Lecture 3

The ELIMINATION Problem

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Problematique

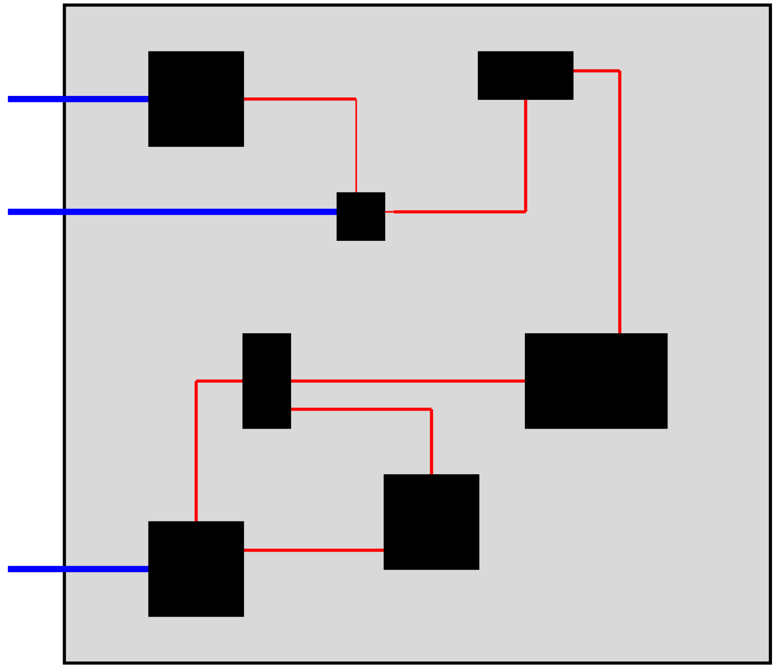
Develop a theory and algorithms for eliminating latent variables

Introduction

First principles model $\rightsquigarrow$ **auxiliary, latent** variables

e.g.

- interconnection variables
- 'theoretical' latent variables:
  momenta, potentials, driving noise, . . .
- state variables
- . . .



Given the behavioral eq'ns for the components,

how do those of the interconnected system look like?

Recall the definitions:

A dynamical system with latent variables **=**

$$\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$$

$\mathbb{T} \subseteq \mathbb{R}$, **the** ***time-axis***
$\mathbb{W}$, **the** ***signal space***
$\mathbb{L}$, **the** ***latent variable space***

$\mathfrak{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$ **:** ***the full behavior***

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$\mathbb{T} \subseteq \mathbb{R}$, **the *time-axis* (= the set of relevant time instances).**
$\mathbb{W}$, **the *signal space* (= the variables that the model aims at).**
$\mathbb{L}$, **the *latent variable space* (= auxiliary modeling variables).**

$$\mathfrak{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}} \text{ : \textit{the full behavior}}$$

(= the pairs $(w, \ell) : \mathbb{T} \to \mathbb{W} \times \mathbb{L}$ that the model declares possible).

The manifest behavior

Call the elements of $\mathbb{W}$ ***'manifest' variables*** ,

those of $\mathbb{L}$ ***'latent' variables*** .

The latent variable system $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ induces the ***manifest system*** $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, with ***manifest behavior***

$$\mathfrak{B} = \{w : \mathbb{T} \to \mathbb{W} \mid \exists\ \ell : \mathbb{T} \to \mathbb{L} \text{ such that } (w, \ell) \in \mathfrak{B}_{\text{full}}\}$$

