Lecture 4:

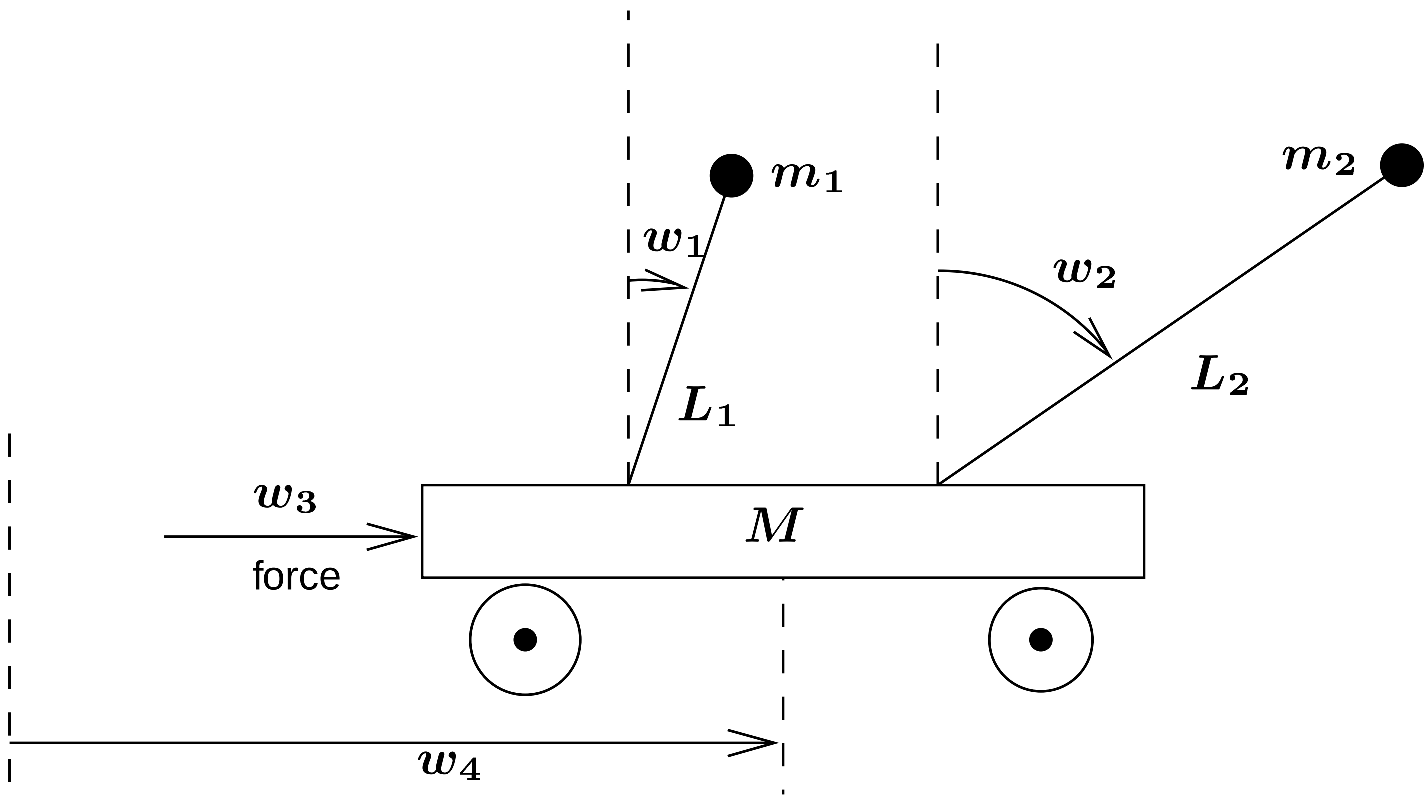
Controllability and observability

Part 1:

Controllability

Example

Two inverted pendula mounted on a chart. Length of the pendula: L_1, L_2, respectively.



