Lecture 7

SYNTHESIS OF DISSIPATIVE SYSTEMS

OUTLINE

Part I: $\mathcal{H}_\infty$ control and the synthesis of dissipative systems

1. **The $\mathcal{H}_\infty$ control problem in a behavioral context**
2. **Generalization: synthesis of dissipative systems**

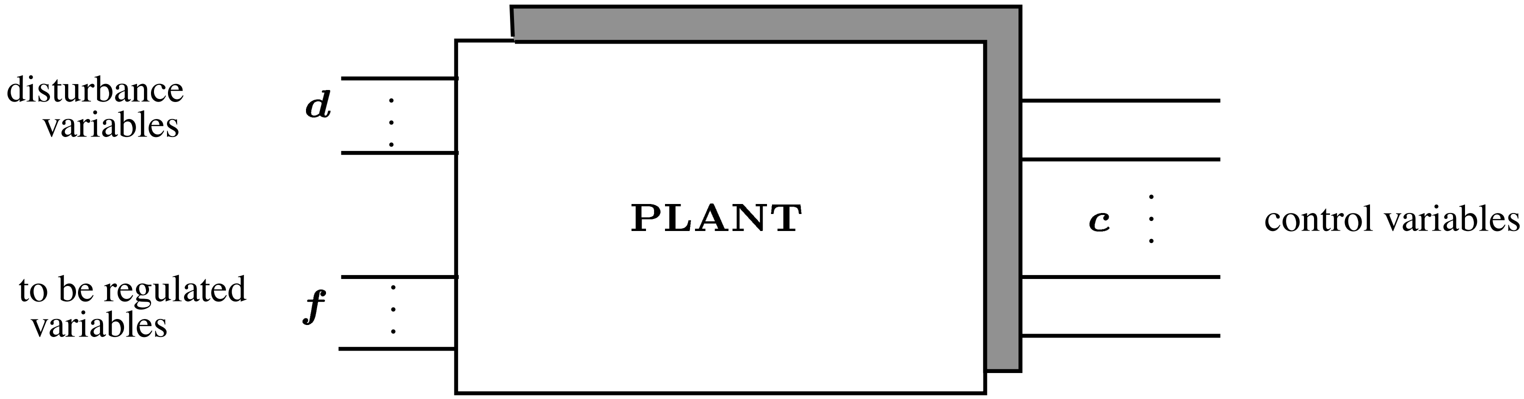
Part II: Special cases

1. **The state space $\mathcal{H}_\infty$ control problem**
2. **The $\mathcal{H}_\infty$ filtering problem**

PART I

$\mathcal{H}_\infty$ CONTROL AND THE SYNTHESIS OF DISSIPATIVE SYSTEMS

$\mathcal{H}_\infty$ CONTROL IN A BEHAVIORAL CONTEXT



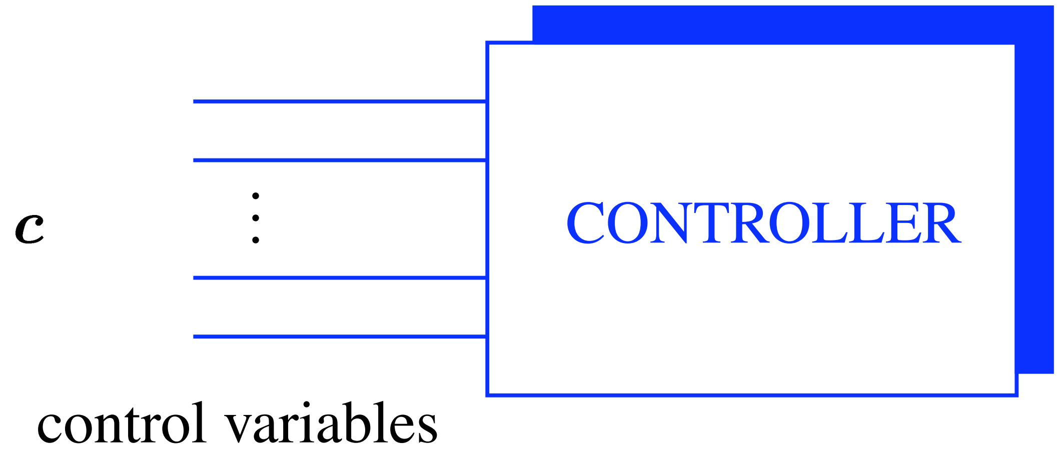
Our plant has three kinds of variables:

- **to be regulated variables f,**
- **exogenous disturbance variables d,**
- **control variables c.**

Full plant behavior:

$$\mathcal{P}_{\text{full}} := \{(d, f, c) \mid (d, f, c) \text{ satisfies the plant equations}\}$$

We assume that $\mathcal{P}_{\text{full}}$ is a linear differential system, i.e. an element of $\mathfrak{L}^{\mathtt{d}+\mathtt{f}+\mathtt{c}}$.



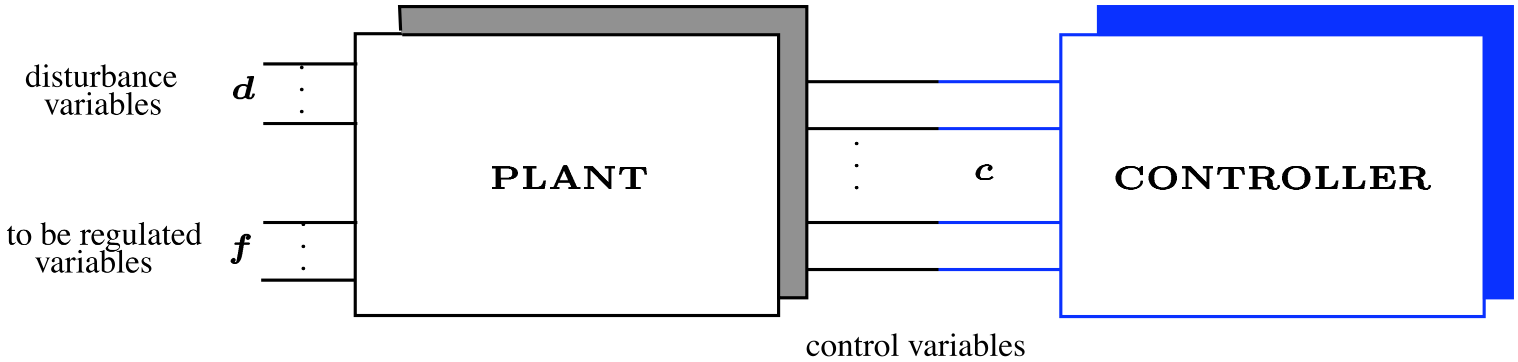
The control variables c are those variables in the full plant that we are allowed to put constraints on. In particular, we allow constraints of the form

$$C(\frac{d}{dt})c = 0,$$

with $C \in \mathbb{R}^{\bullet \times \mathtt{c}}[\xi]$.

