi^2+1)$**. List all the covers of the resulting** $\mathscr{B}$ **in the set of siso systems.**
2. **Generalize to arbitrary** p_1, p_2.
3. **Observe from this example that the list of siso covers of a siso system** $\mathscr{B} \in \mathscr{L}^2$ **is finite. Is this also the case for the set of siso systems for which** $\mathscr{B}$ **is a cover?**

Exercise 9: Minimal kernel representations

1. **Prove that $\mathfrak{U}_{\mathtt{n}} := \{U \in \mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}} \,|\, U$ is unimodular$\}$ forms a multiplicative group.**

2. **Define $\mathscr{K}_{\mathtt{p}}^{\mathtt{w}}$ to be the set of elements of $\mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{w}}$ with $\mathtt{rank} = \mathtt{p}$. Prove that $\mathfrak{U}_{\mathtt{p}}$ induces a transformation group on $\mathscr{K}_{\mathtt{p}}^{\mathtt{w}}$ by premultiplication.**

3. **Show that $\mathtt{kernel}(R(\frac{d}{dt}))$ defines a complete invariant for this transformation group.**

4. **Show that the $\mathtt{p}\times\mathtt{p}$ minors (of the elements of $\mathscr{K}_{\mathtt{p}}^{\mathtt{w}}$) listed in a well-defined order are invariants of this transformation group. In particular, the degree of the $\mathtt{p}\times\mathtt{p}$ minor of highest degree is an invariant. In lecture 4b, we show that this invariant equals $\mathtt{n}$, the dimension of the state space.**

Exercise 10: Elimination in the RLC circuit

Consider the RLC circuit discussed as an example in the lecture about the emergence of latent variables. This exercise ask you to eliminate the latent variables in an *ad hoc* manner.

1. **First eliminate** V_1, V_2, V_3, V_4 **and** I_a, I_b, I_c, I_f **to arrive at**

$$C\frac{d}{dt}V = I_e + CR_C I_e, \;\; V = L\frac{d}{dt}I_d + R_L I_d, \;\; I = I_e + I_d.$$

2. **Next, distinguish two cases to eliminate** I_d, I_e**.**

Exercise 11: Actions on $\mathscr{L}^{\mathtt{w}}$

Let $\mathscr{B}, \mathscr{B}_1, \mathscr{B}_2 \in \mathscr{L}^{\mathtt{w}}$ **and** $F \in \mathbb{R}[\xi]^{\mathtt{w}\times\mathtt{w}}$**.**

1. **Prove that** $(\mathscr{B}_1 + \mathscr{B}_2) \in \mathscr{L}^{\mathtt{w}}$
2. **Prove that** $\mathscr{B}_1 \cap \mathscr{B}_2 \in \mathscr{L}^{\mathtt{w}}$
3. **Prove that** $F(\frac{d}{dt})\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$
4. **Prove that** $F(\frac{d}{dt})^{-1}\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$

Exercise 12: Elimination algorithm

Let $F \in \mathbb{R}[\xi]^{w_1 \times \mathtt{w}_2}$. $\mathscr{N} := \{n \in \mathbb{R}[\xi]^{1 \times \mathtt{w}_1} \,|\, nF = 0\}$ **is an** $\mathbb{R}[\xi]$**-module, called the left syzygy of** F**. Assume that you have available an algorithm that computes a basis of the left syzygy of** F**, and a basis for a complement of** $\mathscr{N}$**, that of the module** $\mathscr{N}'$ **such that** $\mathscr{N} \oplus \mathscr{N}' = \mathbb{R}[\xi]^{1 \times \mathtt{w}_1}$**. Consider**

$$R\left(\frac{d}{dt}\right) w = M\left(\frac{d}{dt}\right) \ell \qquad (\blacktriangledown)$$

We wish to obtain an algorithm to eliminate the latent variables ℓ **from this equation.**

Assume that the rows of N **form a basis for the left syzygy of** M**, and that the rows of** N' **for a basis for complement of this left syzygy.**

Exercise 12: Elimination algorithm (continued)

1. **Prove that** $\begin{bmatrix} N' \\ N \end{bmatrix}$ **is unimodular**

2. **Prove that** $\begin{bmatrix} N' \\ N \end{bmatrix} M$ **is of the form** $\begin{bmatrix} M' \\ 0 \end{bmatrix}$**,** M' **full row rank.**

3. **Deduce an algorithm to compute a kernel representation of the manifest behavior of (▼).**

Conclusion: elimination of latent variables in LTIDSs can be reduced to a standard problem in computer algebra.

Exercise 13: Input/output partition

Consider the electrical circuit discussed in the lecture on elimination of latent variables. The external variables are the port voltage V and the port current I.

