Lecture 7

DISSIPATIVE SYSTEMS

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Theme

A dissipative system absorbs supply, 'globally', over time (+ space).

¿¿ Can this be expressed 'locally', as

rate of change in storage (+ spatial flux) $\leq$ supply rate

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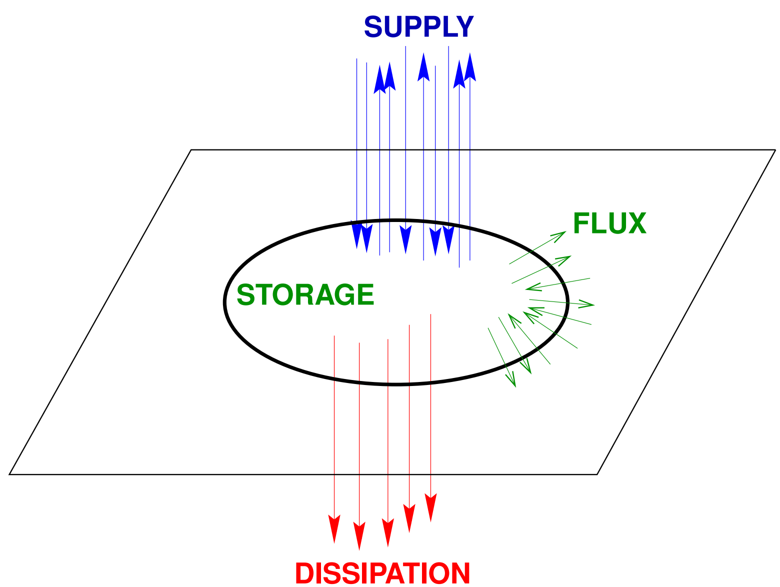

rate of change in storage (+ spatial flux)

= supply rate + (non-negative) dissipation rate ??

The subject in its historical context ...

Lyapunov functions

Consider the classical dynamical system, the ***'flow'***

$$\Sigma : \quad \frac{d}{dt}x = f(x)$$

with $x \in \mathbb{X} = \mathbb{R}^{\mathrm{n}}$**, the** ***state space,*** **and** $f : \mathbb{X} \to \mathbb{X}$**.**

Denote the set of solutions $x : \mathbb{R} \to \mathbb{X}$ **by** $\mathfrak{B}$**, the** ***'behavior'.***

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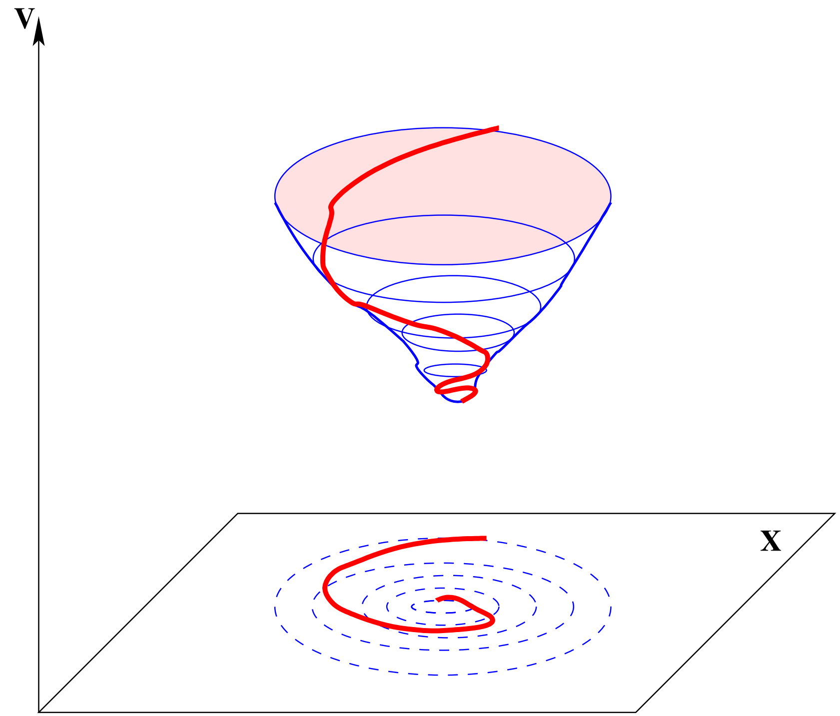
$$\frac{d}{dt} V(x(\cdot)) \leq 0$$

Equivalently, if $\quad \overset{\bullet}{V}^{\Sigma} := \nabla V \cdot f \leq 0.$

Typical Lyapunov 'theorem':



$$V(x) > 0 \text{ and } \overset{\bullet}{V}^{\Sigma}(x) < 0 \text{ for } 0 \neq x \in \mathbb{X}$$

$$\Rightarrow$$

$\forall\ x \in \mathfrak{B},$ there holds $x(t) \to 0$ for $t \to \infty$ **'global stability'**

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Aleksandr Mikhailovich Lyapunov (1857-1918)

Introduced Lyapunov's 'second method' in his Ph.D. thesis (1899).

