Lecture 8

SYNTHESIS OF DISSIPATIVE SYSTEMS

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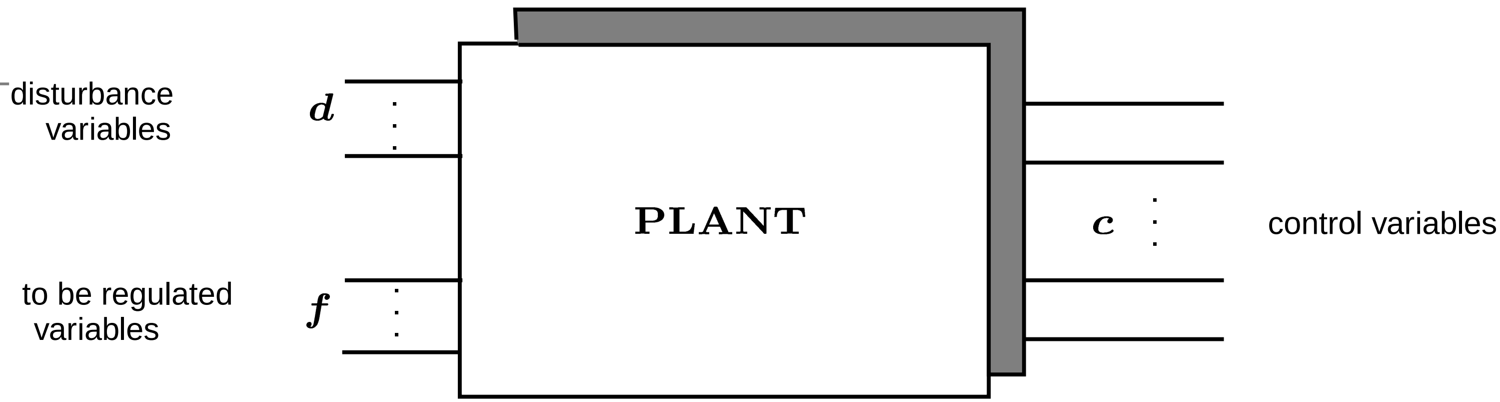
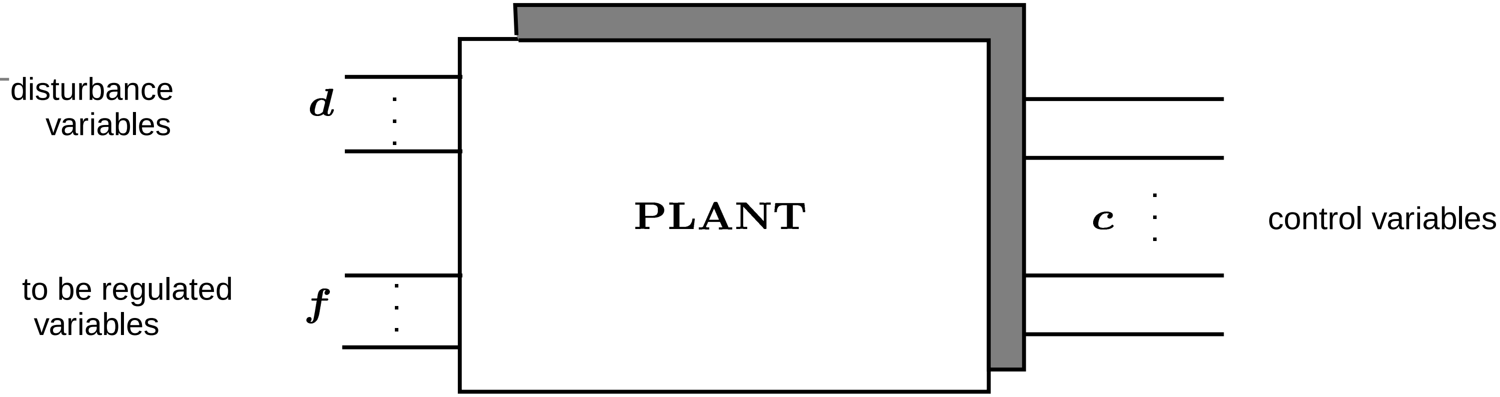