1. **Which of these variables is input, output?**
2. **Does your answer change if you consider only input/output partitions such that the transfer function is proper?**
3. **Assume that we want to use the 'scattering variables' $u = V + \rho I, y = V - \rho I$ as input and output variables. Choose ρ such that the resulting input/output system has a strictly proper transfer function.**

Exercise 14: Input and output cardinality

1. **Consider a polynomial matrix $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ representing a behavior $\mathscr{B}$ in kernel form. Prove that the number of outputs $\mathtt{p}(\mathscr{B})$ equals $\mathtt{rank}(R)$.**
2. **Consider $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$, 'complexified'. For $\lambda \in \mathbb{C}$, define $A_\lambda := \{v \in \mathbb{C}^{\mathtt{w}} \mid ve^{\lambda t} \in \mathscr{B}\}$. Prove that A_λ is a linear subspace of $\mathbb{C}^{\mathtt{w}}$. Let $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ be such that $\mathscr{B} = \ker\, R\left(\frac{d}{dt}\right)$. Prove that $A_\lambda = \mathtt{kernel}(R(\lambda))$.**
3. **Prove that $\mathtt{dimension}(A_\lambda)$ is the same for all except a finite number of $\lambda \in \mathbb{C}$. Prove that this 'normal' dimension is equal to $\mathtt{m}(\mathscr{B})$, the number of input variables of $\mathscr{B}$.**
4. **Prove that $\mathtt{dimension}(A_\lambda)$ is zero for all except a finite number of $\lambda \in \mathbb{C}$ iff $\mathscr{B}$ is autonomous. Prove that $\mathtt{dimension}(A_\lambda)$ is constant iff $\mathscr{B}$ is controllable.**

Exercise 15: Autonomy and orthogonality

Define $\mathscr{B}_1 \in \mathscr{L}^{\mathtt{w}}$ **and** $\mathscr{B}_2 \in \mathscr{L}^{\mathtt{w}}$ **to be orthogonal if** $\int_{-\infty}^{+\infty} w_1(t)^\top w_2(t)\,dt = 0$ **for all** $w_1 \in \mathscr{B}_1, w_2 \in \mathscr{B}_2$ **of compact support**.

1. **Define, using these ideas, the orthogonal complement,** $\mathscr{B}^\perp$ **of** $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$**.**
2. **Prove that the systems defined by the kernel representation** $R\left(\frac{d}{dt}\right) w = 0$ **and the image representation** $w = R^\top\left(-\frac{d}{dt}\right)\ell$ **are orthogonal.**
3. **It can be shown that under certain conditions (controllability) the systems** $R\left(\frac{d}{dt}\right) w = 0$ **and** $w = R^\top\left(-\frac{d}{dt}\right)\ell$ **are orthogonal complements. Assume this to be the case. Prove that** $\mathscr{B} \cap \mathscr{B}^\perp$ **is autonomous but in general not zero.**

Exercise 16: Observable image representation

1. We have seen in the lecture that $\mathscr{B} \in \mathscr{L}^\bullet$ is controllable iff it admits an image representation. Prove that it is controllable iff it admits an **observable image representation**
2. Consider the behavior described in kernel form by

$$p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)\, u$$

with $p, q \in \mathbb{R}[\xi]$.
What does controllability mean in terms of p, q?
Assuming controllability, give an observable image representation.
Give also a non-observable image representation.

Exercise 17: Representations of observable systems

1. **Partition the external variable in $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ as (w_1, w_2). Prove that there exists a polynomial differential operator $F\left(\frac{d}{dt}\right)$ such that $(w_1, w_2) \in \mathscr{B}$ implies $w_2 = F\left(\frac{d}{dt}\right) w_1$ iff w_2 is observable from w_1.**
2. **Partition the external variable in $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ as (w_1, w_2). Assume that w_2 is observable from w_1. Prove that there exists a kernel representation of $\mathscr{B}$ of the form**

$$H\left(\frac{d}{dt}\right) w_1 = 0 \qquad F\left(\frac{d}{dt}\right) w_1 = w_2$$

3. **Can you make this special kernel representation minimal?**

Exercise 18: Stabilizability

Let $\mathscr{B} = \mathtt{kernel}\left(R\left(\frac{d}{dt}\right)\right) \in \mathscr{L}^{\mathtt{w}}$.