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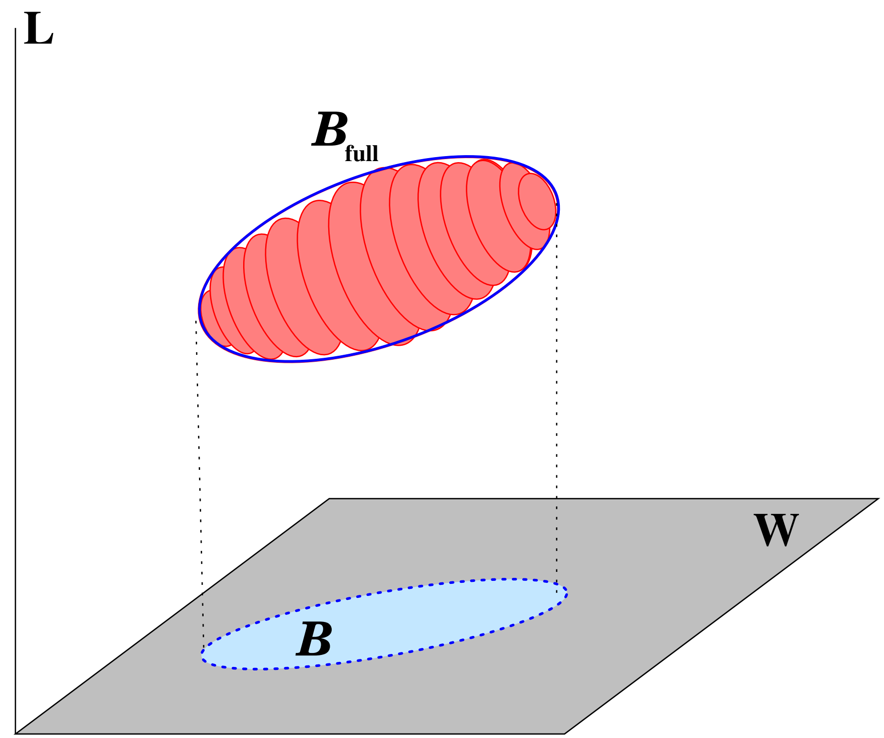
In convenient equations for $\mathfrak{B}$, the latent variables are ***'eliminated'***.
But how do these equations look like, and how are they obtained?



Given mathematical structure for $\mathfrak{B}_{\text{full}}$,

what mathematical structure for $\mathfrak{B}$ emerges?

Examples

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2. **The projection of an algebraic variety is, in general, not an algebraic variety:**

- $$w_1^2 + w_2^2 + \ell^2 = 1 \;\rightsquigarrow\; w_1^2 + w_2^2 \le 1$$

- $$w * \ell = 1 \;\rightsquigarrow\; w \neq 0$$

Examples

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2. The projection of an algebraic variety is, in general, not an algebraic variety:

3. How about the projection of (the sol'n set of) a smooth differential equation?

$$f(w, \frac{d}{dt}w, \ldots, \frac{d^{\mathrm{n}}}{dt^{\mathrm{n}}}w, \ell, \frac{d}{dt}\ell, \ldots, \frac{d^{\mathrm{n}}}{dt^{\mathrm{n}}}\ell) = 0$$

Again a differential eq'n ??

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2. The projection of an algebraic variety is, in general, not an algebraic variety:

3. How about the projection of (the sol'n set of) a smooth differential equation?

4. How about the projection of a constant coefficient linear differential equation?

$$R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$$

¿¿ Again a constant coefficient linear differential eq'n ??

Elimination

First principle models $\leadsto$ latent variables.

For systems described by linear constant coefficient ODE's: $\leadsto$

$$\boxed{R(\frac{d}{dt})w = M(\frac{d}{dt})\ell}$$

with $R, M \in \mathbb{R}^{\bullet\times\bullet}[\xi]$.

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with $R, M \in \mathbb{R}^{\bullet\times\bullet}[\xi]$.

This is the natural model class to start a study of finite dimensional linear time-invariant systems! Much more so than

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du.$$

Is it(s manifest behavior) also a differential system ??

Consider $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$.

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Full behavior:

$$\mathfrak{B}_{\mathrm{full}} = \{(w,\ell) \in \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}+\ell}) \mid \cdots\}.$$

belongs to $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}+\ell}$**, by definition.**

Its manifest behavior equals

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid \ \exists \ \ell \text{ such that } \cdots\}.$$

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Theorem: **It does!**

Proof: **The 'fundamental principle'.**

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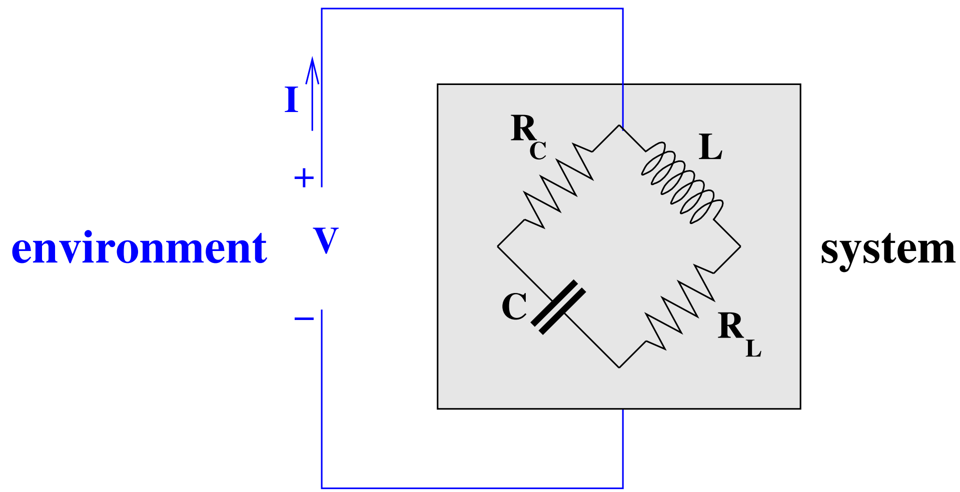
The fundamental principle (for PDE's) states that