disturbance variables
d
to be regulated variables
f
PLANT
c
control variables



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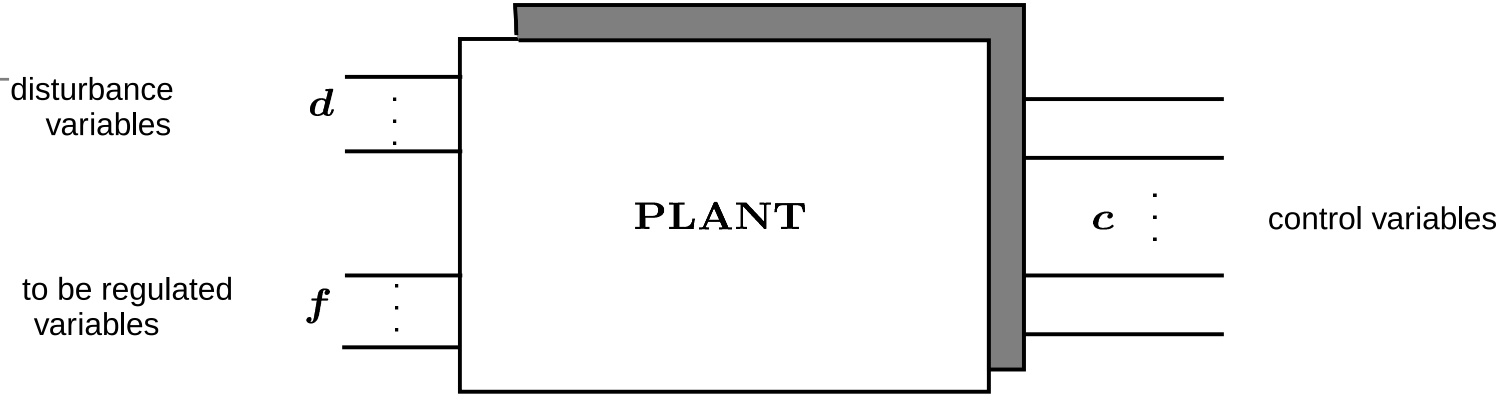

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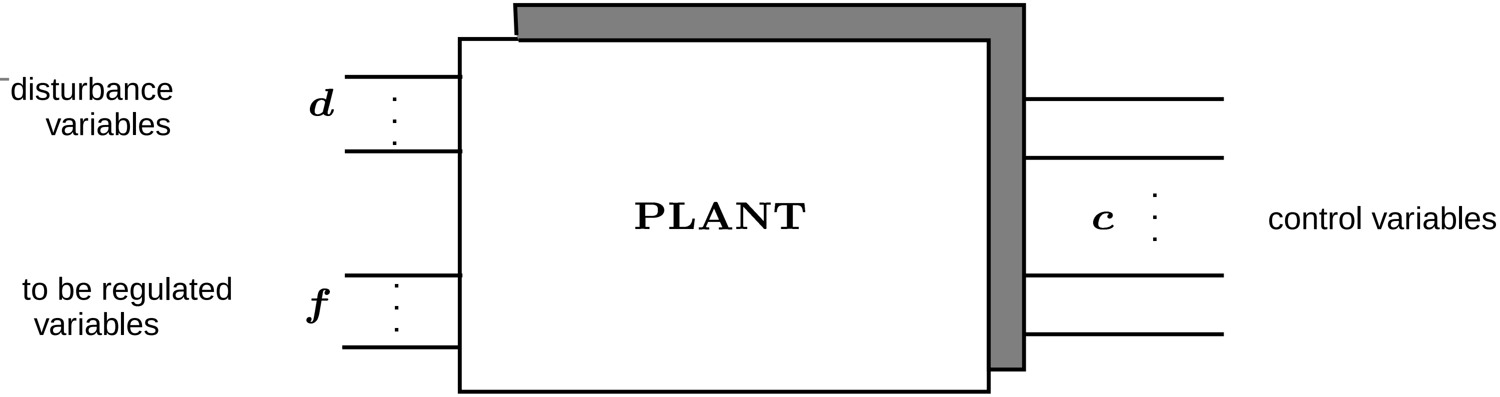