In other words: a controller is a linear differential system $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$, with manifest variable c:

$$\mathcal{C} = \{c \mid c \text{ satisfies the controller equations}\}.$$



Given a plant behavior $\mathcal{P}_{\text{full}}$, and a controller behavior $\mathcal{C}$, the manifest controlled behavior $\mathcal{K}$ is given by

$$\mathcal{K} := \{(d, f) \mid \text{ there exists } c \in \mathcal{C} \text{ such that } (d, f, c) \in \mathcal{P}_{\text{full}}\}.$$

Recall: we say: $\mathcal{C}$ implements $\mathcal{K}$ or: $\mathcal{K}$ is implemented by the controller $\mathcal{C}$.

Also: $\mathcal{K} \in \mathfrak{L}^{\mathrm{d}+\mathrm{f}}$ is called implementable (with respect to $\mathcal{P}_{\text{full}}$) if there exists a controller $\mathcal{C} \in \mathfrak{L}^{\mathrm{c}}$ that implements $\mathcal{K}$.

CONTROL PROBLEMS

Recall: given a full plant $\mathcal{P}_{\text{full}}$, and a set of design specifications, the corresponding control problem is to find conditions for the existence of, and compute, a controller $\mathcal{C}$ such that the resulting manifest controlled behavior $\mathcal{K}$ satisfies the specifications.

Reformulation: given $\mathcal{P}_{\text{full}}$ and a set of design specifications, find conditions for the existence of, and compute, a behavior $\mathcal{K}$ such that

- $\mathcal{K}$ is implementable,
- $\mathcal{K}$ satisfies the specifications.

Of course, after finding an implementable behavior $\mathcal{K}$ that satisfies the specifications, one still needs to **compute an actual controller** $\mathcal{C} \in \mathfrak{L}^{\mathrm{c}}$ that implements $\mathcal{K}$.

$\mathcal{H}_\infty$ SPECIFICATIONS

In an $\mathcal{H}_\infty$-context, we will deal with controllable systems, and the specifications on $\mathcal{K} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}_{\text{cont}}$ are:

- **disturbance attenuation:**

$$\int_{-\infty}^{\infty} |f|^2 - |d|^2 dt \leq 0 \text{ for all } (d, f) \in \mathcal{K} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{f}}),$$

- **stability:**

$$(0, f) \in \mathcal{K} \Rightarrow \lim_{t \to \infty} f(t) = 0,$$

- **'liveness':**

in $\mathcal{K}$, d is free.

Liveness: in the manifest controlled behavior, no direct restrictions on the exogenous disturbances are allowed: every component of d is arbitrary.

$\mathcal{H}_\infty$ SPECIFICATIONS IN TERMS OF KERNEL REPRESENTATIONS

Suppose $\mathcal{K} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}_{\mathrm{cont}}$ is represented by a minimal kernel representation

$$P(\frac{d}{dt})f = Q(\frac{d}{dt})d,$$

with $P \in \mathbb{R}^{\bullet \times \mathtt{f}}[\xi]$ and $Q \in \mathbb{R}^{\bullet \times \mathtt{d}}[\xi]$.

Question: what do the $\mathcal{H}_\infty$ specifications on $\mathcal{K}$ say about this representation?

Proposition 7.1: The following statements are equivalent:

1. $\int_{-\infty}^{\infty} |f|^2 - |d|^2 dt \leq 0$ for all $(d, f) \in \mathcal{K} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{f}})$, $(0, f) \in \mathcal{K} \Rightarrow \lim_{t\to\infty} f(t) = 0$, and in $\mathcal{K}$, d is free,
2. $P(\frac{d}{dt})f = Q(\frac{d}{dt})d$ is an input/output representation, P is Hurwitz, and the transfer matrix $G_{d\mapsto f} := P^{-1}Q$ from d to f satisfies $\|G_{d\mapsto f}\|_{\mathcal{H}_\infty} \leq 1$.

$\mathcal{H}_\infty$ SPECIFICATIONS AND DISSIPATIVITY

The $\mathcal{H}_\infty$ specifications on $\mathcal{K}$ can be restated in terms of dissipativity of $\mathcal{K}$ with respect to the supply rate $Q_\Sigma(d, f) = |d|^2 - |f|^2$ associated with the (constant) two-variable polynomial matrix

$$\Sigma = \begin{bmatrix} I_{\mathtt{d}} & 0 \\ 0 & -I_{\mathtt{f}} \end{bmatrix}$$ **, and the input cardinality of $\mathcal{K}$:**

Proposition 7.2: Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}_{\mathrm{cont}}$. The following statements are equivalent:

1. **$\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$, and $\mathtt{m}(\mathcal{K}) = \mathtt{d}$,**
2. **$\int_{-\infty}^{\infty} |f|^2 - |d|^2 dt \leq 0$ for all $(d, f) \in \mathcal{K} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{f}})$, $(0, f) \in \mathcal{K} \Rightarrow \lim_{t\to\infty} f(t) = 0$, and in $\mathcal{K}$, d is free.**

Recall: $\mathfrak{B}$ is called Σ-dissipative on $\mathbb{R}_-$ if $\int_{-\infty}^{0} Q_\Sigma(w)dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

THE $\mathcal{H}_\infty$ CONTROL PROBLEM

Given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}+\mathtt{c}}$, the $\mathcal{H}_\infty$ control problem is to find conditions for the existence of, and compute, $\mathcal{K} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}_{\text{cont}}$ that satisfies the following three properties:

- $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$ **(implementability),**
- $\mathcal{K}$ **is Σ-dissipative on** $\mathbb{R}_-$ **(dissipativity),**
- $\mathtt{m}(\mathcal{K}) = \mathtt{d}$ **(liveness).**