$$F(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})x = y$$

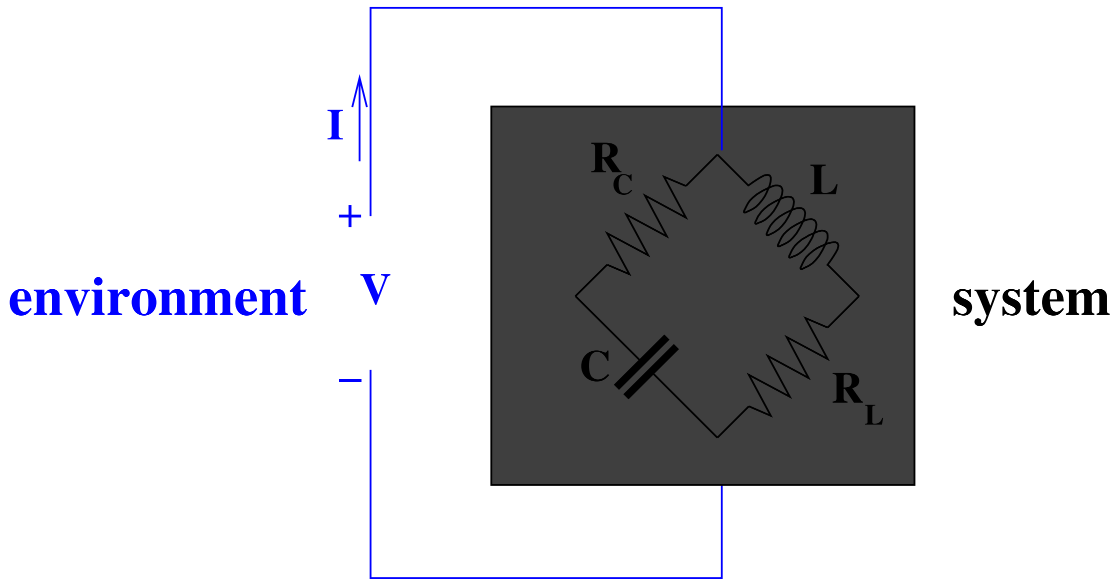
$F \in \mathbb{R}^{\mathrm{n}_1 \times \mathrm{n}_2}[\xi_1, \cdots, \xi_{\mathrm{n}}]$, $y \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathrm{n}_1})$ is solvable for $x \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathrm{n}}, R^{\mathrm{n}_2})$ iff

$$n \in \mathbb{R}^{\mathrm{n}_1}[\xi_1, \cdots, \xi_{\mathrm{n}}] \wedge n^{\top} F = 0 \quad \Rightarrow \quad n^{\top}(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})y = 0.$$

Example: Consider once again our electrical RLC - circuit:



!! Model the relation between manifest V and I !!


I
+
V
−
environment
R
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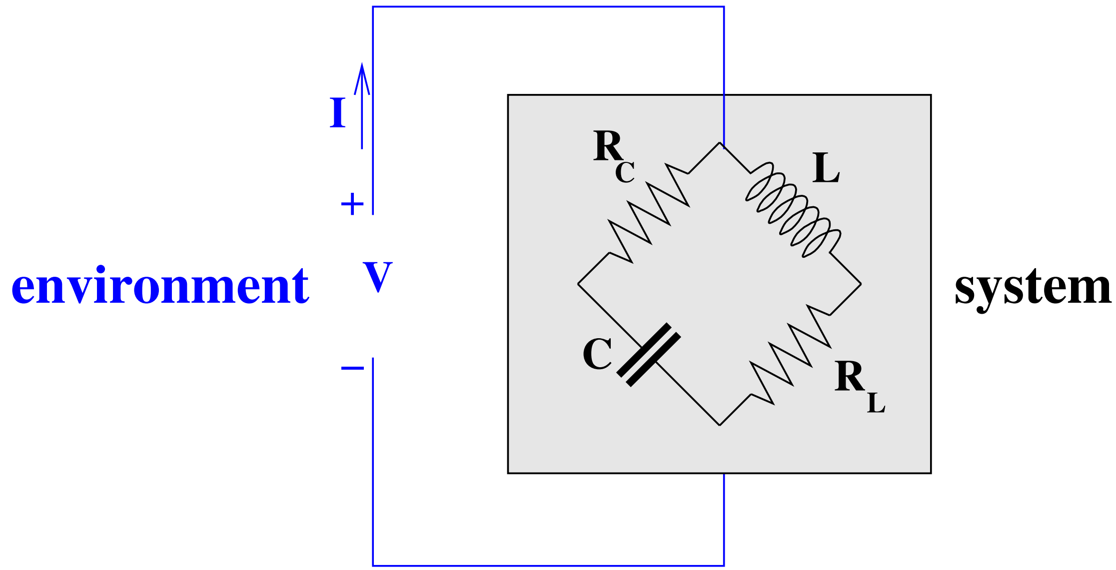