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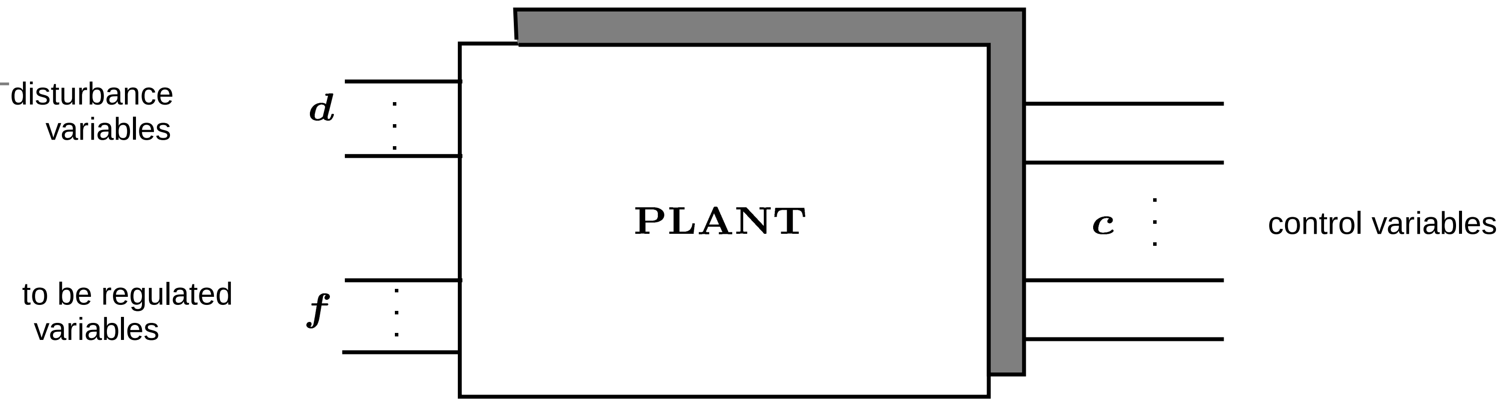

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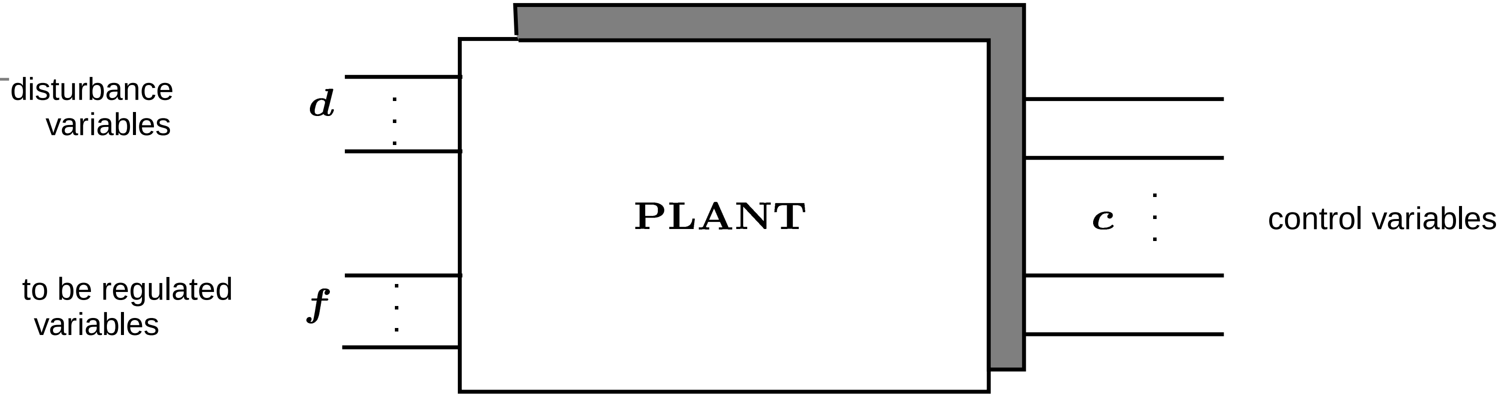