1. **Prove that** $\mathscr{B}$ **is asymptotically stable (i.e., all elements of** $\mathscr{B}$ **go to zero as** $t \to \infty$**) iff** $\mathtt{rank}(R(\lambda)) = \mathtt{w}(\mathscr{B})$ **for all** $\lambda \in \mathbb{C}$ **s.t. Re**$(\lambda) \geq 0$.
2. **Prove that** $\mathscr{B}$ **is stable (i.e., all elements of** $\mathscr{B}$ **are bounded on** $[0, \infty)$ **iff**
   - **(a)** $\mathtt{rank}(R(\lambda)) = \mathtt{w}(\mathscr{B})$ **for all** $\lambda \in \mathbb{C}$ **s.t. Re**$(\lambda) > 0$;
   - **(b)** **for all** $\omega \in \mathbb{R}$, $w(\mathscr{B}) - \mathtt{rank}(R(i\omega))$ **equals the multiplicity of** $i\omega$ **as a root of** $\mathtt{determinant}(R)$.
3. **Prove that** $[\![\mathscr{B}$ **is stabilizable**$]\!] \Leftrightarrow [\![\mathtt{rank}\, R(\lambda) = \mathtt{rank}(R)$ **for all** $\lambda \in \mathbb{C}$ **such that Re**$(\lambda) \geq 0]\!]$.

Exercise 19: Rational representations of controllable siso systems

Consider the system described by the 'ODE'

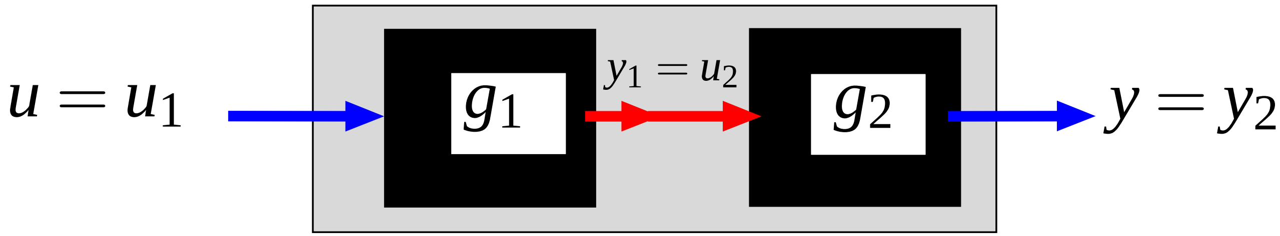
$$g\left(\frac{d}{dt}\right)w = 0 \qquad (\clubsuit)$$

with $0 \neq g \in \mathbb{R}(\xi)^{1\times 2}$**. Let** $g = [g_1 \vdots g_2]$**,** $g_1 = \frac{n_1}{d_1}, g_2 = \frac{n_2}{d_2}$ **with** $n_1, d_1,$ **and** n_2, d_2 **coprime polynomials.**

1. **Give a polynomial based kernel representation of (♣).**
2. **Give conditions of** n_1, d_1, n_2, d_2 **for (♣) to be controllable.**
3. **Prove that if (♣) is controllable, then all controllable rational symbol based kernel representations of this system are obtained by** $g \mapsto fg \qquad 0 \neq f \in \mathbb{R}[\xi]$**.**

Exercise 20: Series connection

Consider the series connection of siso systems with transfer functions g_1 and g_2. Assume that these systems are both controllable.



1. **Give a kernel representation of the series connection.**
2. **Under what conditions is this series connection also controllable?**
3. **What is the controllable part of the series connection, and what is its transfer function?**
4. **Can controllability change if you take the series connection in reverse order.**

Exercise 21: Spectral factorization

Consider the factorization equation

$$h(\xi) = f(-\xi) f(\xi) \qquad (\blacksquare)$$

with $h \in \mathbb{R}[\xi]$ **given, and** $f \in \mathbb{R}[\xi]$ **the unknown.**

1. **Prove that (■) is solvable iff**
   (i) $h(\xi) = h(-\xi)$ **and (ii)** $h(i\omega) \geq 0$ **for all** $\omega \in \mathbb{R}$**.**
2. **How should the above conditions be modified for the existence of a solution** h **that is Hurwitz?**
   (i.e. with its roots in the open left half part of $\mathbb{C}$**)**
3. **Use these results to prove that the controllable system** $\mathscr{B} \in \mathscr{L}^2$ **has a stable norm-preserving image representation.**

Exercise 22: Structure of state equations

Let $\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{x}}, \mathscr{B}_{\text{full}})$ be a discrete-time linear time-invariant latent variable system. Assume that it is *complete*, i.e. that

$$[\![w \in \mathscr{B}]\!] \Leftrightarrow [\![w|_{[t_0,t_1]} \in \mathscr{B}|_{[t_0,t_1]} \text{ for all } -\infty < t_0 \geq t_1 < \infty]\!]$$

Prove that Σ is a state system iff there exist matrices $E, F, G \in \mathbb{R}^{\bullet\times\bullet}$ such that $\mathscr{B}_{\text{full}}$ is described by

$$E\sigma x + Fx + Gw = 0$$

(*Hint:* For the "only if" part, define

$$\mathscr{V} := \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid \exists (x,w) \in \mathscr{B}_{\text{full}} \textbf{ s. t. } \begin{bmatrix} x(1) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\}$$

Obviously $\mathscr{V}$ is a linear space. Deduce E, F, G from $\mathscr{V}$.)