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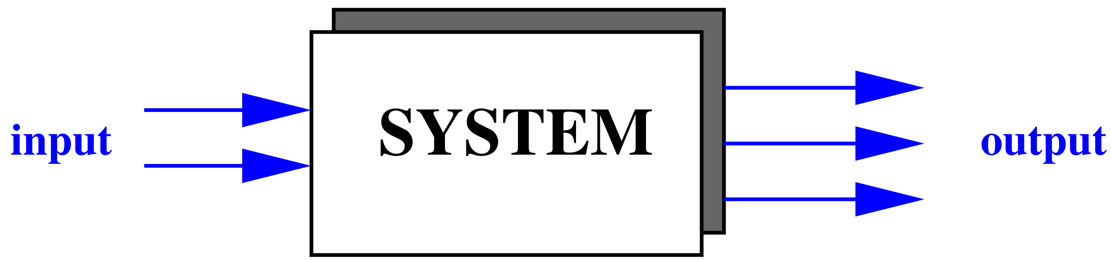

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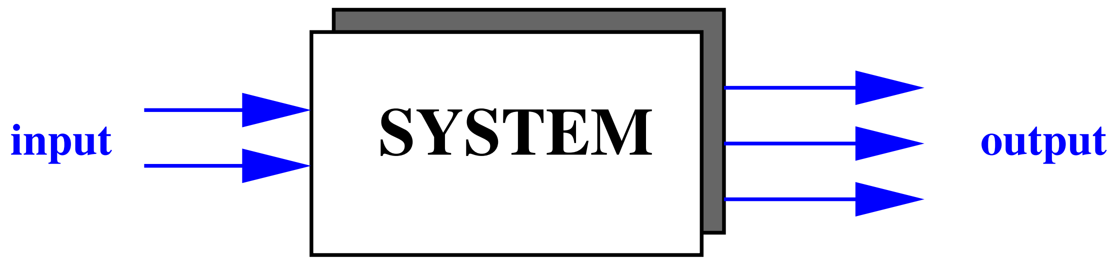

$\rightsquigarrow$ the **‘dynamical system’**

$$\Sigma : \quad \frac{d}{dt}x = f(x,u), \quad y = h(x,u).$$

$u \in \mathbb{U} = \mathbb{R}^{\mathtt{m}}, y \in \mathbb{Y} = \mathbb{R}^{\mathtt{p}}, x \in \mathbb{X} = \mathbb{R}^{\mathtt{n}}$: **input, output, state**.

Behavior $\mathfrak{B} =$ all sol’ns $(u, y, x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$.

Dissipative systems: the classical i/s/o setting

Let $\boxed{s : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}}$ be a function, called the *supply rate*.

Σ is said to be *dissipative* w.r.t. the supply rate s if $\exists$

$$\boxed{V : \mathbb{X} \to \mathbb{R},}$$

called the *storage function*, such that

$$\boxed{\frac{d}{dt} V(x(\cdot)) \leq s(u(\cdot), y(\cdot))}$$

along input/output/state trajectories ($\forall \ (u(\cdot), y(\cdot), x(\cdot)) \in \mathfrak{B}$).

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This inequality is called the *dissipation inequality.*

Equivalent to

$$\overset{\bullet}{V}^{\Sigma}(x,u) := \nabla V(x) \cdot f(x,u) \leq s(u, h(x,u))$$

for all $(u,x) \in \mathbb{U} \times \mathbb{X}$.

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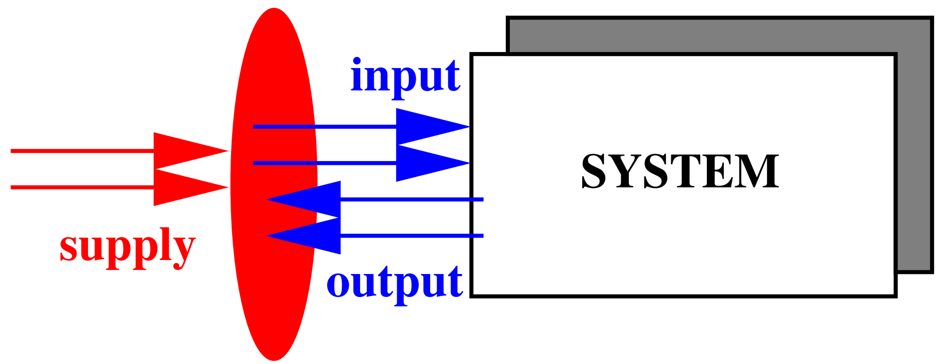
for all $(u, x) \in \mathbb{U} \times \mathbb{X}$.

If equality holds: ‘conservative’ system.

$s(u,y)$ **models something like the power delivered to the system when the input value is u and output value is y.**



$V(x)$ **then models the internally stored energy.**

Dissipativity $:\Leftrightarrow$

rate of increase of internal energy $\leq$ power delivered.