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- **to be regulated variables f,**
- **exogenous disturbance variables d,**
- **control variables c.**

Full plant behavior $\mathcal{P}_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}+\mathtt{c}}$:

$$\mathcal{P}_{\mathrm{full}} := \{(d, f, c) \mid (d, f, c) \text{ satisfies the plant equations}\}$$

Note: to be controlled variable is (d, f).

General control problem

Control: **given a set of design specifications, find conditions for the existence of, and compute, a controller $\mathcal{C}$ such that the resulting manifest controlled behavior $\mathcal{K}$ satisfies the specifications.**

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Reformulation: **given a set of design specifications, find conditions for the existence of, and compute, a behavior $\mathcal{K}$ such that**

- $\mathcal{K}$ **satisfies the specifications,**
- $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$ **(implementability).**

$\mathcal{N}$ **the hidden behavior, $\mathcal{P}$ the manifest plant behavior w.r.th $\mathcal{P}_{\text{full}}$.**

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$\mathcal{N}$ the hidden behavior, $\mathcal{P}$ the manifest plant behavior w.r.th $\mathcal{P}_{\mathrm{full}}$.

Of course, after finding such $\mathcal{K}$ one still needs to compute an actual controller $\mathcal{C} \in \mathfrak{L}^{\mathrm{c}}$ that implements $\mathcal{K}$.

$\mathcal{H}_\infty$ specifications

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Liveness: in the controlled system behavior, no direct restrictions on the exogenous disturbances are allowed: every component of d is arbitrary.

$\mathcal{H}_\infty$ specifications and dissipativity

The $\mathcal{H}_\infty$ specifications on $\mathcal{K}$ can be reformulated in terms of Σ dissipativity of $\mathcal{K}$, with

$$\Sigma = \begin{bmatrix} I_{\mathrm{d}} & 0 \\ 0 & -I_{\mathrm{f}} \end{bmatrix},$$

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Proposition: Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}_{\mathrm{cont}}$. The following statements are equivalent:

1. **$\mathcal{K}$ satisfies the $\mathcal{H}_\infty$ specifications,**
2. **$\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$, and $\mathtt{m}(\mathcal{K}) = \mathtt{d}$,**

Recall: $\mathfrak{B}$ is called Σ-dissipative on $\mathbb{R}_-$ if $\int_{-\infty}^{0} Q_\Sigma(w)dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

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find conditions for the existence of, and compute, $\mathcal{K} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}_{\rm cont}$ that satisfies the following three properties:

- $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$ **(implementability),**
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Recall: $\mathcal{N}$ is the hidden behavior, and $\mathcal{P}$ the manifest plant behavior associated with $\mathcal{P}_{\rm full}$:

$$\mathcal{N} = \{(d, f) \mid (d, f, 0) \in \mathcal{P}_{\rm full}\},$$

$$\mathcal{P} = \{(d, f) \mid \text{ there exists } c \text{ such that } (d, f, c) \in \mathcal{P}_{\rm full}\}.$$

More general problem:

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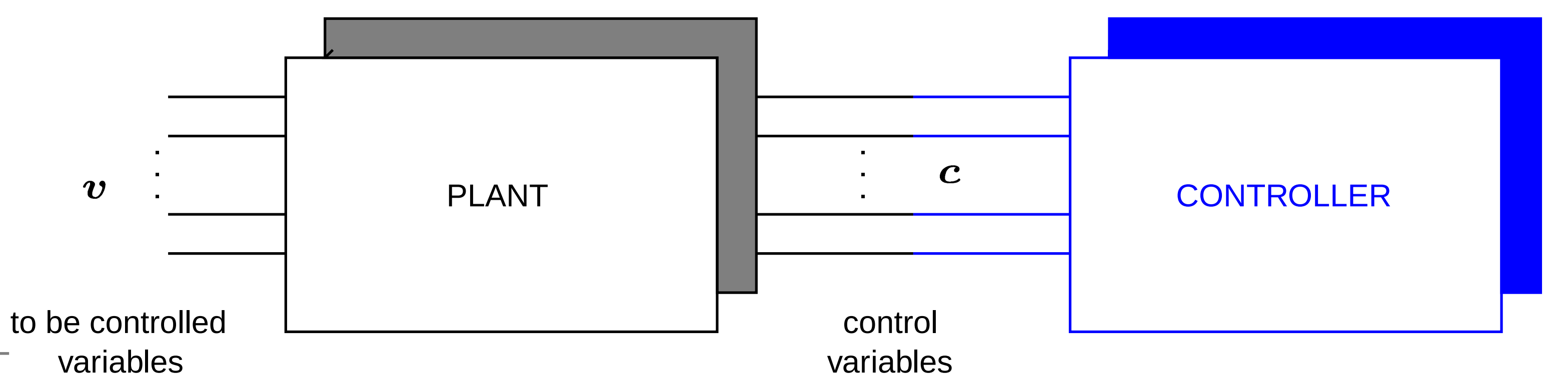
- $\mathtt{m}(\mathcal{K}) = \mathtt{d} \longrightarrow \mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$.



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Given $\mathcal{N}, \mathcal{P} \in \mathfrak{L}^{\mathrm{v}}_{\mathrm{cont}}$ with $\mathcal{N} \subseteq \mathcal{P}$,
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Can be shown: $\mathcal{K}$ is Σ-dissipative $\Rightarrow \mathtt{m}(\mathcal{K}) \leq \sigma_+(\Sigma)$.
Hence: the input cardinality of such $\mathcal{K}$ **attains the upper bound $\sigma_+(\Sigma)$.**

Questions:

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- **If such $\mathcal{K}$ exists, how can it be computed?.**
- **Given such $\mathcal{K}$, how can we compute a controller $\mathcal{C}$ that implements this $\mathcal{K}$?**

Deriving necessary conditions

Assume $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$, $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$, and $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$.

Then $\mathcal{K}$ is Σ-dissipative, so: $\mathcal{N}$ is Σ-dissipative.

Deriving necessary conditions

Since $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$, we have: $\mathcal{K}$ is Σ-dissipative $\Leftrightarrow$ $(\Sigma\mathcal{K})^\perp$ is $(-\Sigma)$-dissipative.

Since $\mathcal{K} \subseteq \mathcal{P}$, we have $(\Sigma\mathcal{P})^\perp \subseteq (\Sigma\mathcal{K})^\perp$, whence:

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For a given $\mathfrak{B}$, $\mathfrak{B}^\perp$ is the orthogonal behavior of $\mathfrak{B}$, defined by

$$\mathfrak{B}^\perp = \{ w \mid \int_{-\infty}^{+\infty} w^T w' dt = 0 \text{ for all } w' \in \mathfrak{B} \cap \mathfrak{D} \}.$$

It can be shown: $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}} \Rightarrow \mathfrak{B}^\perp \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$.

So far, we have derived **two** necessary conditions. We can obtain a set of necessary **and sufficient** conditions by adding a **third** condition.

This conditions deals with the existence of certain **storage functions** for $\mathcal{N}$ and $(\Sigma\mathcal{P})^{\perp}$.

