Lecture 2

LINEAR DIFFERENTIAL SYSTEMS

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Part 1: Generalities

Linear differential systems

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3. ***differential,*** meaning
   $\mathfrak{B}$ consists of the solutions of a system of differential equations.

Linear constant coefficient differential equations

Variables: $w_1, w_2, \ldots w_{\mathtt{w}}$, **up to** $\mathtt{n}$ **times differentiated,** $\mathtt{g}$ **equations.**

$$\begin{array}{rcl}
\Sigma_{j=1}^{\mathtt{w}} R_{1,j}^{0} w_j + \Sigma_{j=1}^{\mathtt{w}} R_{1,j}^{1} \frac{d}{dt} w_j + \cdots + \Sigma_{j=1}^{\mathtt{w}} R_{1,j}^{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w_j & = & 0 \\
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$j = 1, \ldots, w$ **: for the variable** w_j **involved,**

$k = 1, \ldots, n$ **: for the order** $\frac{d^k}{dt^k}$ **of differentiation.**

In vector/matrix notation:

$$w = \begin{bmatrix} w_1 \\ w_2, \\ \vdots \\ w_{\mathtt{w}} \end{bmatrix}, \quad R_{\mathtt{k}} = \begin{bmatrix} R^{\mathtt{k}}_{1,1} & R^{\mathtt{k}}_{1,2} & \cdots & R^{\mathtt{k}}_{1,\mathtt{w}} \\ R^{\mathtt{k}}_{2,1} & R^{\mathtt{k}}_{2,2} & \cdots & R^{\mathtt{k}}_{2,\mathtt{w}} \\ \vdots & \vdots & \cdots & \vdots \\ R^{\mathtt{k}}_{\mathtt{g},1} & R^{\mathtt{k}}_{\mathtt{g},2} & \cdots & R^{\mathtt{k}}_{\mathtt{g},\mathtt{w}} \end{bmatrix}.$$

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Yields

$$\boxed{R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0,}$$

with $R_0, R_1, \cdots, R_{\mathtt{n}} \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}$.

Combined with the polynomial matrix (in the indeterminate ξ)

$$R(\xi) = R_0 + R_1\xi + \cdots + R_\mathrm{n}\xi^\mathrm{n},$$

we obtain for this the short notation

$$\boxed{R(\tfrac{d}{dt})w = 0.}$$

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Including latent variables $\rightsquigarrow$

$$\boxed{R(\tfrac{d}{dt})w = M(\tfrac{d}{dt})\ell}$$

with $R, M \in \mathbb{R}^{\bullet\times\bullet}[\xi]$.

Polynomial matrices

A **polynomial matrix** is a polynomial with matrix coefficients:

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We may view $P(\xi)$ also as a matrix of polynomials:

$$P(\xi) = \begin{bmatrix} P_{1,1}(\xi) & P_{1,2}(\xi) & \cdots & P_{1,\mathtt{n}_2}(\xi) \\ P_{2,1}(\xi) & P_{2,2}(\xi) & \cdots & P_{2,\mathtt{n}_2}(\xi) \\ \vdots & \vdots & \cdots & \vdots \\ P_{\mathtt{n}_1,1}(\xi) & P_{\mathtt{n}_1,2}(\xi) & \cdots & P_{\mathtt{n}_1,\mathtt{n}_2}(\xi) \end{bmatrix},$$

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Notation: $\mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_2}[\xi]$, $\mathbb{R}^{\bullet \times \mathtt{n}}[\xi]$, $\mathbb{R}^{\mathtt{n} \times \bullet}[\xi]$, $\mathbb{R}^{\bullet \times \bullet}[\xi]$.

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Distributional solutions?

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Transmits main ideas, easier to handle, easy theory, sometimes (too) restrictive.

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R determines $\mathfrak{B}$ uniquely, the converse is not true!

Notation and nomenclature

$\mathfrak{L}^\bullet$: all such systems (with any - finite - number of variables)

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Elements of $\mathfrak{L}^\bullet$: ***linear differential systems***

$R(\frac{d}{dt})w = 0$ 'has' behavior $\mathfrak{B}$

Σ or $\mathfrak{B}$: the system ***induced*** by $R \in \mathbb{R}^{\bullet \times \bullet}[\xi]$

Part 2: Inputs and outputs

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Idea: **there are degrees of freedom in the differential equation** $R(\frac{d}{dt})w = 0$**. In other words: the requirement** $w \in \mathfrak{B}$ **leaves some of the components** $w_1, w_2, \ldots, w_{\mathtt{w}}$ **unconstrained. These components are arbitrary functions.**

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Example

Position $q(t) \in \mathbb{R}^3$ of point mass M subject to a force $F(t) \in \mathbb{R}^3$:

$$\Sigma = (\mathbb{R}, \mathbb{R}^6, \mathfrak{B}),$$

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Also: $(q, F) \in \mathfrak{B}$ does not put constraints on q: q is allowed to be any function. After choosing q, F is determined uniquely.