Recall: $\mathcal{N}$ is the hidden behavior, and $\mathcal{P}$ the manifest plant behavior associated with $\mathcal{P}_{\text{full}}$:

$$\mathcal{N} = \{(d, f) \mid (d, f, 0) \in \mathcal{P}_{\text{full}}\},$$

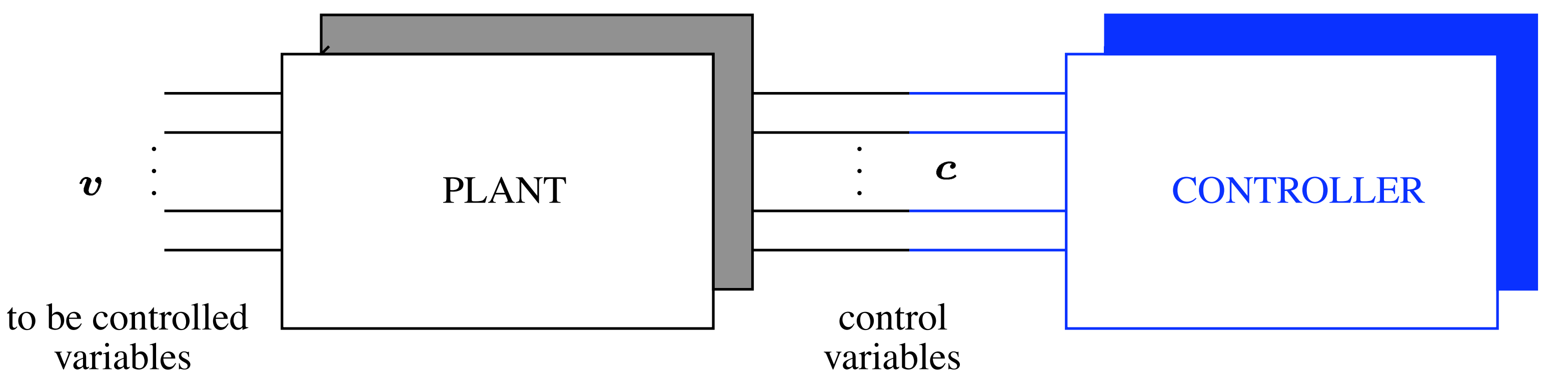
$$\mathcal{P} = \{(d, f) \mid \text{ there exists } c \text{ such that } (d, f, c) \in \mathcal{P}_{\text{full}}\}.$$

SYNTHESIS OF DISSIPATIVE SYSTEMS

More general problem:

- in $\mathcal{P}_{\text{full}}$ to be controlled variable (d, f) $\longrightarrow$ in $\mathcal{P}_{\text{full}}$ to be controlled variable v,
- $\Sigma = \begin{bmatrix} I_{\mathtt{d}} & 0 \\ 0 & -I_{\mathtt{f}} \end{bmatrix}$ defining $|d|^2 - |f|^2$ $\longrightarrow$ general

  $\Sigma = \Sigma^* \in \mathbb{R}^{\mathtt{v} \times \mathtt{v}}[\zeta, \eta]$ defining the QDF $Q_\Sigma(v)$
- $\mathtt{m}(\mathcal{K}) = \mathtt{d}$ $\longrightarrow$ $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$



GENERAL PROBLEM FORMULATION

General problem:

Let $\mathcal{N}, \mathcal{P} \in \mathfrak{L}^{\mathtt{v}}_{\text{cont}}$ and $\Sigma = \Sigma^* \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}[\zeta, \eta]$, with $\mathcal{N} \subset \mathcal{P}$, and Σ non-degenerate; Σ is called the weighting functional

The problem is to find $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}_{\text{cont}}$ such that:

1. $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$ **(implementability),**
2. $\mathcal{K}$ **is Σ-dissipative on $\mathbb{R}_-$ (dissipativity),**
3. $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$ **(liveness).**

It can be shown: $\mathcal{K}$ is Σ-dissipative $\Rightarrow \mathtt{m}(\mathcal{K}) \leq \sigma_+(\Sigma)$.

Hence the input cardinality of such $\mathcal{K}$ attains the upper bound $\sigma_+(\Sigma)$.

Questions:

- **What are necessary and sufficient conditions, in terms of $\mathcal{N}$, $\mathcal{P}$, and Σ, for the existence of such $\mathcal{K}$?**
- **If such $\mathcal{K}$ exists, how can it be computed?.**
- **Given such $\mathcal{K}$, how can we compute a controller $\mathcal{C}$ that implements this $\mathcal{K}$?**

deriving necessary conditions

Assume $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, $\mathcal{K}$ Σ-dissipative on $\mathbb{R}_-$, and $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$.

Then $\mathcal{K}$ is Σ-dissipative, so: $\boxed{\mathcal{N} \text{ is } \Sigma\text{-dissipative.}}$

Since $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$, we have: $\mathcal{K}$ is Σ-dissipative $\Leftrightarrow$ $\mathcal{K}^{\perp_\Sigma}$ is $(-\Sigma)$-dissipative.

Since $\mathcal{K} \subset \mathcal{P}$, we have $\mathcal{P}^{\perp_\Sigma} \subset \mathcal{K}^{\perp_\Sigma}$, whence:
$\boxed{\mathcal{P}^{\perp_\Sigma} \text{ is } (-\Sigma)\text{-dissipative.}}$

Recall: For a given $\mathfrak{B}$, $\mathfrak{B}^{\perp_\Phi}$ is the Φ-orthogonal behavior of $\mathfrak{B}$, defined by

$$\mathfrak{B}^{\perp_\Phi} = \{ w \mid \int_{-\infty}^{+\infty} L_\Phi(w, w')\, dt = 0 \text{ for all } w' \in \mathfrak{B} \cap \mathfrak{D} \}.$$

So far, we have derived two necessary conditions. We can obtain a set of necessary and sufficient conditions by adding a third condition.

This conditions deals with the existence of certain storage functions for $\mathcal{N}$ and $\mathcal{P}^{\perp_\Sigma}$.

Since $\mathcal{N} \subset \mathcal{P}$ we have $\mathcal{P}^{\perp_\Sigma} \subset \mathcal{N}^{\perp_\Sigma}$, so $\mathcal{N} \perp_\Sigma \mathcal{P}^{\perp_\Sigma}$.