Special case: 'closed' system: $s = 0$ then

dissipativeness $\leftrightarrow$ V is a Lyapunov function.

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Stability for closed systems $\simeq$ Dissipativity for open systems.

The construction of storage functions

Basic question:

Given (a representation of) Σ, the dynamics,
and given s, the supply rate,
is the system dissipative w.r.t. s, i.e.,
does there exist a storage function V such that
the dissipation inequality holds?

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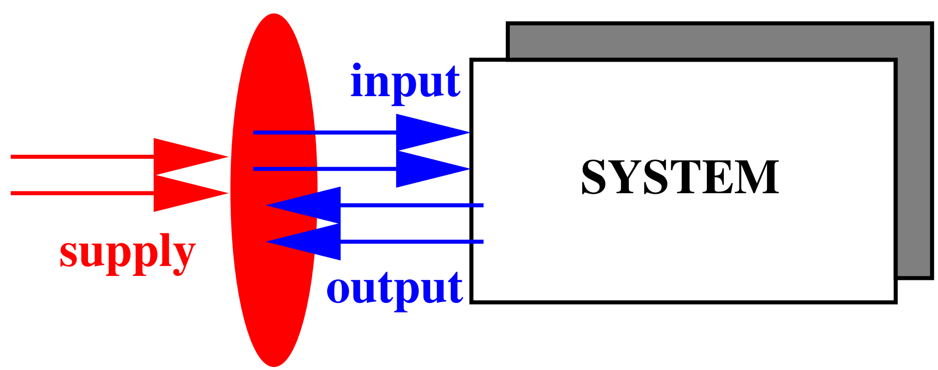

Assume s 'power', known dynamics, what is the internal stored energy?

The construction of storage f'ns is very well understood, particularly for linear i/s/o systems and quadratic supply rates.

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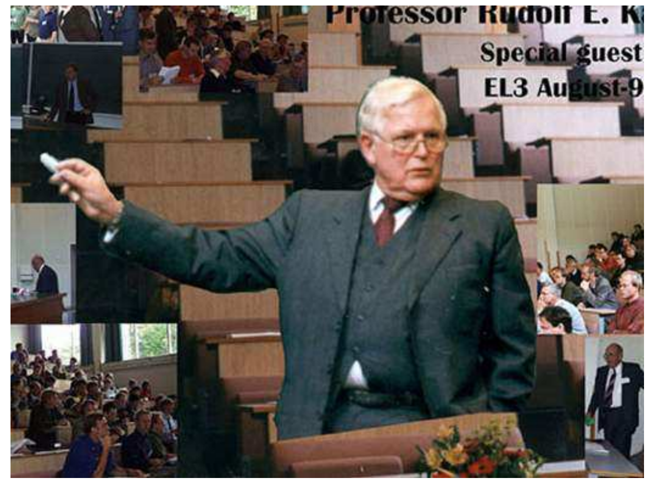
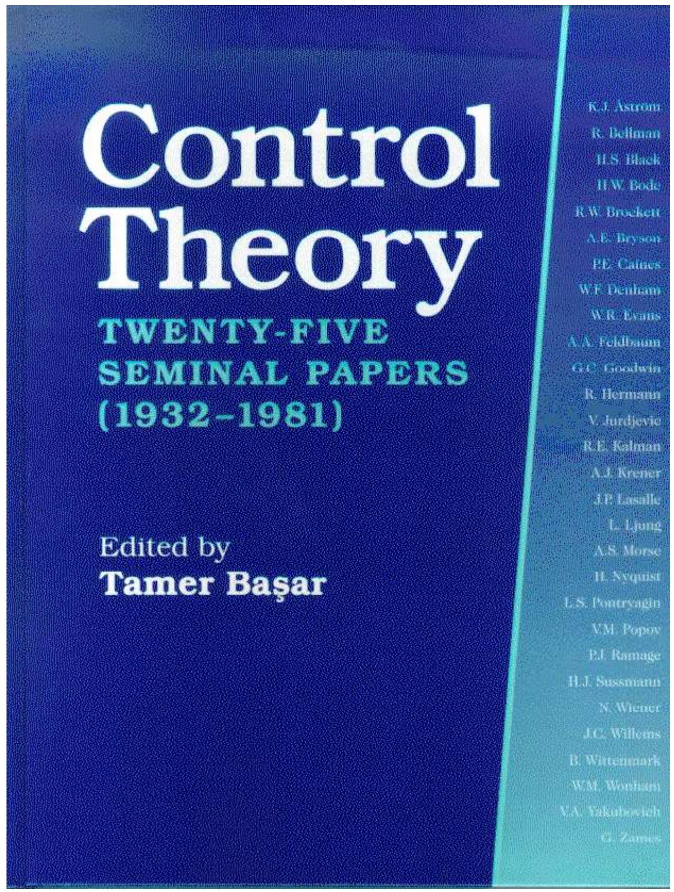


Leads to the KYP-lemma, LMI's, ARIneq, ARE, semi-definite programming, spectral factorization, Lyapunov functions, $\mathcal{H}_\infty$ and robust control, positive and bounded real functions, electrical circuit synthesis, stochastic realization theory.