Since $\mathcal{N} \subseteq \mathcal{P}$ we have $(\Sigma\mathcal{P})^{\perp} \subseteq (\Sigma\mathcal{N})^{\perp}$

This is used to prove the existence of a two-variable polynomial matrix $\Psi \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}[\zeta,\eta]$ such that

$$\frac{d}{dt}L_{\Psi}(v_1,v_2) = L_{\Sigma}(v_1,v_2), \text{ for } v_1 \in \mathcal{N}, v_2 \in (\Sigma\mathcal{P})^{\perp}.$$

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The BDF $L_{\Psi}(v_1,v_2)$ is the so called adapted bilinear differential form. It is unique on $\mathcal{N} \times (\Sigma\mathcal{P})^{\perp}$.

Bilinear differential forms (BDF's)

Two-variable polynomial matrix:

$$\Phi(\zeta,\eta) = \sum_{h,k=0}^{N} \Phi_{h,k}\zeta^h\eta^k,$$

$\Phi_{h,k} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$, N is a nonnegative integer. Φ induces a bilinear functional, acting on infinitely differentiable trajectories, as follows:

$$L_\Phi : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_2}) \longrightarrow \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}),$$

$$L_\Phi(w_1,w_2) = \sum_{h,k=0}^{N} (\frac{d^h w_1}{dt^h})^T \Phi_{h,k} \frac{d^k w_2}{dt^k}.$$

This functional is called the bilinear differential form induced Φ

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1. $\mathcal{N}$ **is** Σ**-dissipative,**
2. $(\Sigma\mathcal{P})^{\perp}$ **is** $(-\Sigma)$**-dissipative,**
3. **there exist** $\Psi_{\mathcal{N}}, \Psi_{(\Sigma\mathcal{P})^{\perp}} \in \mathbb{R}^{\mathrm{v}\times\mathrm{v}}[\zeta,\eta]$**, defining**
   - **a storage function** $Q_{\Psi_{\mathcal{N}}}$ **for** $\mathcal{N}$ **as a** Σ**-dissipative system,**
   - **a storage function** $Q_{\Psi_{(\Sigma\mathcal{P})^{\perp}}}$ **for** $(\Sigma\mathcal{P})^{\perp}$ **as a** $(-\Sigma)$**-dissipative system,**

   such that that the QDF

$$Q_{\Psi_{\mathcal{N}}}(v_1) - Q_{\Psi_{(\Sigma\mathcal{P})^{\perp}}}(v_2) + 2L_{\Psi}(v_1, v_2)$$

   is non-negative for $v_1 \in \mathcal{N}$ **and** $v_2 \in (\Sigma\mathcal{P})^{\perp}$**.**

The coupling condition

Surprising condition is the non-negativity:

$$Q_{\Psi_{\mathcal{N}}}(v_1) - Q_{\Psi_{(\Sigma\mathcal{P})^{\perp}}}(v_2) + 2L_{\Psi}(v_1, v_2) \geq 0$$

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Generalization of the well-known coupling condition of state space $\mathcal{H}_\infty$-theory involving solutions of algebraic Riccati equations.

From general result to particular representations

Statement of the main result does not use representations of $\mathcal{N}$ and $\mathcal{P}$. Hence: applicable independent of the particular representation by which the full plant $\mathcal{P}_{\text{full}}$ is given. Each 'numerical' specification of $\mathcal{P}_{\text{full}}$ leads to a 'numerical' verification of the conditions, and a 'numerical' computation of the controlled behavior and the controller.