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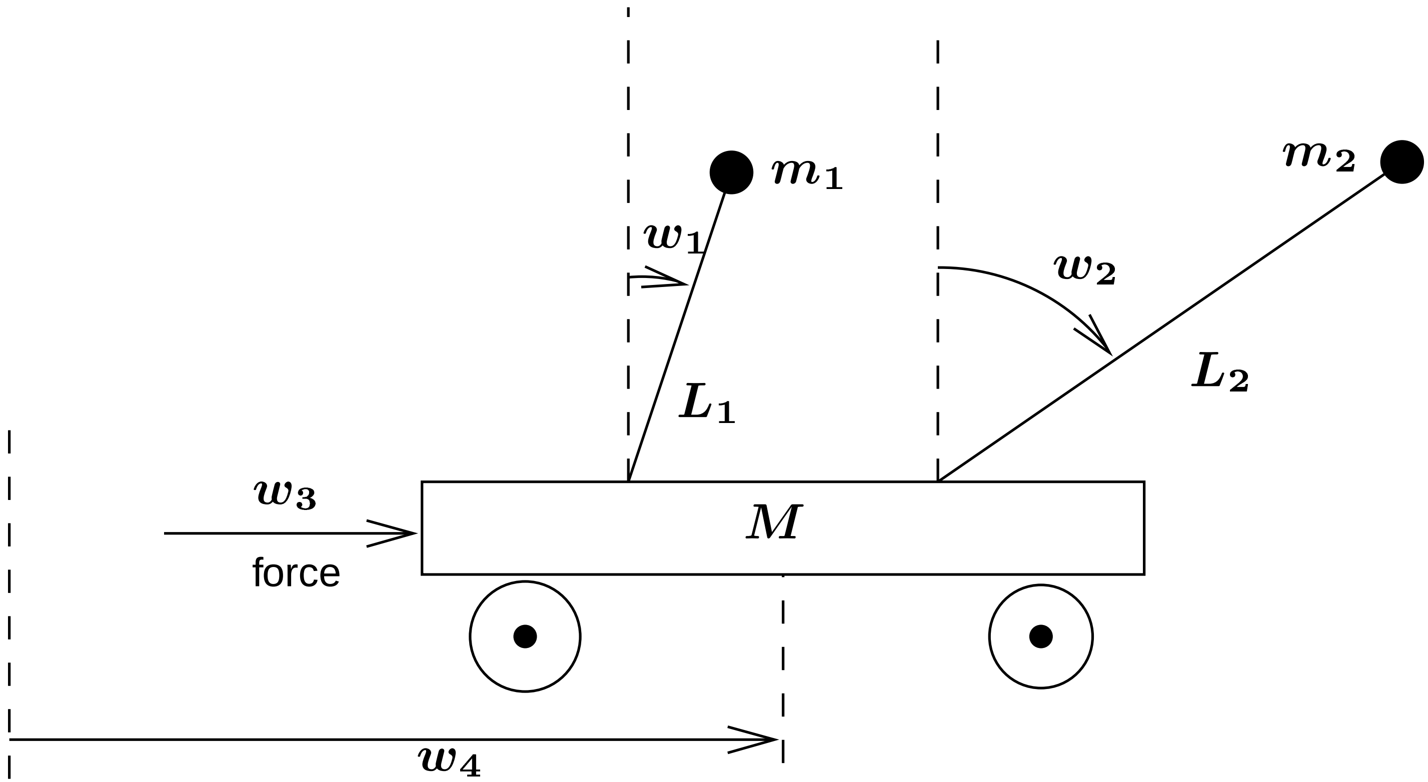

Defines a system with behavior

$$\mathfrak{B} = \{(w = (w_1, w_2, w_3, w_4) \mid w \text{ satisfies Newton's laws}\}$$

By physical reasoning: if $L_1 = L_2$, then $w_1 - w_2$ does not depend on the external force w_3: if $w_1(t) = w_2(t)$ for $t < 0$, then also $w_1(t) = w_2(t)$ for $t \geq 0$, regardless of the external force w_3.

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Hence: there is no $w \in \mathfrak{B}$ with $w_1|_{(-\infty,0)} = w_2|_{(-\infty,0)}$ while at the same time $w_1|_{[0,\infty)} \neq w_2|_{[0,\infty)}$.

No trajectory $w \in \mathfrak{B}$ with $w_1|_{(-\infty,0)} = w_2|_{(-\infty,0)}$ can be 'steered' to a future trajectory with $w_1|_{[0,\infty)} \neq w_2|_{[0,\infty)}$.