Exercise 23: The state dimension

The McMillan degree of a system $\mathscr{B} \in \mathscr{L}^\bullet$ is the dimension of the minimal state space of $\mathscr{B}$, denoted by $\mathtt{n}(\mathscr{B})$. Let $R\left(\frac{d}{dt}\right) w = 0$ be a minimal kernel representation of $\mathscr{B}$.

1. **Prove that $\mathtt{n}(\mathscr{B})$ is equal to the maximum of the degrees of the $\mathtt{w}(\mathscr{B}) \times \mathtt{w}(\mathscr{B})$ minors of R. You may use the following fact: there exists a unimodular matrix U such that UR is row-reduced. A polynomial matrix is *row reduced* if the matrix formed by the coefficients of the highest degrees of the rows is of full row rank.**

2. **Prove that $\mathtt{n}(\mathscr{B}) = \mathtt{degree}(\mathtt{determinant}(P))$ with P such that**

$$\begin{bmatrix} P & -Q \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = 0$$

is an input/output representation of $\mathscr{B}$ with $P^{-1}Q$ proper.

Exercise 24: State construction for siso systems

Consider the behavior described in kernel form by

$$p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)\, u$$

with

$$\begin{aligned} p(\xi) &= p_0 + p_1\xi + \ldots + p_n\xi^n \\ q(\xi) &= q_0 + q_1\xi + \ldots + q_n\xi^n \end{aligned}$$

1. **Write the polynomial matrix** $X \in \mathbb{R}^{n\times 2}[\xi]$ **obtained by applying the shift-and-cut map to the matrix**

$$\begin{bmatrix} p(\xi) & -q(\xi) \end{bmatrix}$$

2. **Is** $X(\xi)$ **obtained in this way a minimal state map? Explain.**

Exercise 24: State construction for siso systems (continued)

3. **Verify that the matrices A, B, C, and D corresponding to this state map are**

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & -\frac{p_{n-1}}{p_n} \\ 0 & 1 & 0 & \dots & 0 & -\frac{p_{n-2}}{p_n} \\ \vdots & \ddots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -\frac{p_1}{p_n} \\ 0 & 0 & 0 & \dots & 1 & -\frac{p_0}{p_n} \end{bmatrix} \qquad B = \begin{bmatrix} q_{n-1} - \frac{p_{n-1}q_n}{p_n} \\ q_{n-2} - \frac{p_{n-2}q_n}{p_n} \\ \vdots \\ q_1 - \frac{p_1 q_n}{p_n} \\ q_0 - \frac{p_0 q_n}{p_n} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & \frac{1}{p_n} \end{bmatrix} \qquad D = \frac{q_n}{p_n}$$

Exercise 25: Construction of Lyapunov function

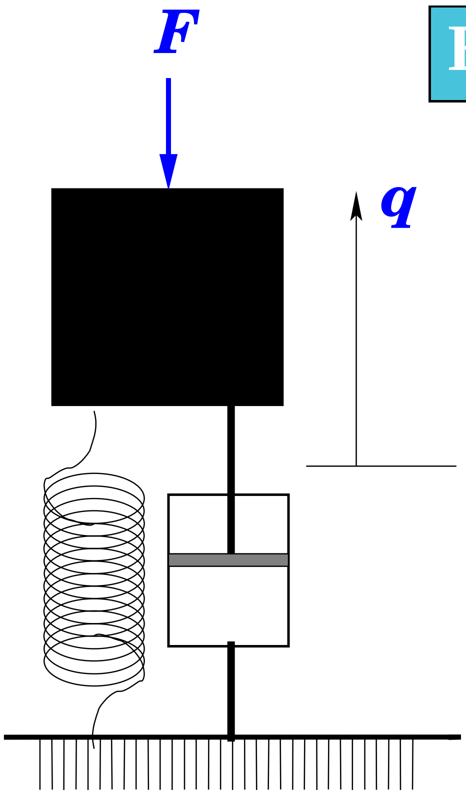
Consider the scalar system

$$w + 2\frac{d}{dt}w + 2\frac{d^2}{dt^2}w + \frac{d^3}{dt^3}w = 0.$$

1. **Prove that this polynomial is Hurwitz.**
2. **Use the calculus of QDFs to find a Lyapunov function with derivative**

$$-w^2 - \left(\frac{d}{dt}w\right)^2 - \left(\frac{d^2}{dt^2}w\right)^2 .$$

Exercise 26: Power, energy and dissipation



$q =$ distance from equilibrium
$F =$ force exerted

Assume all elements linear choose the units so that the spring constant, the mass, and the damping constant are all $= 1$