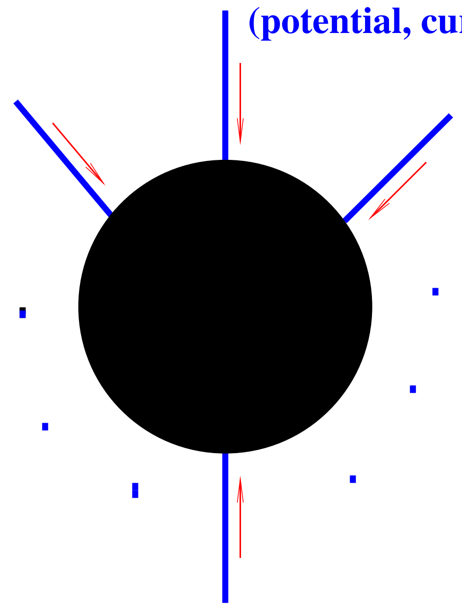
Dissipative systems play a remarkably central role in the field.



The behavioral point of view

Physical examples

Electrical circuit:



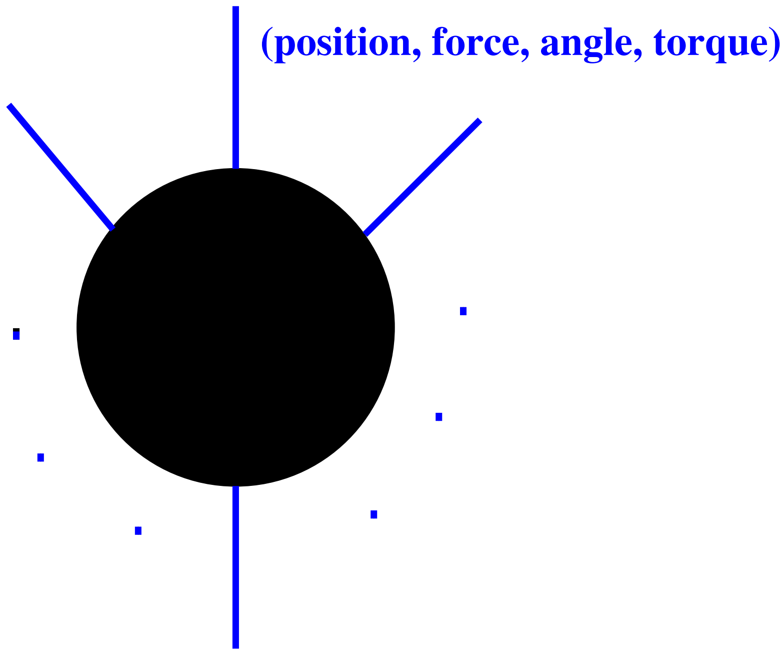
Dissipative w.r.t. $\Sigma_{\ell=1}^{N} V_\ell I_\ell$ **(electrical power).**

Physical examples

System	Supply	Storage
Electrical circuit	$V^\top I$ V : voltage I : current	energy in capacitors & inductors

Physical examples

Mechanical device:



Dissipative w.r.t. $\Sigma_{\ell=1}^{\mathtt{N}}((\frac{d}{dt}q_\ell)^\top F_\ell + (\frac{d}{dt}\theta_\ell)^\top T_\ell)$ **(mech. power).**

Physical examples

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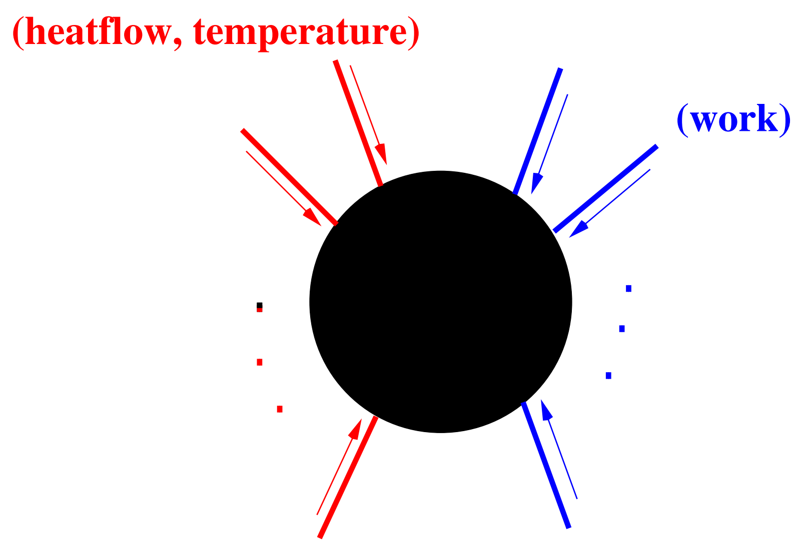
Electrical circuit	$V^\top I$ V : voltage I : current	energy in capacitors & inductors
Mechanical system	$F^\top v + (\frac{d}{dt}\theta)^\top T$ F : force, v : velocity θ: angle, T : torque	potential + kinetic energy

Physical examples

Thermodynamic system:



Conservative w.r.t. $\Sigma_{\ell=1}^{\mathrm{N}} \; Q_\ell + \Sigma_{\ell=1}^{\mathrm{N}'} \; W_\ell$,

Dissipative w.r.t. $-\,\Sigma_{\ell=1}^{\mathrm{N}} \; \frac{Q_\ell}{T\ell}$.