Assume now that the lengths of the pendula are unequal:

$$L_1 \neq L_2.$$

It turns out (more difficult to prove) that in that case it is possible to connect any past trajectory with any future trajectory:

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Given $w', w'' \in \mathfrak{B}$, there exists $w \in \mathfrak{B}$ and $T \geq 0$ such that

$$\begin{aligned} w|_{(-\infty,0)} &= w'|_{(-\infty,0)} \\ w|_{[T,\infty)} &= w''|_{[T,\infty)} \end{aligned}$$

Controllability

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ **is called** **controllable** **if for all** $w_1, w_2 \in \mathfrak{B}$ **there exists** $w \in \mathfrak{B}$ **and** $T \geq 0$ **such that**

$$w(t) = \begin{cases} w_1(t) & t < 0 \\ w_2(t) & t \geq T \end{cases}$$

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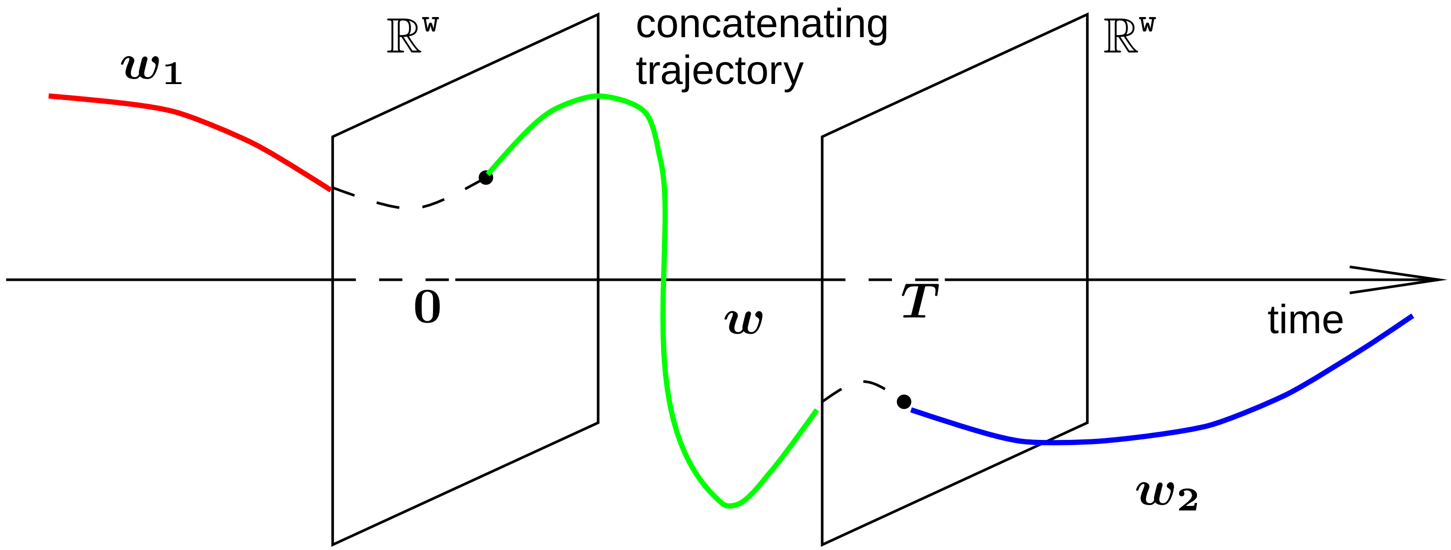

Controllability in terms of kernel representations

Suppose $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is represented in kernel representation by $R(\frac{d}{dt})w = 0$.

How to decide whether $\mathfrak{B}$ is controllable?

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Theorem: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, and let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ be such that $R(\frac{d}{dt})w = 0$ is a kernel representation of $\mathfrak{B}$. Then $\mathfrak{B}$ is controllable if and only if

$$\operatorname{rank}(R(\lambda)) = \operatorname{rank}(R) \text{ for all } \lambda \in \mathbb{C},$$

equivalently, if and only if $\operatorname{rank}(R(\lambda))$ is the same for all $\lambda \in \mathbb{C}$.

Note: $\operatorname{rank}(R)$ is the rank of R as a matrix of polynomials.