Hence, there exists a two-variable polynomial matrix $\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})} \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}[\zeta,\eta]$ such that

$$\frac{d}{dt}L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}}(v_1,v_2) = L_\Sigma(v_1,v_2), \text{ for } v_1 \in \mathcal{N}, v_2 \in \mathcal{P}^{\perp_\Sigma}.$$

The BLDF $L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}}(v_1,v_2)$ is the so called $(N, \mathcal{P}^{\perp_\Sigma}, \Sigma)$ adapted bilinear differential form. It is unique on $\mathcal{N} \times \mathcal{P}^{\perp_\Sigma}$.

FORMULATION OF THE MAIN RESULT

Theorem 7.3: $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}_{\text{cont}}$ **described in the problem formulation exists if and only if the following conditions are satisfied:**

1. $\mathcal{N}$ **is** Σ**-dissipative,**
2. $\mathcal{P}^{\perp_\Sigma}$ **is** $(-\Sigma)$**-dissipative,**
3. **there exist** $\Psi_{\mathcal{N}}, \Psi_{\mathcal{P}^{\perp_\Sigma}} \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}[\zeta,\eta]$**, defining**
   - **a storage function** $Q_{\Psi_{\mathcal{N}}}$ **for** $\mathcal{N}$ **as a** Σ**-dissipative system, i.e.,** $\frac{d}{dt}Q_{\Psi_{\mathcal{N}}}(v_1) \leq Q_\Sigma(v_1)$ **for** $v_1 \in \mathcal{N}$**,**
   - **a storage function** $Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}$ **for** $\mathcal{P}^{\perp_\Sigma}$ **as a** $(-\Sigma)$**-dissipative system, i.e.,** $\frac{d}{dt}Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v_2) \leq -Q_\Sigma(v_2)$ **for** $v_2 \in \mathcal{P}^{\perp_\Sigma}$**,**

   such that that the QDF

$$Q_{\Psi_{\mathcal{N}}}(v_1) - Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v_2) + 2L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}}(v_1, v_2)$$

   is non-negative for $v_1 \in \mathcal{N}$ **and** $v_2 \in \mathcal{P}^{\perp_\Sigma}$**.**

- **Surprising condition is the non-negativity:**

$$Q_{\Psi_{\mathcal{N}}}(v_1) - Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v_2) + 2L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}}(v_1, v_2) \geq 0$$

  for $v_1 \in \mathcal{N}$ and $v_2 \in \mathcal{P}^{\perp_\Sigma}$. This condition is called the coupling condition. It expresses that the storage functions $Q_{\Psi_{\mathcal{N}}}$ and $Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}$ should be coupled via the $(N, \mathcal{P}^{\perp_\Sigma}, \Sigma)$ adapted bilinear differential form.

  Generalization of the well-known coupling condition of state space $\mathcal{H}_\infty$-theory involving solutions of algebraic Riccati equations.

- **Note that, for a given $\mathcal{P}_{\text{full}}$, the conditions are purely in terms of properties of the hidden behavior $\mathcal{N}$ and manifest plant behavior $\mathcal{P}$. No representations are involved.**

Outline of proof:

Fact: the case of a general, non-degenerate $\Sigma = \Sigma^* \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}[\zeta,\eta]$ can be reduced to the case that Σ is a constant, non-singular, symmetric matrix. Assume this holds.

(only if) Let $X, Z \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ be such that $X(\frac{d}{dt})$ is a minimal state map for $\mathcal{K}$, $Z(\frac{d}{dt})$ a minimal state map for $\mathcal{K}^\perp$ and such that

$$\frac{d}{dt}(X(\frac{d}{dt})v_1)^T(Z(\frac{d}{dt})v_2) = v_1^T v_2 \quad v_1 \in \mathcal{K}, v_2 \in \mathcal{K}^{\perp_\Sigma}.$$

(i.e. (X, Z) is a matched pair of minimal state maps for $\mathcal{K}$ and $\mathcal{K}^\perp$.)

1. $\mathcal{K}$ Σ-dissipative on $\mathbb{R}_-$ and $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma) \Rightarrow$ every storage function of $\mathcal{K}$ is of the form $|X(\frac{d}{dt})v_1|^2_K$, with the matrix $K = K^T > 0$.

2. Hence $-|Z(\frac{d}{dt})\Sigma v_2|^2_{K^{-1}}$ is a storage function of $\mathcal{K}^{\perp_\Sigma}$ as a $(-\Sigma)$-dissipative system.

3. We have, for all $v_1 \in \mathcal{K}$ and $v_2 \in \mathcal{K}^{\perp\Sigma}$:

$$\frac{d}{dt}|X(\frac{d}{dt})v_1|_K^2 \leq |v_1|_\Sigma^2 \quad \text{(dissipation inequality)},$$

$$-\frac{d}{dt}|Z(\frac{d}{dt})\Sigma v_2|_{K^{-1}}^2 \leq -|v_2|_\Sigma^2 \quad \text{(dissipation inequality)},$$

$$\frac{d}{dt}(X(\frac{d}{dt})v_1)^T(Z(\frac{d}{dt})\Sigma v_2) = v_1^T \Sigma v_2 \quad \text{(matching condition)},$$

Since $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, whence $\mathcal{P}^{\perp\Sigma} \subset \mathcal{K}^{\perp\Sigma}$, these relations also hold for $v_1 \in \mathcal{N}$ and $v_2 \in \mathcal{P}^{\perp\Sigma}$.

4. **Note that**

$$|X(\frac{d}{dt})v_1|^2_K + |Z(\frac{d}{dt})\Sigma v_2|^2_{K^{-1}} + 2(X(\frac{d}{dt})v_1)^T(Z(\frac{d}{dt})\Sigma v_2)$$
$$= |X(\frac{d}{dt})v_1 + K^{-1}Z(\frac{d}{dt})\Sigma v_2|^2_K.$$

for $v_1 \in \mathcal{N}$ and $v_2 \in \mathcal{P}^{\perp_\Sigma}$. Since K is positive definite, this QDF is non-negative.