Free variables

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, $\quad w = (w_1, w_2, \ldots, w_{\mathtt{w}})$.

Let $I = \{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, \mathtt{w}\}$,

The functions $w' = (w_{i_1}, w_{i_2}, \ldots, w_{i_k})$ obtained by selecting from $w = (w_1, w_2, \ldots, w_{\mathtt{w}}) \in \mathfrak{B}$ only the components in the index set I, **form again a linear differential system** (***the elimination theorem, see lecture 3***).

Denote it by $P_I\mathfrak{B}$.

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The set of variables $\{w_{i_1}, w_{i_2}, \ldots, w_{i_k}\}$ is called **free in $\mathfrak{B}$** if

$$P_I\mathfrak{B} = \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{|I|}),$$

where $|I| = k$, the cardinality of the set I.

Maximally free

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, $w = (w_1, w_2, \ldots, w_{\mathtt{w}})$.

Let $I = \{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, \mathtt{w}\}$

The set of variables $\{w_{i_1}, w_{i_2}, \ldots, w_{i_k}\}$ is called **maximally free in $\mathfrak{B}$** if it is free, and if for any $I' \subseteq \{1, 2, \ldots, \mathtt{w}\}$ such that $I \subsetneqq I'$ we have

$$P_{I'}\mathfrak{B} \subsetneqq \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{|I'|}).$$

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So: if $\{w_{i_1}, w_{i_2}, \ldots, w_{i_k}\}$ is maximally free, then any set of variables obtained by adding to this set one or more of the remaining variables is no longer free.

Input/output partition

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, $\quad w = (w_1, w_2, \ldots, w_{\mathtt{w}})$.

Possibly after permutation of its components, a partition of w into $w = (w^{(1)}, w^{(2)})$, with $w^{(1)} = (w_1, w_2, \ldots, w_{\mathtt{m}})$ and $w^{(2)} = (w_{\mathtt{m}+1}, w_{\mathtt{m}+2}, \ldots, w_{\mathtt{w}})$, is called **an input/output partition in $\mathfrak{B}$** if $\{w_1, w_2, \ldots, w_{\mathtt{m}}\}$ is maximally free.

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Non-uniqueness: for given $\mathfrak{B}$, the manifest variable w in general allows more than one input/output partition.

Input/output representations

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be the system with kernel representation

$$P(\frac{d}{dt})w_2 = Q(\frac{d}{dt})w_1, \quad w = (w_1, w_2),$$

where $P \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, and $Q \in \mathbb{R}^{\bullet\times\bullet}[\xi]$.

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The representation $P(\frac{d}{dt})w_2 = Q(\frac{d}{dt})w_1$ is then called an input/output representation of $\mathfrak{B}$.

The rational matrix $P^{-1}Q$ is called **the transfer matrix of $\mathfrak{B}$ w.r.t. the given input/output partition**

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Theorem: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, with manifest variable w. There exists (possibly after permutation of the components) a componentwise partition of w into $w = (u, y)$, and polynomial matrices $P \in \mathbb{R}^{\mathtt{y} \times \mathtt{y}}[\xi], \det(P) \neq 0, Q \in \mathbb{R}^{\mathtt{y} \times \mathtt{u}}[\xi]$, such that

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There even exists such a partition such that $P^{-1}Q$ is proper.

Input and output cardinality

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ **has many input/output partitions** $w = (u, y)$**. However: the number of input components in any input/output partition of** $\mathfrak{B}$ **is fixed. This number is denoted by** $\mathtt{m}(\mathfrak{B})$ **and is called the input cardinality of** $\mathfrak{B}$**:**

$$\mathtt{m}(\mathfrak{B}) := \max\{k \in \mathbb{N} | \{w_{i_1}, w_{i_2}, \ldots, w_{i_k}\} \text{ is free in } \mathfrak{B}\}.$$

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For $\mathfrak{B} = \ker(R(\frac{d}{dt}))$: $\mathtt{p}(\mathfrak{B}) = \operatorname{rank}(R)$.

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- For a given system $\mathfrak{B}$, $w = (u, y)$ is an input/output partition if the set of components of u is maximally free: these components can be chosen arbitrarily. The components of y are then determined up to initial conditions.
- An input/output representation of $\mathfrak{B}$ is a special kind of kernel representation: $P(\frac{d}{dt})y = Q(\frac{d}{dt})u$, $w = (u, y)$, with $\det(P) \neq 0$. Equivalent with: (u, y) is an input/output partition.

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- **Given $\mathfrak{B}$, the number of components of u is the same in any i/o partition $w = (u, y)$. This number is called the input cardinality $\mathtt{m}(\mathfrak{B})$ of $\mathfrak{B}$.**

End of Lecture 2