Examples

1. $\mathfrak{B} \in \mathfrak{L}^2$ **represented by** $p(\frac{d}{dt})y = q(\frac{d}{dt})u$, $w = (y, u)$ **(single input/single output system). Here,** $p, q \in \mathbb{R}[\xi]$, $p, q \neq 0$**.** $\mathfrak{B}$ **is controllable if and only if**

$$\operatorname{rank}([p(\lambda) \;\; q(\lambda)]) = 1 \text{ for all } \lambda \in \mathbb{C},$$

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equivalently, the polynomials p, q are coprime.

2. $\mathfrak{B} \in \mathfrak{L}^{\mathtt{n}+\mathtt{m}}$ represented by $\frac{d}{dt}x = Ax + Bu$, $w = (x, u)$.
Obviously, this is a kernel representation, with
$R(\xi) = [\xi I - A \;\; B]$.
$\mathfrak{B}$ is controllable if and only if

$$\text{rank}([\lambda I - A \;\; B]) = \mathtt{n} \;\; \text{for all } \lambda \in \mathbb{C}$$

(Hautus test).

Controllability and image representations

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and let $M \in R^{\mathtt{w}\times\mathtt{l}}[\xi]$. If

$$\mathfrak{B} = \{w \mid \text{ there exists } \ell \text{ such that } w = M(\tfrac{d}{dt})\ell\}$$

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Question: Which $\mathfrak{B}$'s in $\mathfrak{L}^{\mathtt{w}}$ have an image representation?

Theorem: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. $\mathfrak{B}$ has an image representation if and only if $\mathfrak{B}$ is controllable.

Note: Relation with the notion of flat system.

Part 2

Observability

Example

Consider a point mass M with position vector $q(t)$, moving under influence of a force vector $F(t)$. This defines a system $\mathfrak{B} \in \mathfrak{L}^6$, represented by

$$M\frac{d^2q}{dt^2} = F, \quad w = (q, F).$$

For a given F, many q's will satisfy the system equation: the actual q will of course depend on $q(0)$ and $\frac{dq}{dt}(0)$.

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For a **given** F, many q's will satisfy the system equation: the actual q will of course depend on $q(0)$ and $\frac{dq}{dt}(0)$.

In other words: F does not determine q uniquely. This is expressed by saying that

in $\mathfrak{B}$, q is not observable from F.

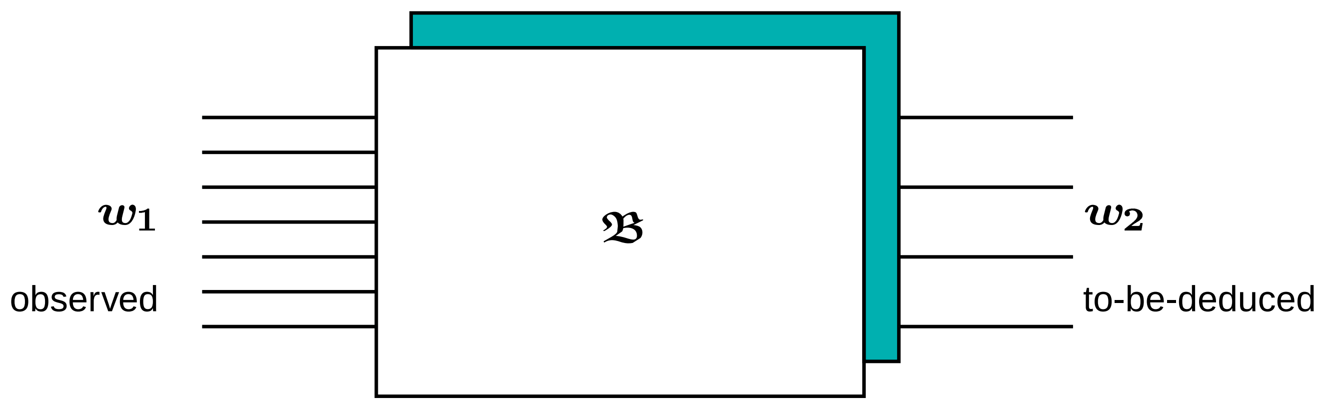
Observability

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, and $w = (w_1, w_2)$ be a partition of the manifest variable w. We will say that

in $\mathfrak{B}$, the component w_2 is observable from the component w_1 if

w_2 is uniquely determined by w_1, i.e., if

$$(w_1, w_2'), (w_1, w_2'') \in \mathfrak{B} \quad \Rightarrow \quad w_2' = w_2''.$$



Example

Let $p \in \mathbb{R}[\xi]$, $p \neq 0$.