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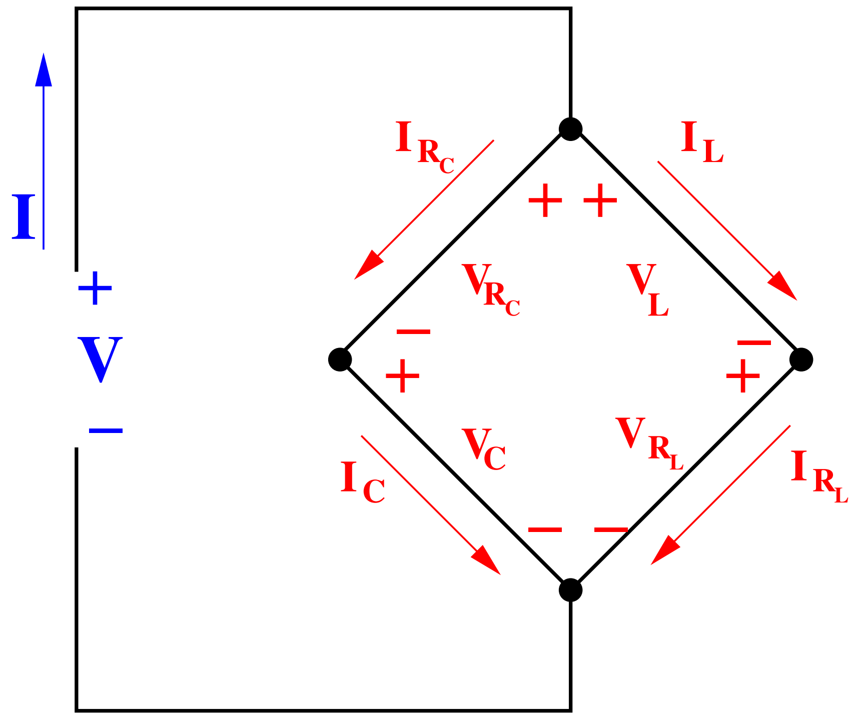

The circuit graph

Introduce the latent variables:

the voltage across and the current in each branch:

$V_{R_C}, I_{R_C}, V_C, I_C, V_{R_L}, I_{R_L}, V_L, I_L.$

System equations

Constitutive equations (CE):

$$V_{R_C} = R_C I_{R_C}, \; V_{R_L} = R_L I_{R_L}, \; C\frac{d}{dt}V_C = I_C, \; L\frac{d}{dt}I_L = V_L$$

Kirchhoff's voltage laws (KVL):

$$V = V_{R_C} + V_C, \; V = V_L + V_{R_L}, \; V_{R_C} + V_C = V_L + V_{R_L}$$

Kirchhoff's current laws (KCL):

$$I = I_{R_C} + I_L, \; I_{R_C} = I_C, \; I_L = I_{R_L}, \; I_C + I_{R_L} = I$$

After elimination, we obtain the following explicit differential equation the between V and I:

<u>**Case 1**</u>**:** $CR_C \neq \frac{L}{R_L}$.

$$\left(\frac{R_C}{R_L} + \left(1 + \frac{R_C}{R_L}\right)CR_C\frac{d}{dt} + CR_C\frac{L}{R_L}\frac{d^2}{dt^2}\right)V$$
$$= \left(1 + CR_C\frac{d}{dt}\right)\left(1 + \frac{L}{R_L}\frac{d}{dt}\right)R_C I.$$

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Case 2: $CR_C = \frac{L}{R_L}$.

$$\boxed{\left(\frac{R_C}{R_L} + CR_C \frac{d}{dt}\right) V = \left(1 + CR_C \frac{d}{dt}\right) R_C I}$$

These are the exact relations between V and I !

First principles modeling ($\cong$ CE's, KVL, & KCL)

$\rightsquigarrow$ **15 behavioral equations.**

Include both the port and the branch voltages and currents.