5. **Since**

$$\frac{d}{dt}(X(\frac{d}{dt})v_1)^T(Z(\frac{d}{dt})\Sigma v_2) = v_1^T\Sigma v_2 = L_\Sigma(v_1, v_2)$$

for $v_1 \in \mathcal{N}$ and $v_2 \in \mathcal{P}^{\perp_\Sigma}$, the $(N, \mathcal{P}^{\perp_\Sigma}, \Sigma)$ adapted bilinear differential form is equal to

$$L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}}(v_1, v_2) = (X(\frac{d}{dt})v_1)^T(Z(\frac{d}{dt})\Sigma v_2).$$

6. **Finally define the required storage functions:**

$$Q_{\Psi_{\mathcal{N}}}(v_1) := |X(\frac{d}{dt})v_1|^2_K, v_1 \in \mathcal{N},$$

$$Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v_2) := -|Z(\frac{d}{dt})\Sigma v_2|^2_{K^{-1}}, v_2 \in \mathcal{P}^{\perp_\Sigma}.$$

These storage functions, together with the $(N, \mathcal{P}^{\perp_\Sigma}, \Sigma)$ adapted bilinear differential form $L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}}(v_1, v_2)$, yield the coupling condition:

$$\boxed{Q_{\Psi_{\mathcal{N}}}(v_1) - Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v_2) + 2L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}}(v_1, v_2) \geq 0}$$

for $v_1 \in \mathcal{N}$ and $v_2 \in \mathcal{P}^{\perp_\Sigma}$.

(if) Involves a construction of $\mathcal{K}$. Ingredients are the fine points of dissipative systems.

NON-UNIQUENESS OF STORAGE FUNCTIONS

Note: our main theorem is stated as an existence result: it requires the existence of storage functions that satisfy the coupling condition.

Storage functions of a given system are essentially non-unique.

Question: which storage functions of $\mathcal{N}$ and $\mathcal{P}^{\perp_\Sigma}$ are most likely to satisfy the coupling condition

$$Q_{\Psi_{\mathcal{N}}}(v_1) - Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v_2) + 2L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}}(v_1, v_2) \geq 0?$$

Answer:

- **the largest storage function $Q_{\Psi_{\mathcal{N}}^{\sup}}(v_1)$ for $\mathcal{N}$ (the required supply),**
- **the smallest storage function $Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}^{\inf}}(v_2)$ of $\mathcal{P}^{\perp_\Sigma}$ (the available storage).**

ALTERNATIVE FORMULATION OF THE MAIN RESULT

Theorem 7.4: $\mathcal{K} \in \mathfrak{L}^{\mathrm{v}}_{\mathrm{cont}}$ **described in the problem formulation exists if and only if the following conditions are satisfied:**

1. $\mathcal{N}$ **is** Σ**-dissipative,**
2. $\mathcal{P}^{\perp_\Sigma}$ **is** $(-\Sigma)$**-dissipative,**
3. **the QDF**

$$Q_{\Psi^{\mathrm{sup}}_{\mathcal{N}}}(v_1) - Q_{\Psi^{\mathrm{inf}}_{\mathcal{P}^{\perp_\Sigma}}}(v_2) + 2L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}}(v_1, v_2)$$

is non-negative for all $v_1 \in \mathcal{N}$ **and** $v_2 \in \mathcal{P}^{\perp_\Sigma}$**.**

Here, $Q_{\Psi^{\mathrm{sup}}_{\mathcal{N}}}$ **is the largest storage function of** $\mathcal{N}$ **as a** Σ**-dissipative system,** $Q_{\Psi^{\mathrm{inf}}_{\mathcal{P}^{\perp_\Sigma}}}$ **is the smallest storage function of** $\mathcal{P}^{\perp_\Sigma}$ **as a** $(-\Sigma)$**-dissipative system, and (as before)** $L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}}$ **is the** $(\mathcal{N}, \mathcal{P}^{\perp_\Sigma}, \Sigma)$**-adapted bilinear differential form.**

FROM GENERAL RESULT TO PARTICULAR REPRESENTATIONS

Statement of the main result does not use representations of $\mathcal{N}$ and $\mathcal{P}$. Hence: applicable to any particular representation of the full plant $\mathcal{P}_{\text{full}}$.

Procedure:

- **for a given representation of $\mathcal{P}_{\text{full}}$, compute representations of its hidden behavior $\mathcal{N}$ and its manifest plant behavior $\mathcal{P}$.**
- **Next: express the representation-free conditions of the main result in terms of the parameters of these representations. In general, these conditions will only involve basic matrix computations, ARE's, LMI's, etc.**
- **Use the general construction of the controlled behavior $\mathcal{K}$ to set up algoritms in terms of the parameters of these representations.**

Illustration: full plant $\mathcal{P}_{\text{full}}$ given in state space form:

PART II

SPECIAL CASES

APPLICATION: THE STATE SPACE $\mathcal{H}_\infty$ CONTROL PROBLEM

Assume $\mathcal{P}_{\text{full}}$ represented in input/state/output form by

$$\begin{array}{rcccccc} \frac{d}{dt}x & = & Ax & + & Bu & + & Gd \\ y & = & Cx & + & & + & Dd \\ f & = & Hx & + & Ju & & \end{array}$$

Problem: **find a controller in input/state/output representation, with y as input and u as output:**

$$\begin{array}{rcccc} \frac{d}{dt}x_c & = & A_c x_c & + & B_c y \\ u & = & C_c x_c & + & D_c y \end{array}$$

such that the controlled system is internally stable and its transfer matrix $G_{d\mapsto f}$ from d to f satisfies $\|G_{d\mapsto f}\|_{\mathcal{H}_\infty} \leq 1$.

Control variable: $c = (u, y)$, **to be controlled variable** (d, f).

Weighting functional $\Sigma = \begin{bmatrix} I_{\mathtt{d}} & 0 \\ 0 & -I_{\mathtt{f}} \end{bmatrix}$.

We already know: the following are equivalent:

1. there exists an implementable $\mathcal{K} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ such that in $\mathcal{K}$, d is input, f is output, the transfer matrix $G_{d \mapsto f}$ is **externally** stable, and satisfies $\|G_{d \mapsto f}\|_{\mathcal{H}_\infty} \leq 1$
2. $\mathcal{N}$ is Σ-dissipative, $\mathcal{P}^{\perp_\Sigma}$ is $(-\Sigma)$-dissipative, and the largest storage function of $\mathcal{N}$ and smallest storage function of $\mathcal{P}$ satisfy the coupling condition.