1. Let $\mathfrak{B} \in \mathfrak{L}^2$ be represented by $p(\frac{d}{dt})w_1 + w_2 = 0$, $w = (w_1, w_2)$. Clearly, in $\mathfrak{B}$, w_2 is observable from w_1: for given w_1, w_2 is given by $w_2 = -p(\frac{d}{dt})w_1$.

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2. Let $\mathfrak{B} \in \mathfrak{L}^2$ be represented by $p(\frac{d}{dt})w_1 + \frac{d}{dt}w_2 = 0$, $w = (w_1, w_2)$. This time, in $\mathfrak{B}$, w_2 is not observable from w_1: w_1 determines only $\frac{d}{dt}w_2$, so w_2 up to a constant.

Observability in terms of kernel representations

Suppose $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is represented in kernel representation by $R(\frac{d}{dt})w = 0$, with $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$. Partition $w = (w_1, w_2)$. Accordingly, partition $R = [R_1 \;\; R_2]$, so that $\mathfrak{B}$ is represented by $R_1(\frac{d}{dt})w_1 + R_2(\frac{d}{dt})w_2 = 0$.

How do we check whether, in $\mathfrak{B}$, w_2 is observable from w_1?

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Theorem: in $\mathfrak{B}$, w_2 is observable from w_1 if and only if

$$\mathrm{rank}(R_2(\lambda)) = \mathtt{w}_2 \text{ for all } \lambda \in \mathbb{C},$$

i.e., $R_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$.

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In that case, there exists $L \in \mathbb{R}^{\mathtt{w}_2 \times \bullet}[\xi]$ such that $LR_2 = I_{\mathtt{w}_2}$ (i.e. a polynomial left inverse of R_2), and we have

$$(w_1, w_2) \in \mathfrak{B} \quad \Rightarrow \quad w_2 = L(\frac{d}{dt})R_1(\frac{d}{dt})w_1.$$

Example

Consider the system $\mathfrak{B}$, with $w = (u, y, x)$, represented by

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx + Du. \end{aligned}$$

Under what conditions is x observable from (u, y)?

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Hence: x observable from $(u, y) \Leftrightarrow \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ full column rank for all $\lambda \in \mathbb{C}$. **(Hautus test)**

Part 3:

Stabilizability and detectability

Stabilizability

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called **stabilizable** if for all $w \in \mathfrak{B}$ there exists $w' \in \mathfrak{B}$ such that

- $w'(t) = w(t)$ for $t < 0$,
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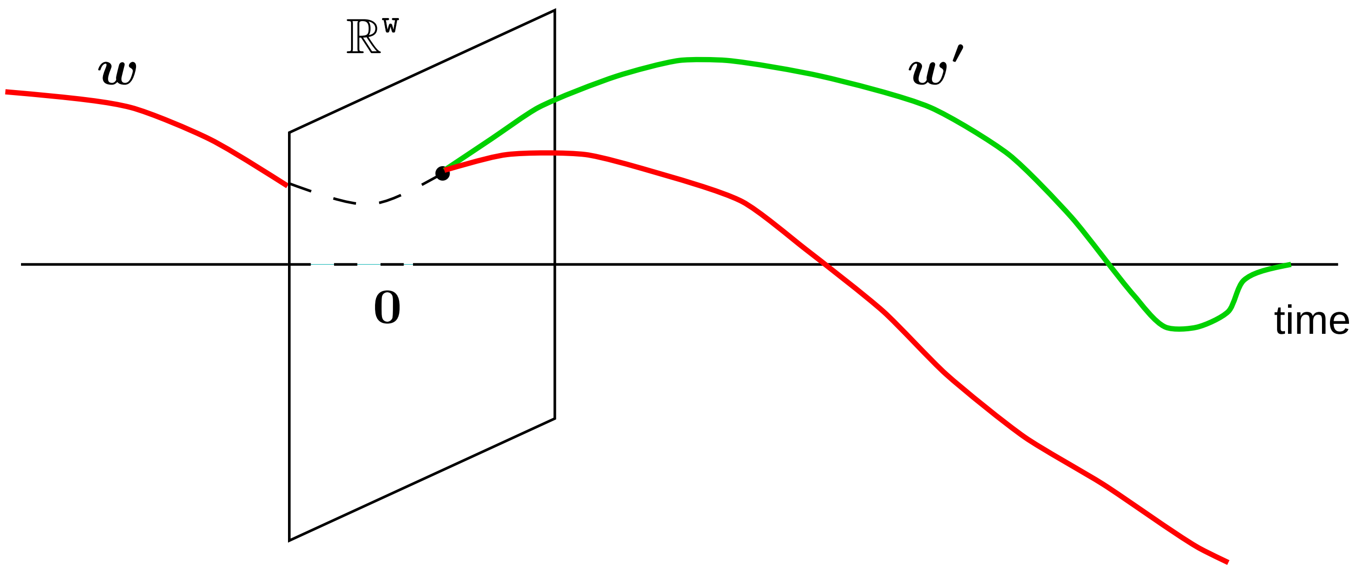