Physical examples

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Mechanical system	$F^\top v + (\frac{d}{dt}\theta)^\top T$ F : force, v : velocity θ: angle, T : torque	potential + kinetic energy
Thermodynamic system	$Q + W$ Q : heat, W : work	internal energy
Thermodynamic system	$-Q/T$ Q : heat, T : temp.	entropy
etc.	etc.	etc.

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$\rightsquigarrow$ **Behavioral systems!**

We will only treat linear time-inv. diff. systems and quadratic differential forms (QDF’s) as supply rates and storage functions.

QDF's

The quadratic map acting on $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ and its derivatives, defined by

$$w \mapsto \sum_{k,\ell} (\frac{d^k}{dt^k} w)^\top \Phi_{k,\ell} (\frac{d^\ell}{dt^\ell} w)$$

is called *quadratic differential form* (QDF) on $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.
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$\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$; WLOG: $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$.

Introduce the 2-variable polynomial matrix Φ

$$\Phi(\zeta, \eta) = \sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell.$$

Denote the QDF as Q_Φ. QDF's are parametrized by $\mathbb{R}^{\bullet \times \bullet}[\zeta, \eta]$.

Dissipative behavioral systems

Definition: $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be **dissipative** w.r.t. the **supply rate** Q_Φ with **storage function** Q_Ψ if the **dissipation inequality**

$$\boxed{\frac{d}{dt} Q_\Psi(\ell) \leq Q_\Phi(w)}$$

for all $(w, \ell) \in \mathfrak{B}_{\mathrm{full}}$, a latent variable representation of $\mathfrak{B}$. If equality holds: **'conservative'**.

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If the storage function acts on w, i.e.,

$$\boxed{\frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w)}$$

for all $w \in \mathfrak{B}$, then we call the storage function **observable**.

Theorem: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, controllable, Q_Φ a QDF, the supply rate.
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$$\int_{-\infty}^{+\infty} Q_\Phi(w)\, dt \geq 0$$

for all $w \in \mathfrak{B}$ of compact support.

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4. $$M^\top(-i\omega)\Phi(-i\omega,\omega)M(i\omega) \geq 0$$

   for all $\omega \in \mathbb{R}$, with $w = M(\frac{d}{dt})\ell$ any image repr. of $\mathfrak{B}$.

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<u>**Theorem**</u>: **Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, controllable, Q_Φ a QDF, the supply rate. The following are equivalent:**

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Theorem: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, controllable, Q_Φ a QDF, the supply rate. The following are equivalent:

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Remarks:

1. The condition: Given $R(\frac{d}{dt})w = 0$ and Φ, $\exists$ Ψ such that

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1. The condition: Given $R(\frac{d}{dt})w = 0$ and Φ, $\exists\ \Psi$ such that

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2. It can be shown that every observable storage function is a **memoryless state function**!

3. The set of observable storage functions is

convex, compact, and attains its maximum and minimum:

$$Q_{\Psi_{\text{available}}}(w) \leq Q_{\Psi}(w) \leq Q_{\Psi_{\text{required}}}(w)$$

for all $w \in \mathfrak{B}$, with

$$Q_{\Psi_{\text{available}}}(w)(0) := \text{supremum}\{-\int_0^{\infty} Q_{\Phi}(\hat{w})\, dt\}$$

$$Q_{\Psi_{\text{required}}}(w)(0) := \text{infimum}\{-\int_{-\infty}^{0} Q_{\Phi}(\hat{w})\, dt\}$$

with the sup and inf over all $\hat{w}$ such that the concatenations,

$$\hat{w} \wedge_0 w,\, w \wedge_0 \hat{w} \in \mathfrak{B}.$$

The need for introducing non-observable storage f'ns is very real:

1. Theoretical example with behavior consisting of the signals (w_1, w_2), with w_1 free and w_2 governed by

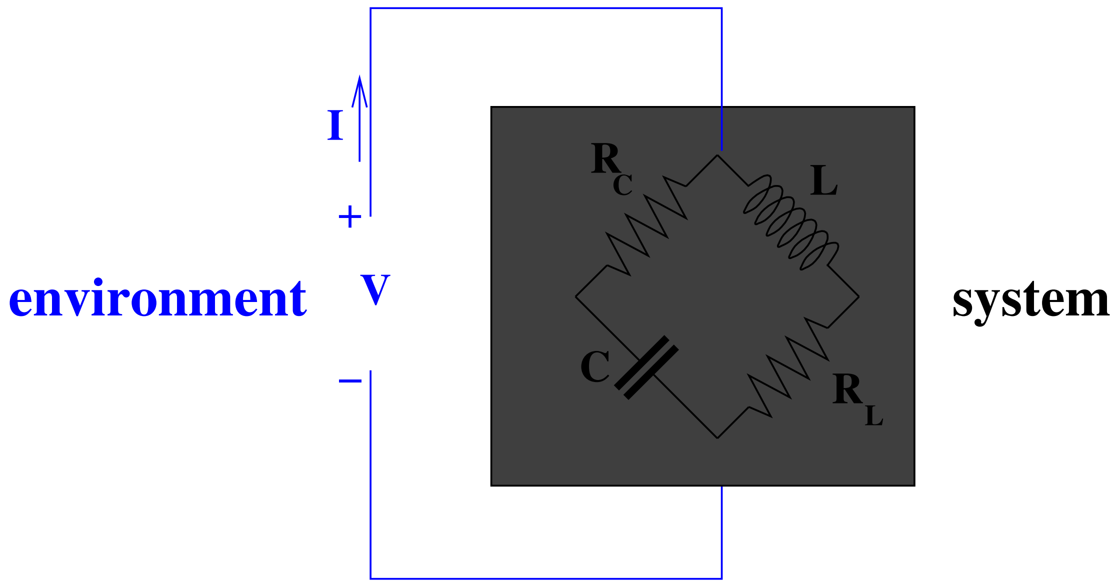
$$\frac{d}{dt} w_2 = \alpha w_2,$$

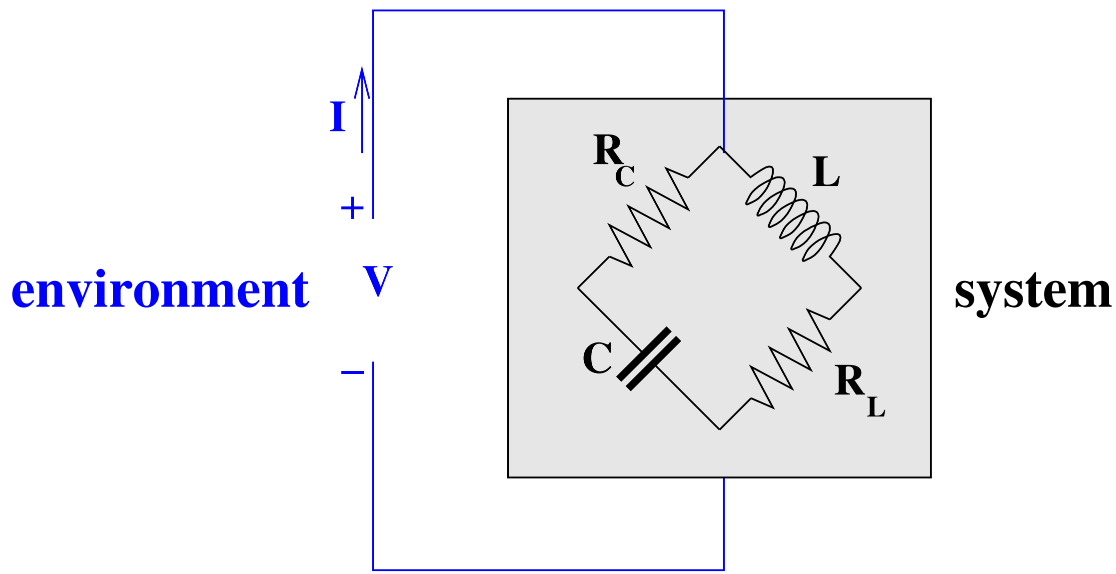
and supply rate $w_1 w_2$. $\nexists$ an observable storage f'n, but the (unobservable) latent variable representation

$$\frac{d}{dt} x = -\alpha x + w_1$$

$\rightsquigarrow$ **the storage f'n $\boxed{x w_2}$. We call this system dissipative!**