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- **Next: express the conditions of the main result in terms of the parameters of these representations. This will typically result in ARE's, LMI's, factorizability of polynomial matrices, etc.**
- **Use the general construction of the controlled behavior $\mathcal{K}$ to set up synthesis algorithms in terms of the parameters of these representations.**

Illustration: Plant in state space representation

$\mathcal{P}_{\text{full}}$ **represented by**

$$
\begin{array}{rcccccc}
\frac{d}{dt}x & = & Ax & + & Bu & + & Gd_1 \\
y & = & Cx & + & & + & d_2 \\
f & = & Hx & & & &
\end{array}
$$

Control variable: $c = (u, y)$**, to be controlled variable** (d_1, d_2, u, f)**. Weighting functional** $\Sigma = \begin{bmatrix} I_{\text{d}} & 0 \\ 0 & -I_{\text{f}} \end{bmatrix}$**.**

We want $\mathcal{N}$ **and** $\mathcal{P}$ **controllable. For this, assume** (A, G) **controllable,** (H, A) **observable.**

Verification of the conditions

Dissipativity of $\mathcal{N}$

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State space representation hidden behavior $\mathcal{N}$:

$$\mathcal{N} = \{(d_1, -C\bar{x}, 0, H\bar{x}) \mid \frac{d}{dt}\bar{x} = A\bar{x} + Gd_1\}$$

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Fact: $\mathcal{N}$ is Σ-dissipative if and only if the Riccati inequality

$$-A^T K_{\mathcal{N}} - K_{\mathcal{N}} A - H^T H + K_{\mathcal{N}} G G^T K_{\mathcal{N}} + C^T C \geq 0$$

has a real symmetric solution $K_{\mathcal{N}}$.

Dissipativity of $(\Sigma\mathcal{P})^{\perp}$

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$$(\Sigma\mathcal{P})^\perp = \{(G^T\bar{x}, 0, -B^T\bar{x}, v) \mid \frac{d}{dt}\bar{x} = -A^T\bar{x} + H^T v\}$$

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Fact: $(\Sigma\mathcal{P})^{\perp}$ is $(-\Sigma)$-dissipative if and only if the Riccati inequality

$$AK_{\mathcal{P}} + K_{\mathcal{P}}A^T - GG^T - K_{\mathcal{P}}H^T HK_{\mathcal{P}} + BB^T \geq 0$$

has a real symmetric solution $K_{\mathcal{P}}$.

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This non-negativity is equivalent to the combined conditions

1. $K_{\mathcal{N}} > 0$,
2. $K_{\mathcal{P}} < 0$,
3. $K_{\mathcal{N}} \geq (-K_{\mathcal{P}})^{-1}$.

Solution for the state space case

Theorem: Consider the plant $\mathcal{P}_{\text{full}}$ represented in input/state/output representation. Then the following statements are equivalent:

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Theorem: **Consider the plant $\mathcal{P}_{\text{full}}$ represented in input/state/output representation. Then the following statements are equivalent:**

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Solution for the state space case

Theorem: Consider the plant $\mathcal{P}_{\text{full}}$ represented in input/state/output representation. Then the following statements are equivalent:

1. there exists an implementable $\mathcal{K}$ that satisfies the $\mathcal{H}_\infty$ specifications.
2. there exist real symmetric solutions $K_\mathcal{N} > 0$ and $K_\mathcal{P} < 0$ of the Riccati inequalities

$$-A^T K_\mathcal{N} - K_\mathcal{N} A - H^T H + K_\mathcal{N} G G^T K_\mathcal{N} + C^T C \geq 0,$$

$$A K_\mathcal{P} + K_\mathcal{P} A^T - G G^T - K_\mathcal{P} H^T H K_\mathcal{P} + B B^T \geq 0,$$

such that $K_\mathcal{N} \geq (-K_\mathcal{P})^{-1}$.

Also: formulas for input/state/output representations of the required feedback controllers.

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- These conditions are in terms of the existence of certain storage functions associated with the hidden behavior and manifest plant behavior. In particular, these storage functions should satisfy a coupling condition.

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- These conditions are, representation free, and are hence applicable to any particular 'numerical' representation of the full plant $\mathcal{P}_{\mathrm{full}}$.
- As an illustration we have derived conditions for the 'classical' state space $\mathcal{H}_\infty$ control problem.

End of Lecture 8