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equivalently, if and only if $\mathrm{rank}(R(\lambda))$ is the same for all $\lambda \in \mathbb{C}^+$ ($\mathbb{C}^+ := \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq 0\}$).

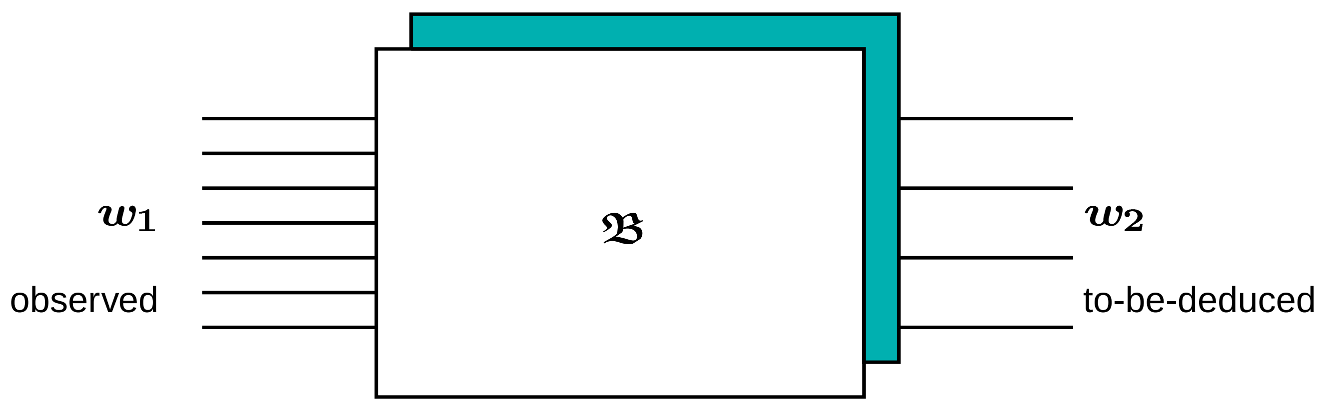
Detectability

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, and $w = (w_1, w_2)$ be a partition of the manifest variable w. We will say that

in $\mathfrak{B}$, the component w_2 is detectable from the component w_1 if

$$(w_1, w_2'), (w_1, w_2'') \in \mathfrak{B} \quad \Rightarrow \quad \lim_{t \to \infty} (w_2'(t) - w_2''(t)) = 0.$$

If w_2 is detectable from w_1, then w_1 determines w_2 **asymptotically.**



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- Given a system $\mathfrak{B}$ and a partition $w = (w_1, w_2)$, w_2 is called observable from w_1 if the condition $(w_1, w_2) \in \mathfrak{B}$ determines w_2 uniquely.

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- A system $\mathfrak{B}$ is controllable if the past and the future of any two trajectories in $\mathfrak{B}$ can be concatenated to obtain a trajectory in $\mathfrak{B}$.
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- Given a system $\mathfrak{B}$ and a partition $w = (w_1, w_2)$, w_2 is called observable from w_1 if the condition $(w_1, w_2) \in \mathfrak{B}$ determines w_2 uniquely.
- Observability is a property of the system and a partition of its variables. Given a kernel representation of the system, observability can be effectively tested.

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- **Given a system $\mathfrak{B}$ and a partition $w = (w_1, w_2)$, w_2 is called detectable from w_1 if the condition $(w_1, w_2) \in \mathfrak{B}$ determines w_2 asymptotically as $t \longrightarrow \infty$.**