1. **Write the behavioral differential equation relating F and q in kernel and in image form.**
2. **Express the power delivered, the stored energy, and the dissipation rate in terms of QDFs acting both on (F, q), and on the latent variable ℓ of your image representation.**
3. **Verify the identity relating the power delivered, the stored energy, and the dissipation rate.**

Exercise 27: SOS

Let $\Phi \in \mathbb{R}[\zeta,\eta]^{\mathtt{w}\times\mathtt{w}}$ be symmetric. We have seen in the lecture that $Q_\Phi \geq 0$ iff there exists $D \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$ such that

$$\Phi(\zeta,\eta) = D^\top(\zeta)D(\eta), \qquad \text{i.e.} \qquad Q_\Phi(w) = |D\left(\tfrac{d}{dt}\right)w|^2$$

1. **Prove this, and interpret it as '*a QDF is nonnegative iff it is a sum-of-squares*'.**
2. **In the case $\mathtt{w} = 1$ give a bound on the number of squares and on the degree of D in terms of the degree on Φ.**
3. **State and prove the analogous statement '*a QDF is nonnegative along $\mathscr{B}$ iff it is a sum-of-squares on $\mathscr{B}$*' for a controllable $\mathscr{B} \in \mathscr{L}^\bullet$.**

Exercise 28: Computation of storage functions

Consider the system

$$\left(\frac{d}{dt}+1\right)w_1 = \left(\frac{d}{dt}+4\right)w_2$$

and the supply rate $w_1\ w_2$**.**

1. **Write this system in image form.**
2. **Express what condition a QDF in the latent variable of the image representation has to satisfy in order to qualify as a storage.**
3. **Use polynomial factorization to compute 2 distinct storages.**

Exercise 29: When is a QDF a derivative of a QDF?

We have seen in the lecture that for a given $\Phi \in \mathbb{R}[\zeta,\eta]^{\mathtt{w}_1 \times \mathtt{w}_2}$, there exists $\Psi \in \mathbb{R}[\zeta,\eta]^{\mathtt{w}_1 \times \mathtt{w}_2}$ such that

$$\Phi(\zeta,\eta) = (\zeta+\eta)\Psi(\zeta,\eta)$$

iff $\Phi(-\xi,\xi) = 0$. The 'only if' part is clear. The aim of this exercise is to give a matrix-based proof of the 'if' part.

1. **Prove that you may restriction attention to $\mathtt{w}_1 = \mathtt{w}_2 = 1$.**
2. **Prove that you may restrict attention to homogeneous scalar two-variable polynomials**

$$\Phi(\zeta,\eta) = a_0\zeta^{\mathtt{n}} + a_0\zeta^{\mathtt{n}-1}\eta + \cdots + a_{\mathtt{n}-1}\zeta\eta^{\mathtt{n}-1} + a_{\mathtt{n}}\eta^{\mathtt{n}}.$$

3. **Prove that if Φ is homogeneous of degree $\mathtt{n}$, then Ψ must be homogeneous of degree $\mathtt{n}-1$.**

Exercise 29: When is a QDF a derivative of a QDF? (continued)

4. **Consider the equation**

$$a_0\zeta^{\mathtt{n}} + a_1\zeta^{\mathtt{n}-1}\eta + \cdots + a_{\mathtt{n}-1}\zeta\eta^{\mathtt{n}-1} + a_{\mathtt{n}}\eta^{\mathtt{n}}$$
$$= (\zeta+\eta)(b_0\zeta^{\mathtt{n}-1} + b_1\zeta^{\mathtt{n}-2}\eta + \cdots + b_{\mathtt{n}-2}\zeta\eta^{\mathtt{n}-2} + b_{\mathtt{n}-1}\zeta^{\mathtt{n}-1}). \quad (\nabla)$$

   Write this as a matrix equation $a = Mb$ **with** $a \in \mathbb{R}^{\mathtt{n}+1}, b \in \mathbb{R}^{\mathtt{n}}$ **defined in the obvious way.** **What is** M**? Prove that it has rank** $\mathtt{n}$**.**

5. **Prove that** $\mathtt{image}\left(\begin{bmatrix}1 & -1 & 1 & -1 & \cdots\end{bmatrix}\right)$ **is the left kernel of** M**. Deduce that (∇) has a solution iff**

$$a_0 + a_2 + \cdots = a_1 + a_3 + \cdots .$$

6. **Show that this means** $\Phi(-\xi,\xi) = 0$**. Conclude the 'if' part.**

Exercise 30: LMIs

Consider the system $\frac{d}{dt}x = Ax + Bu$ **and the QDF supply rate** $x^\top Qx + 2x^\top Su + u^\top Ru$**. Assume** $Q = Q^\top, R = R^\top$**.**