Note: provided $\mathcal{N}$ and $\mathcal{P}$ controllable. For this, assume (A, G) controllable pair, (H, A) an observable pair.

In order to simplify notation: $DD^T = I$ and $DG^T = 0$, $J^T J = I$ and $J^T H = 0$.

VERIFICATION OF THE CONDITIONS

Dissipativity of $\mathcal{N}$

Output nulling representation hidden behavior $\mathcal{N} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$:

$$\frac{d}{dt}x = Ax + \begin{bmatrix} G & 0 \end{bmatrix} \begin{bmatrix} d \\ f \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C \\ H \end{bmatrix} x + \begin{bmatrix} D & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} d \\ f \end{bmatrix}.$$

Fact: $\mathcal{N}$ is Σ-dissipative if and only if the Riccati inequality

$$-A^T K_{\mathcal{N}} - K_{\mathcal{N}} A - H^T H - K_{\mathcal{N}} G G^T K_{\mathcal{N}} + C^T C \geq 0$$

has a real symmetric solution $K_{\mathcal{N}}$.

All storage functions for $\mathcal{N}$: $x^T K_{\mathcal{N}} x$.

Largest storage function for $\mathcal{N}$: $x^T K_{\mathcal{N}}^+ x$, with $K_{\mathcal{N}}^+$ the largest real symmetric solution of the algebraic Riccati equation

$$-A^T K_{\mathcal{N}} - K_{\mathcal{N}} A - H^T H - K_{\mathcal{N}} G G^T K_{\mathcal{N}} + C^T C = 0.$$

Dissipativity of $\mathcal{P}^{\perp_\Sigma}$

Driving variable representation manifest plant behavior $\mathcal{P} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$:

$$\begin{aligned} \frac{d}{dt}x &= Ax + \begin{bmatrix} B & G \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix}, \\ \begin{bmatrix} d \\ f \end{bmatrix} &= \begin{bmatrix} 0 \\ H \end{bmatrix} x + \begin{bmatrix} 0 & I \\ J & 0 \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix}. \end{aligned}$$

Output nulling representation of $\mathcal{P}^{\perp_\Sigma}$ (with variable (d', f')):

$$\begin{aligned} \frac{d}{dt}z &= -A^T z + \begin{bmatrix} 0 & -H^T \end{bmatrix} \begin{bmatrix} d' \\ f' \end{bmatrix}, \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} B^T \\ G^T \end{bmatrix} z + \begin{bmatrix} 0 & J^T \\ -I & 0 \end{bmatrix} \begin{bmatrix} d' \\ f' \end{bmatrix}. \end{aligned}$$

Fact: $\mathcal{P}^{\perp_\Sigma}$ is $(-\Sigma)$-dissipative if and only if the Riccati inequality

$$AK_\mathcal{P} + K_\mathcal{P}A^T - GG^T - K_\mathcal{P}H^T H K_\mathcal{P} + BB^T \geq 0$$

has a real symmetric solution $K_\mathcal{P}$.

All storage functions for $\mathcal{P}^{\perp_\Sigma}$: $z^T K_\mathcal{P} z$.

Smallest storage function for $\mathcal{P}^{\perp_\Sigma}$: $z^T K_{\mathcal{P}}^- z$, with $K_{\mathcal{P}}^-$ the smallest real symmetric solution of the algebraic Riccati equation

$$AK_{\mathcal{P}} + K_{\mathcal{P}}A^T - GG^T - K_{\mathcal{P}}H^T H K_{\mathcal{P}} + BB^T = 0.$$

The coupling condition

x state of $\mathcal{N}$, z state of $\mathcal{P}^{\perp_\Sigma}$. Direct computation shows:

$$\frac{d}{dt} x^T z = d^T d' - f^T f'.$$

Hence: the $(\mathcal{N}, \mathcal{P}^{\perp_\Sigma}, \Sigma)$-adapted bilinear differential form is $x^T z$

Coupling condition: $x^T K_{\mathcal{N}} x - z^T K_{\mathcal{P}} z + 2x^T z \geq 0$ along state trajectories x of $\mathcal{N}$ and z of $\mathcal{P}^{\perp_\Sigma}$.

Trimness of representations ⇒ coupling condition becomes

$$\begin{bmatrix} K_{\mathcal{N}} & I \\ I & -K_{\mathcal{P}} \end{bmatrix} \geq 0.$$

This non-negativity is equivalent to the combined conditions

1. $K_{\mathcal{N}} > 0$,
2. $K_{\mathcal{P}} < 0$,
3. $K_{\mathcal{N}} \geq (-K_{\mathcal{P}})^{-1} \quad (\Leftrightarrow \rho(K_{\mathcal{N}} K_{\mathcal{P}}) \geq 1)$.

Here, ρ denotes the spectral radius.

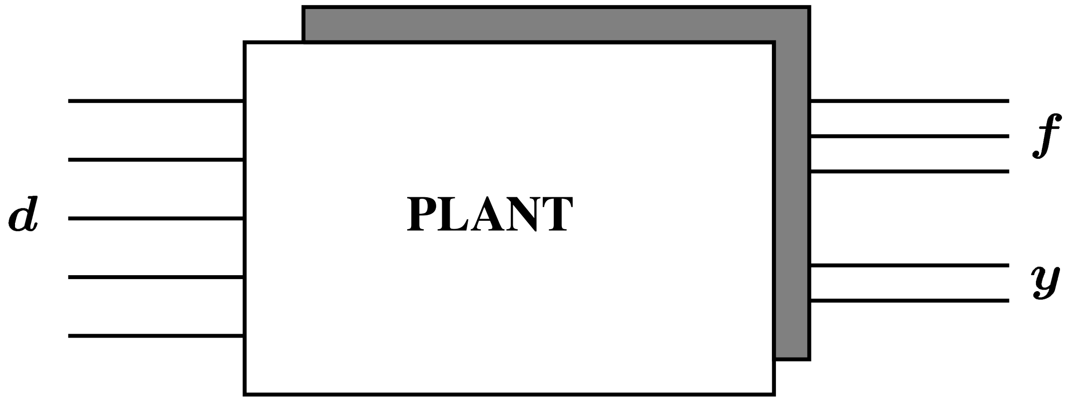
SOLUTION OF THE STATE SPACE $\mathcal{H}_\infty$ CONTROL PROBLEM

Theorem 7.5: **The following statements are equivalent:**

1. **There exists a feedback controller such that the controlled system is internally stable, and the closed loop transfer matrix $G_{d \mapsto f}$ satisfies $\|G_{d \mapsto f}\|_{\mathcal{H}_\infty} \leq 1$,**
2. **there exist real symmetric solutions of the algebraic Riccati equations, and the largest real symmetric solution $K_{\mathcal{N}}^{+}$ and smallest real symmetric solution $K_{\mathcal{P}}^{-}$ satisfy $K_{\mathcal{N}}^{+} > 0$, $K_{\mathcal{P}}^{-} < 0$, and $K_{\mathcal{N}}^{+} \geq (-K_{\mathcal{P}}^{-})^{-1}$.**

Also: formulas for input/state/output representations of the required feedback controllers.