2. Our favorite RLC circuit

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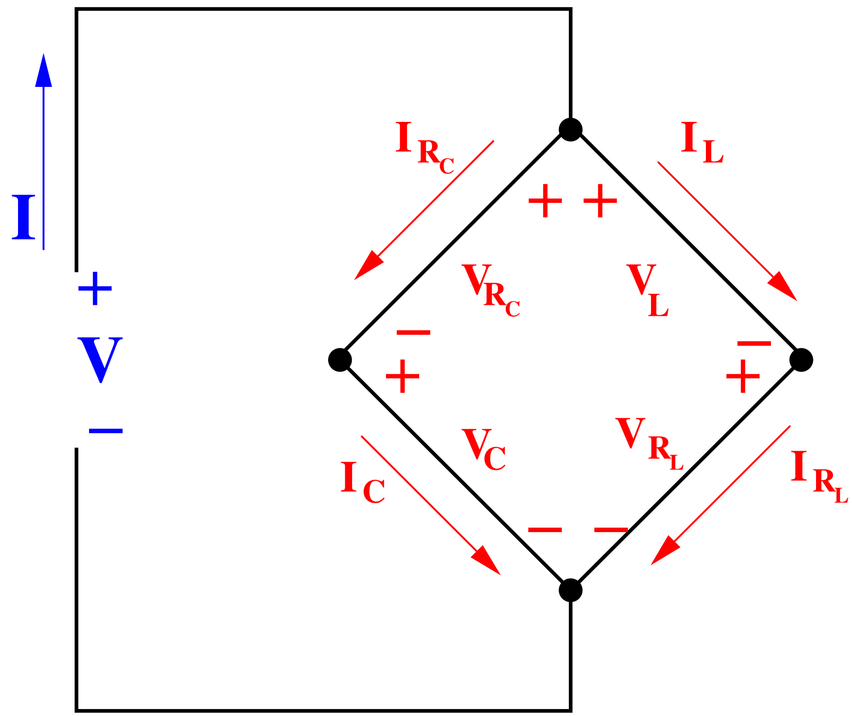
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There are many such examples.

3. For PDE's an observable storage function may not exist at all!

Example: **Maxwell's eq'ns:**

dissipative (in fact, conservative) w.r.t. the QDF $- \vec{E} \cdot \vec{j}$.

Maxwell's equations

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dissipative (in fact, conservative) w.r.t. the QDF $-\vec{E}\cdot\vec{j}$.

In other words, if $\vec{E},\vec{j}$ is of compact support and satisfies

$$\begin{aligned} \varepsilon_0 \frac{\partial}{\partial t}\nabla\cdot\vec{E} + \nabla\cdot\vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2 \nabla\times\nabla\times\vec{E} + \frac{\partial}{\partial t}\vec{j} &= 0, \end{aligned}$$

then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} (-\vec{E}\cdot\vec{j}) \ dxdydz \right) dt = 0.$$

The ***stored energy density,*** S, and

the ***energy flux density*** (the ***Poynting vector***), $\vec{F}$,

$$S(\vec{E},\vec{B}) := \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{\varepsilon_0 c^2}{2}\vec{B}\cdot\vec{B},$$

$$\vec{F}(\vec{E},\vec{B}) := \varepsilon_0 c^2 \vec{E}\times\vec{B}.$$

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$$\vec{F}(\vec{E}, \vec{B}) := \varepsilon_0 c^2 \vec{E}\times\vec{B}.$$

lead to the local conservation law:

$$\boxed{\frac{\partial}{\partial t}S(\vec{E}, \vec{B}) + \nabla\cdot\vec{F}(\vec{E}, \vec{B}) = -\vec{E}\cdot\vec{j}.}$$

The ***stored energy density,*** S, and

the ***energy flux density*** (the ***Poynting vector***), $\vec{F}$,

$$S(\vec{E},\vec{B}) := \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{\varepsilon_0 c^2}{2}\vec{B}\cdot\vec{B},$$

$$\vec{F}(\vec{E},\vec{B}) := \varepsilon_0 c^2 \vec{E}\times\vec{B}.$$

lead to the local conservation law:

$$\boxed{\frac{\partial}{\partial t}S(\vec{E},\vec{B}) + \nabla\cdot\vec{F}(\vec{E},\vec{B}) = -\vec{E}\cdot\vec{j}.}$$

Involves $\vec{B}$, unobservable from the energy variables $\vec{E}$ and $\vec{j}$.
An observable stored energy does not exist al all!

Outline of the proof

Using controllability and image representations, we assume, WLOG:

$$\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$$

Outline of the proof

$$\int_{\mathbb{R}} Q_{\Phi}(w) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$$\Updownarrow \quad \textbf{(Parseval)}$$

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}$$

Outline of the proof

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}$$

$$\Updownarrow$$

(Factorization equation)

$$\exists \; D : \quad \Phi(-\xi, \xi) = D^\top(-\xi) D(\xi)$$

Outline of the proof

$$\exists\ D: \quad \Phi(-\xi,\xi) = D^\top(-\xi)D(\xi)$$

$\Updownarrow$ **(easy)**

$$\exists\ \Psi: \quad (\zeta+\eta)\Psi(\zeta,\eta) = \Phi(\zeta,\eta) - D^\top(\zeta)D(\eta)$$

Outline of the proof

$$\exists\ \Psi : \quad (\zeta+\eta)\Psi(\zeta,\eta) = \Phi(\zeta,\eta) - D^\top(\zeta)D(\eta)$$

$\Updownarrow$ **(clearly)**

$$\exists\ \Psi : \quad \frac{d}{dt}Q_\Psi(w) \leq Q_\Phi(w) \text{ for all } w \in \mathfrak{C}^\infty$$

Assuming factorizability:

Global dissipation $:\Leftrightarrow$

$$\int_{\mathbb{R}^n} Q_\Phi(w) \geq 0 \text{ for all } w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$$

$$\Updownarrow$$

$$\exists\ \Psi : \quad \frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w) \text{ for all } w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$$

$\Leftrightarrow$: **Local dissipation**

The proof thus completely hinges on the factorization eq'n.