1. **Prove that** $x^\top Kx$ **with** $K = K^\top$ **is a storage iff**

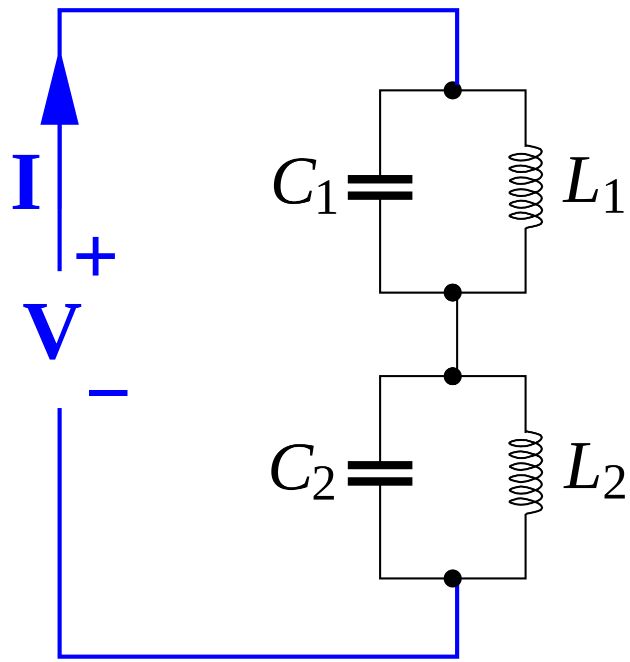
$$\begin{bmatrix} Q - A^\top K - KA & S - KB \\ S^\top - B^\top K & R \end{bmatrix} \geq 0. \qquad \text{(LMI)}$$

   This inequality is a ***linear matrix inequality*** **(LMI).**

2. **Prove that (LMI) has a solution** K **if** $\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \geq 0$**.**

3. **Prove that the set of solutions** K **is always convex (for fixed** A, B, R, S, Q**).**

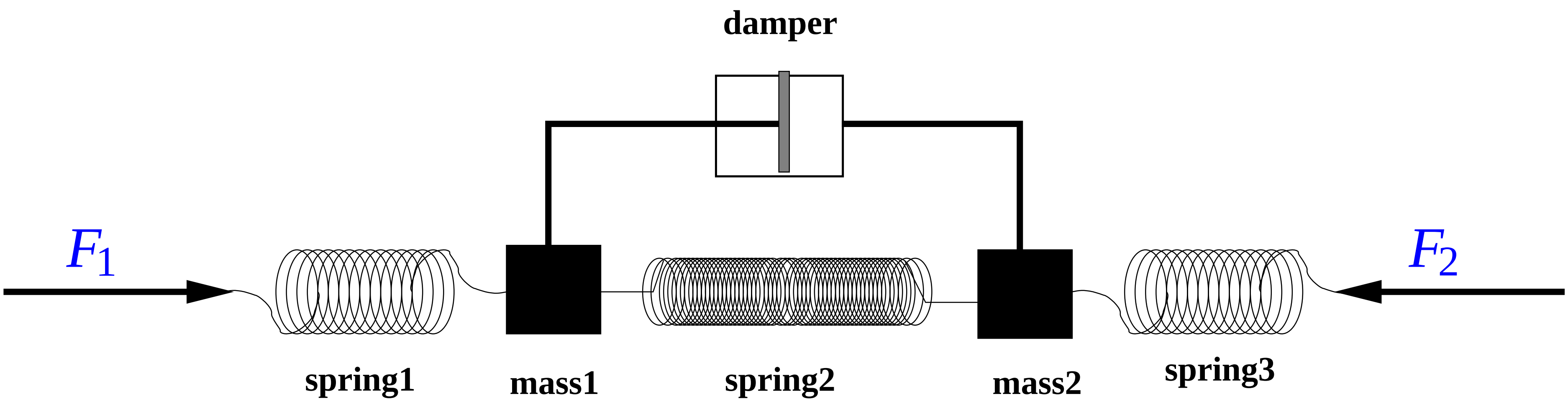
Exercise 31: Modeling an LC circuit



1. **Model the port behavior of the above circuit in a systematic way by tearing, zooming, and linking.**
2. **For which values of the L's and C's (all assumed positive) is the port behavior controllable?**
3. **For which values of the L's and C's (all assumed positive) are the branch currents observable from the port variables?**

Exercise 32: Modeling a mass-spring-damper system

Consider the mass-spring-damper system shown below.