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Why can the port behavior be described by a system of linear constant coefficient differential equations?

Because:

1. **The CE's, KVL, & KCL are all linear constant coefficient differential equations.**
2. **The elimination theorem†.**

† capacitor $\longrightarrow \frac{1}{Cs}$, inductor $\longrightarrow Ls$, series, parallel, may give erroneous results

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Why is there *exactly one* equation? Passivity!

Remarks:

▶ **Number of equations after elimination (constant coeff. lin. ODE's)**

$\leq$ number of variables.

Elimination $\Rightarrow$ fewer, higher order equations.

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▶ $\exists$ **effective computer algebra/Gröbner bases type algorithms for elimination**

$$(R, M) \mapsto R'$$

Remarks:

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$\leq$ number of variables.

▶ $\exists$ effective computer algebra/Gröbner bases type algorithms for elimination

▶ Depends on sol'n smoothness!
$\nexists$ elimination theorem on $\mathfrak{D}(\mathfrak{C}^{\infty}$ with compact support):

$$\frac{d}{dt}w = f \rightsquigarrow \int_{-\infty}^{+\infty} w(t)\, dt = 0$$

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▶ **Depends on sol'n smoothness!**

▶ **Not generalizable to smooth nonlinear systems.**

Why are differential equations models so prevalent?

Remarks:

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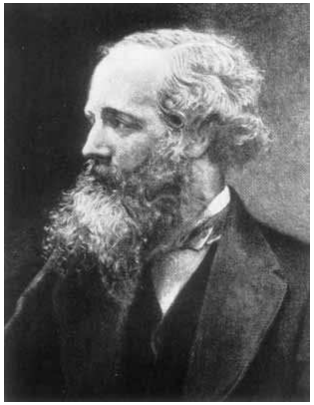
- ▶ **Not generalizable to smooth nonlinear systems.**

- ▶ **Generalizable to linear constant coefficient PDE's.**

Example: Maxwell's equations



$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0} \rho \,, \\ \nabla \times \vec{E} &= -\frac{\partial}{\partial t} \vec{B}, \\ \nabla \cdot \vec{B} &= 0 \,, \\ c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0} \vec{j} + \frac{\partial}{\partial t} \vec{E}. \end{aligned}$$

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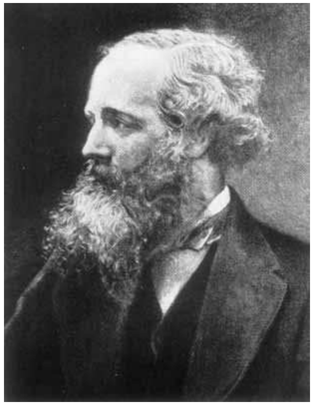

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$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ **(time and space),**

$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

(electric field, magnetic field, current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$,

$\mathfrak{B}$ = **set of solutions to these PDE's.**

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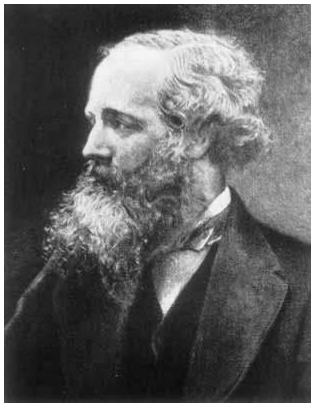

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$\mathfrak{B}$ = set of solutions to these PDE's.

Note: 10 variables, 8 equations! $\Rightarrow \exists$ free variables.

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\end{aligned}
$$

Elimination theorem $\Rightarrow$

this exercise is exact & successful (+ gives algorithm).

It follows from all this that $\mathfrak{L}^\bullet$ has very nice properties. It is closed under:

- **Intersection:** $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}) \Rightarrow (\mathfrak{B}_1 \cap \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}})$.
- **Addition:** $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}) \Rightarrow (\mathfrak{B}_1 + \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}})$.
- **Projection:** $(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}) \Rightarrow (\Pi_{w_1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1})$.
- **Action of a linear differential operator:**
$$(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}, P \in \mathbb{R}^{\mathtt{w}_2\times\mathtt{w}_1}[\xi]) \Rightarrow (P(\tfrac{d}{dt})\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2}).$$
- **Inverse image of a linear differential operator:**
$$(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2}, P \in \mathbb{R}^{\mathtt{w}_2\times\mathtt{w}_1}[\xi]) \Rightarrow ((P(\tfrac{d}{dt}))^{-1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}).$$

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Equally important as elimination is introducing convenient latent variables:

- **state models, first order equations**
- **image representations and controllability**
- **. . .**

End of Lecture 3