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}$$

$$\Updownarrow \qquad \boxed{\text{(Factorization equation)}}$$

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$$\Updownarrow \qquad \boxed{\text{(Factorization equation)}}$$

$$\exists\ D: \quad \Phi(-\xi, \xi) = D^\top(-\xi) D(\xi)$$

This is a classical problem. We sketch the proof also for polynomial matrices in many variables (since it is relevant in the PDE case).

The factorization equation

Consider

$$\boxed{X^\top(-\xi)X(\xi) = Y(\xi)}$$

with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ given, and X the unknown. *Solvable??*

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Under what conditions on Y does there exist a solution X?

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Under what conditions on Y does there exist a solution X?

Scalar case: !! write the real polynomial Y as a sum of squares

$$Y = x_1^2 + x_2^2 + \cdots + x_k^2.$$

$$X^\top(\xi)X(\xi) = Y(\xi)$$

Y is a given polynomial matrix; X is the unknown.

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For $\mathtt{n} = 1$ and $Y \in \mathbb{R}[\xi]$, solvable (for $X \in \mathbb{R}^2[\xi]$) iff

$$Y(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}.$$

$$X^\top(\xi)X(\xi) = Y(\xi)$$

Y **is a given polynomial matrix;** X **is the unknown.**

For $\mathtt{n} = 1$**, and** $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$**, it is well-known (but non-trivial) that this factorization equation is solvable (with** $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ **!) iff**

$$Y(\alpha) = Y^\top(\alpha) \geq 0 \qquad \textbf{for all } \alpha \in \mathbb{R}.$$

$$X^\top(\xi)X(\xi) = Y(\xi)$$

Y is a given polynomial matrix; X is the unknown.

For $\mathtt{n} > 1$, and under this obvious symmetry and positivity requirement,

$$Y(\alpha) = Y^\top(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}^{\mathtt{n}},$$

this equation can nevertheless in general not be solved over the polynomial matrices, for $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$,

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this equation can nevertheless in general not be solved over the polynomial matrices, for $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, but it can be solved over the matrices of rational functions, i.e., for $X \in \mathbb{R}^{\bullet\times\bullet}(\xi)$.

This factorizability is a simple consequence of Hilbert's 17-th pbm!

!! Solve $p = p_1^2 + p_2^2 + \cdots + p_k^2, \ p$ **given**

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!! Solve $p = p_1^2 + p_2^2 + \cdots + p_k^2,\ p$ **given**

A polynomial $p \in \mathbb{R}[\xi_1, \cdots, \xi_n]$, **with** $p(\alpha_1, \ldots, \alpha_n) \geq 0$ **for all** $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ **can in general not be expressed as a sum of squares of polynomials, with the** p_i**'s** $\in \mathbb{R}[\xi_1, \cdots, \xi_n]$.

This factorizability is a simple consequence of Hilbert's 17-th pbm!

!! Solve $p = p_1^2 + p_2^2 + \cdots + p_k^2,$ p **given**

But a rational function (and hence a polynomial) $p \in \mathbb{R}(\xi_1, \cdots, \xi_n)$, **with** $p(\alpha_1, \ldots, \alpha_n) \geq 0,$ **for all** $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n,$ **can be expressed as a sum of squares of** $(k = 2^n)$ **rational functions, with the** p_i**'s** $\in \mathbb{R}(\xi_1, \cdots, \xi_n)$**.**

$\Rightarrow$ **solvability of the factorization eq'n**

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^{\mathrm{n}}$$

$$\Updownarrow \qquad \boxed{\text{(Factorization equation)}}$$

$$\exists\ D : \quad \Phi(-\xi, \xi) = D^{\top}(-\xi)D(\xi)$$

over the rational functions, i.e., with D a matrix with elements in $\mathbb{R}(\xi_1, \cdots, \xi_{\mathrm{n}})$.

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The need to introduce rational functions in this factorization and an image representation of $\mathfrak{B}$ (to reduce the pbm to $\mathfrak{C}^\infty$) are the causes of the unavoidable presence of (possibly unobservable, i.e., 'hidden') latent variables in the local dissipation law for PDE's.

Recapitulation

- **A dissipative system involves the system dynamics, the supply rate, and the storage f'n, related through the dissipation inequality. A storage f'n may involve unobservable latent variables.**

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- A dissipative system involves the **system dynamics,** the **supply rate**, and the **storage f'n**, related through the **dissipation inequality**. A storage f'n may involve **unobservable latent variables**.
- A dissipative system is a natural generalization to **open systems** of a Lyapunov function.
- The theory of dissipative systems centers around
  - conditions for dissipativity in terms of system repr.
  - the construction of the storage function ($\cong$ the factorization eq'n)
- Allowing **unobservable** storage functions is important
- Neither **controllability** nor **observability** are good generic system theoretic assumptions for physical models

End of Lecture 7