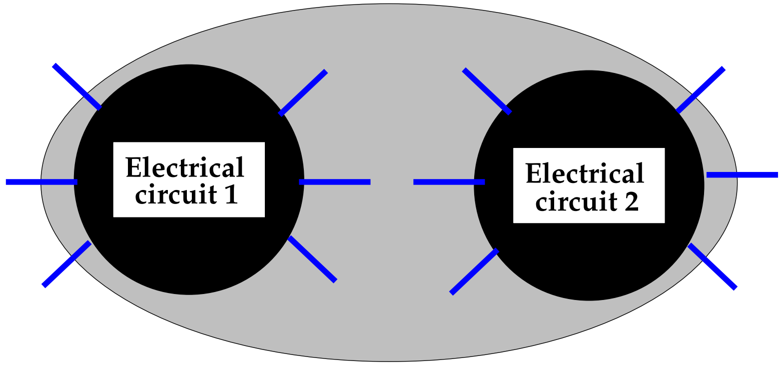
View this system as a interconnection of 3 springs, 2 masses, and 1 damper. There are 2 external forces.

Exercise 32: Modeling a mass-spring-damper system (continued)

1. **Draw the associated graph with leaves.**
2. **Assume all elements to be linear. Take as parameters of the modules (in the obvious notation) $k_1, k_2, k_3, m_1, m_2, d$. Associate with each terminal a position and a force. Write behavioral equations for each of the subsystems.**
3. **Write equations for the interconnections.**
4. **Take as manifest variables the total force $F_{\text{total}} = F_1 + F_2$, and the position of the center of gravity, $\bar{q} = \frac{m_1 q_1 + m_2 q_2}{m_1 + m_2}$. Write the equations for the manifest variable assignment.**
5. **Eliminate the latent variables and obtain the differential equation governing the manifest variables.**

Exercise 33: Ports and terminals

Consider the two circuits shown below. Number the terminals of the first circuit by $1, 2, 3, 4, 5, 6$ **and of the second circuit by** $7, 8, 9, 10, 11, 12$**. Assume that all the terminals together of the individual circuits form ports.**



1. **Interconnect terminal** $(4, 7), (5, 8), (6, 9)$**. Prove that the external terminals of the interconnected circuit also forms a port.**
2. **Assume that terminals** $1, 2, 3$ **of circuit 1 form a port. Prove that terminals** $(10, 11, 12)$ **form a port for the interconnected circuit.**

Exercise 34: Boundedness of solutions

Consider the setting of the stabilization theorem.

1. **Give n.a.s.c. on the invariant factors of $\mathscr{P}$ for the existence of a controller $\mathscr{C}$ such that the all solutions of $\mathscr{K}$ are bounded on $[0, \infty)$. This property is often called 'stability'. What we called 'stability' in the stabilization theorem is then called 'asymptotic stability'.**
2. **Prove that there exists a controller $\mathscr{C}$ such that the all solutions of $\mathscr{K}$ are bounded on $(-\infty, +\infty)$ iff all the invariant factors of $\mathscr{P}$ are even and have simple roots.**

Exercise 35: Pole placement for siso systems

Consider Consider, for
$p, q \in \mathbb{R}[\xi]$, **coprime, with** $\mathtt{degree}(q) < \mathtt{degree}(p) = \mathtt{n}$,
the *Bézout* equation $\quad px + qy = z, \quad x, y, z \in \mathbb{R}[\xi]$.

1. **Prove that for** z **with** $\mathtt{degree}(z) < 2\mathtt{n}$**, there exist** x, y **with** $\mathtt{degree}(x), \mathtt{degree}(y) < \mathtt{n}$ **that satisfy the Bézout equation. <u>Hint</u>: Prove that the map** $(x, y) \mapsto z$ **viewed as a map from the polynomials of degree** $< \mathtt{n}$ **to the polynomials of degree** $< 2\mathtt{n}$ **is injective, hence surjective.**
2. **Prove that**
$\llbracket \mathtt{degree}(z) = 2\mathtt{n} - 1 \rrbracket \Rightarrow \llbracket \mathtt{degree}(y) \leq \mathtt{degree}(x) = \mathtt{n} - 1 \rrbracket$.
3. **Deduce a sharp pole placement result, including properness of the transfer function of the controller, for the siso plant**

$$p\left(\frac{d}{dt}\right) y = q\left(\frac{d}{dt}\right) u.$$

Exercise 36: Implementability

Consider the setting of the implementability theorem. Assume fifull control $c = w$, and controllers $\mathscr{C}$ acting on c.

1. **Prove that the set of implementable controlled behaviors obtained by controllers that act on c is in general smaller than the set of implementable controlled behaviors obtained with full control $c = w$.**
2. **Under what conditions are these sets equal?**

Exercise 37: Output measurements

In the lecture, we assumed that the measurements are elements of the universum $\mathscr{U}$ of events. However, in applications, measurements may be functions of the events. Assume that there is a (known) map $f : \mathscr{U} \to M$, with M the set where the measurements take on their values. Assume that the measurements are collected in a subset $\mathbb{D}$ of M.