THE $\mathcal{H}_\infty$ FILTERING PROBLEM



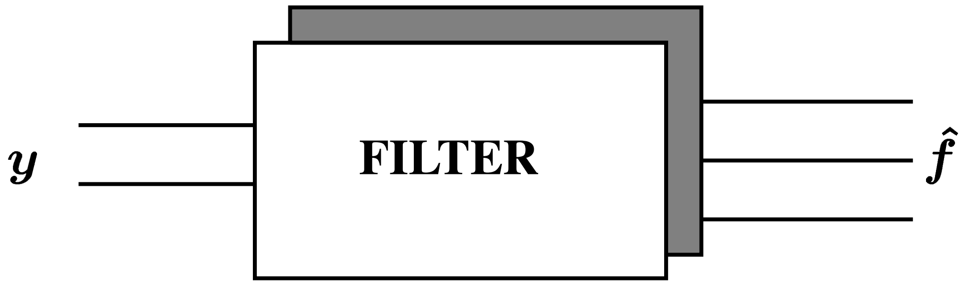
Our plant has three kinds of variables:

- **exogenous disturbance variables d,**
- **to be estimated variables f,**
- **measured variables y.**

Full plant behavior:

$$\mathcal{P}_{\text{full}} := \{(d, f, y) \mid (d, f, y) \text{ satisfies the plant equations}\}$$

We assume that $\mathcal{P}_{\text{full}}$ is a linear differential system, i.e. in $\mathfrak{L}^{\mathtt{d}+\mathtt{f}+\mathtt{y}}$.



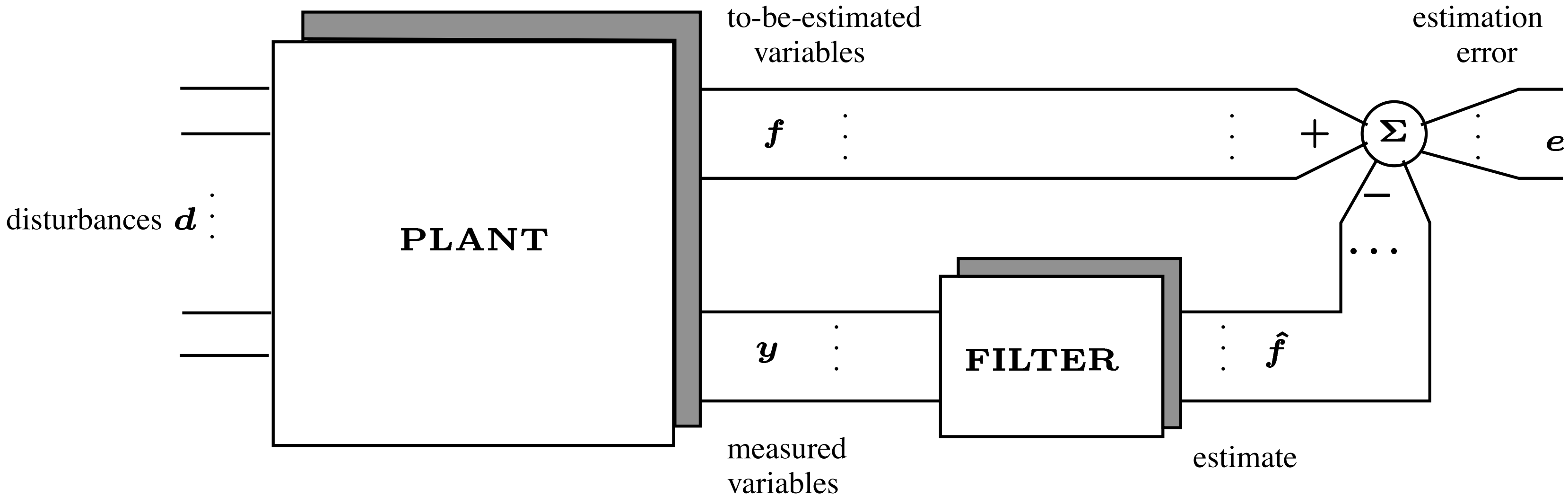
A **filter** is a system with manifest variable $(y, \hat{f})$, where y has the dimension of the measured plant variable, and $\hat{f}$ has the dimension of the to be estimated plant variable f.

We allow filters represented by equations of the form

$$F_1(\frac{d}{dt})y = F_2(\frac{d}{dt})\hat{f}.$$

In other words: a filter is a linear differential system $\mathcal{F} \in \mathfrak{L}^{\mathtt{y}+\mathtt{f}}$.

We aim at finding filters $\mathcal{F}$ such that in the interconnection through y of $\mathcal{P}_{\text{full}}$ and $\mathcal{F}$, $\hat{f}$ is an estimate of f (in a sense to be explained).



Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}+\mathtt{y}}$. For a given filter $\mathcal{F} \in \mathfrak{L}^{\mathtt{y}+\mathtt{f}}$, we define the estimation error behavior as the behavior $\mathcal{E} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ given by:

$$\mathcal{E} = \{(d, e) \mid \text{ there exists } (f, y, \hat{f}) \text{ such that } (d, f, y) \in \mathcal{P}_{\text{full}}, (y, \hat{f}) \in \mathcal{F}, e = f - \hat{f}\}.$$

We say: $\mathcal{F}$ implements $\mathcal{E}$ or: $\mathcal{E}$ is implemented by the filter $\mathcal{F}$.