1. **Explain what you mean by 'unfalsified' and 'the MPUM' in this situation.**

Assume $\mathscr{U} = \mathbb{R}^{\mathtt{w}}$, $\mathscr{M}$ = the set of linear subspaces of $\mathbb{R}^{\mathtt{w}}$, $M = \mathbb{R}^{\mathtt{m}}$, and f linear.

2. **Prove that the MPUM exists in this case.**
3. **Explain how to compute it using $\mathbb{D}$ and f.**

Exercise 38: Exponential interpolation

Let $\lambda \in \mathbb{C}, 0 \neq v \in \mathbb{C}^{\mathtt{w}}$, and consider the vector exponential $t \in \mathbb{R} \mapsto e^{\lambda t} v \in \mathbb{C}^{\mathtt{w}}$.

1. **Prove that $R\left(\frac{d}{dt}\right) w = 0$ is unfalsified by (or 'interpolates') this exponential iff $R(\lambda)v = 0$.**
2. **Prove that $\frac{vv^\top}{v^\top v}\left(\frac{d}{dt} - \lambda\right) w = 0$ defines the MPUM that interpolates this exponential.**
3. **Generalize this to obtain a recursive algorithm for $\mathtt{n}$ exponentials, by computing the 'error' exponential $R(\lambda_{\mathtt{k}}) v_{\mathtt{k}} e^{\lambda_{\mathtt{k}} t}$ at each stage, the MPUM of this exponential, and the recursion $R \mapsto ER$.**

Exercise 39: Autonomous behavior

1. **Compute, using the recursive algorithm discussed in the lecture, the MPUM for the Fibonacci series**

$$0, 1, 1, 2, 3, 5, 8, \ldots$$

Leonardo Fibonacci
ca. 1170 – ca. 1250

2. **Repeat for**

$$\underbrace{1, 1, \ldots, 1}_{\mathtt{n}\ \text{times}}, 0, 0, \ldots$$

3. **Prove that the MPUM for $w(0), w(1), w(2), \ldots$ is autonomous iff the data Hankel matrix has finite rank.**

4. **Assume that the data Hankel matrix has finite rank $\mathtt{n}$. Obtain a kernel representation of the MPUM starting from any rank $\mathtt{n}$ submatrix of the Hankel matrix.**

Exercise 40: Stationarity and minimality

Consider the quadratic expression

$$\int_{-\infty}^{+\infty} \left[w^2 + \left(\frac{d}{dt} w \right)^2 \right] dt.$$

1. **Determine the ODE that gives the stationary trajectories.**
2. **Are these local minima?**
3. **Spectral factor** $1 + \xi^4$**.**
4. **Determine the ODE that gives the local minima w.r.t. one-sided variations.**

Exercise 41: Stationarity for siso systems

Consider the siso system with kernel representation

$$p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u,$$

with $p, q \in \mathbb{R}[\xi]$ **coprime. Consider the quadratic expressions**

$$\int_{-\infty}^{+\infty} (u^2 + y^2)\, dt \quad \textbf{and} \quad \int_{-\infty}^{+\infty} (u^2 - y^2)\, dt.$$

1. **Determine the stationary trajectories.**
2. **Prove that for the first case the stationary trajectories are local minima.**
3. **Determine for the second case a condition on the modulus of the transfer function** $g = \frac{q}{p}$ **for local minimality.**

Exercise 42: Feedback control for siso systems

Consider the siso system with kernel representation

$$p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u,$$

with $p, q \in \mathbb{R}[\xi]$ **coprime, and the quadratic functional**

$$\int_{-\infty}^{+\infty} \left(u^2 + y^2\right)\, dt$$

1. **Prove that there exists a unique Hurwitz polynomial** h **such that**

$$p(-\xi)p(\xi) + q(-\xi)q(\xi) = h(-\xi)h(\xi)$$

2. **What can you say about the degree of** h **in terms of the degrees of** p, q**?**

Assume henceforth $\texttt{degree}(q) < \texttt{degree}(p)$ **and** p, h **monic.**

Exercise 42: Feedback control for siso systems (continued)

3. Prove that

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \left(\frac{d}{dt} \right) \ell$$

is an image representation of the system.

4. Prove that the one-sided local minima are obtained by

$$h \left(\frac{d}{dt} \right) \ell = 0.$$

5. Prove that these traj. are generated by the control law

$$u = \left[p \left(\frac{d}{dt} \right) - h \left(\frac{d}{dt} \right) \right] \ell.$$

6. Prove that this control law is a static gain acting on the state of the system.