Also: $\mathcal{E} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ is called implementable (with respect to $\mathcal{P}_{\text{full}}$).

FILTERING PROBLEMS

Given a full plant $\mathcal{P}_{\mathrm{full}}$, and a set of design specifications, the corresponding filtering problem is to find conditions for the existence of, and compute, a filter $\mathcal{F}$ such that the resulting estimation error behavior $\mathcal{E}$ satisfies the specifications.

Reformulation: given $\mathcal{P}_{\mathrm{full}}$ and a set of design specifications, find conditions for the existence of, and compute, a behavior $\mathcal{E} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ such that

- **$\mathcal{E}$ is implementable,**
- **$\mathcal{E}$ satisfies the specifications.**

Of course, after finding an implementable behavior $\mathcal{E}$ that satisfies the specifications, one still needs to compute an actual filter $\mathcal{F}$ that implements $\mathcal{E}$.

IMPLEMENTABILITY

Let $\mathcal{P}_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}+\mathtt{y}}$. Characterize all $\mathcal{E}$'s that are implementable.

It turns out that here the hidden behavior $\mathcal{N}$ associated with $\mathcal{P}_{\mathrm{full}}$ is crucial. Recall:

$$\mathcal{N} = \{(d, f) \mid (d, f, 0) \in \mathcal{P}_{\mathrm{full}}\}.$$

Theorem: $\mathcal{E} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ is implementable by a filter $\mathcal{F} \in \mathfrak{L}^{\mathtt{y}+\mathtt{f}}$ if and only if

$$\mathcal{N} \subset \mathcal{E}.$$

Moreover, if $\mathcal{E}$ is implementable, then it can be implemented by a filter $\mathcal{F} \in \mathfrak{L}^{\mathtt{y}+\mathtt{f}}$ such that in $\mathcal{F}$, y is input and $\hat{f}$ output.

$\mathcal{H}_\infty$ SPECIFICATIONS

In an $\mathcal{H}_\infty$-context, we will deal with controllable systems, and the specifications on $\mathcal{E} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}_{\text{cont}}$ are:

- **disturbance attenuation:**

$$\int_{-\infty}^{\infty} |e|^2 - |d|^2 dt \leq 0 \text{ for all } (d, e) \in \mathcal{E} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{f}}),$$

- **stability:**

$$(0, e) \in \mathcal{E} \Rightarrow \lim_{t \to \infty} e(t) = 0,$$

- **'liveness':**

in $\mathcal{E}$, d is free.

Liveness: in the estimation error behavior, no direct restrictions on the exogenous disturbances are allowed: every component of d is arbitrary.

It can be shown that these three specifications on $\mathcal{E}$ are equivalent to the the two following:

- $\mathcal{E}$ is Σ-dissipative on $\mathbb{R}_-$ (where $\Sigma = \text{blockdiag}(I_{\mathtt{d}}, -I_{\mathtt{f}})$,
- $\mathtt{m}(\mathcal{E}) = \mathtt{d}$.

Recall the implementability condition

- $\mathcal{N} \subset \mathcal{E}$.

Given $\mathcal{P}_{\text{full}}$, we define the $\mathcal{H}_\infty$ filtering problem as the problem of finding $\mathcal{E} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$, that satisfy these three conditions.

SOLUTION TO THE $\mathcal{H}_\infty$-FILTERING PROBLEM

Note that this problem is a special case of our general result, with $\mathcal{P}$ taken equal to $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{f}})$. Hence $\mathcal{P}^{\perp_\Sigma} = 0$, so (-Σ)-dissipativity of $\mathcal{P}^{\perp_\Sigma}$ is trivially satisfied.

The coupling condition degenerates to the condition that $\mathcal{N}$ should have a non-negative storage function.

Theorem: **Assume that $\mathcal{N} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}_{\mathrm{cont}}$. Then there exists $\mathcal{E} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}_{\mathrm{cont}}$ that satisfies the conditions of the $\mathcal{H}_\infty$-filtering problem if and only if**

$$\mathcal{N} \text{ is } \Sigma\text{-dissipative on } \mathbb{R}_-,$$

equivalently, there exists $\Psi \in \mathbb{R}^{(\mathtt{d}+\mathtt{f})\times(\mathtt{d}+\mathtt{f})}[\zeta, \eta]$, such that

$$Q_\Psi(d,e) \geq 0, \text{ and } \frac{d}{dt}Q_\Psi(d,e) \leq |d|^2 - |e|^2, \ (d,e) \in \mathcal{N}.$$

- **No representations are involved.** **The result can be applied to any particular representation of the full plant $\mathcal{P}_{\text{full}}$.**

- **For particular representations, checking the conditions comes down to basic matrix computations, ARE's, and LMI's.**

- **As an example,** **if the full plant is represented by $\dot{x} = Ax + Gd$, $y = Cx + Dd$, $f = Hx$, (assuming $DD^T = 0$, $C^T D = 0$, (H, A) observable) then the result can be applied to obtain that an $\mathcal{H}_\infty$ filter exists if and only the algebraic Riccati equation**

$$-A^T K - KA - H^T H - KGG^T K + C^T C = 0.$$

has a solution $K = K^T > 0$.

RECAP

- **We have given a representation free formulation of the $\mathcal{H}_\infty$ control problem.**
- **This problem has been generalized to a general problem on the existence of a dissipative behavior wedged in between two given behaviors, having maximal input cardinality.**
- **Necessary and sufficient conditions for the existence of such behavior have been found.**
- **These conditions are in terms of the existence of certain storage functions associated with the hidden behavior and manifest plant behavior. In particular, these storage functions should satisfy a coupling condition.**
- **The conditions are, again, representation free, and are hence applicable to any particular representation of the full plant $\mathcal{P}_{\text{full}}$.**

- We have applied our general result to the 'classical' state space $\mathcal{H}_\infty$ control problem, and have re-derived a version of the well-known solution in terms of two Riccati equations plus coupling condition.
- We have shown that our general problem formulation also has a class of $\mathcal{H}_\infty$ filtering problems as a special case.
- We have applied our general result to derive necessary and sufficient conditions for the existence of $\mathcal{H}_\infty$ filters.
- Again, these conditions are representation free, and are hence applicable to any particular representation of the full plant $\mathcal{P}_{\text{full}}$.